Refined tropical enumerative invariants

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Overview

(1) Refined enumerative invariants are functions (or just formal expressions) that depend on one parameter and are related to certain enumerative problems.
(2) The first examples were discovered by Göttsche-Shende (2014) and Block-Göttsche (2014). These refined invariants appeared as one-parameter deformations of some complex enumerative invariants.
(3) It was observed that, in some cases, the refined invariants interpolate between complex and real enumerative invariants. Since then the study of refined invariants and their interpretation have taken a central place in the tropical enumerative geometry.
(4) We review Block-Göttsche and refined broccoli invariants with a focus on their complex, real, and tropical enumerative meaning. Some challenging open problems will be discussed as well.

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Block-Göttsche refinement

Initial enumerative problems:

Over $\mathbb{C}$: Given a toric surface X (e.g., the plane) and a very ample divisor $D \in \mathrm{Pic}(X)$, compute the number of complex curves $C \in |D|$ of genus g passing through $-DK_X + g - 1$ generic points in X (i.e., the Gromov-Witten invariant $GW_g(X, D)$)

Over $\mathbb{R}$: In case of toric del Pezzo X, compute the number of real rational curves $C \in |D|$ passing through $-DK_X - 1$ generic real points in X and counted with appropriate weights ± 1 (i.e., the totally real Welschinger invariant $W_0(X, D)$)

Note: no enumerative invariant counting real curves $C \subset X$ of any positive genus g is known!

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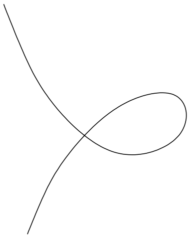
Welschinger sign of a real nodal curve

$$w(C) = (-1)^{s(C)}$$

$s(C)$ - the number of real elliptic nodes of C



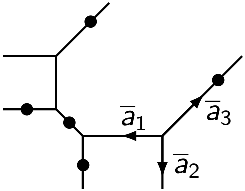
$$\left.\begin{matrix}\cdot\\ \cdot\end{matrix}\right\} s(C)$$

A tropical solution (Mikhalkin, 2002-05)

$$\text{Over } \mathbb{C}: \quad GW_g(X,D) = GW_g^{trop}(X,D) = \sum_{T \in \mathcal{T}(D^{tr},g,\boldsymbol{x})} M(T),$$

where $\mathcal{T}(D^{tr},g,\boldsymbol{x})$ is the set of plane trivalent tropical curves of class D^{tr} and genus g, passing through a configuration $\boldsymbol{x} \subset \mathbb{R}^2$ of $-DK_X + g - 1$ generic points, and

$$M(T) = \prod_{V \in \mathrm{Vert}(T)} \mu(T,V), \quad \mu(T,V) = |\overline{a}_1 \wedge \overline{a}_2|$$



$$\overline{a}_1 + \overline{a}_2 + \overline{a}_3 = 0$$

$$\overline{a}_1, \overline{a}_2, \overline{a}_3 \in \mathbb{Z}^2 \setminus \{0\}$$

$$\text{Over } \mathbb{R}: \quad W_0(X,D) = W_0^{trop}(X,D) = \sum_{T \in \mathcal{T}(D^{tr},0,\boldsymbol{x})} W(T),$$

$$W(T) = \prod_{V \in \mathrm{Vert}(T)} w(T,V),$$

$$w(T,V) = \begin{cases} (-1)^{(\mu(T,V)-1)/2}, & \text{if } \mu(T,V) \text{ is odd,} \\ 0, & \text{if } \mu(T,V) \text{ is even} \end{cases}$$

Theorem (Itenberg-Kharlamov-Sh., 2009)

For any toric surface X, a very ample divisor $D \in \mathrm{Pic}(X)$, and $g \geq 0$,

$$W_g^{trop}(X, D) \stackrel{def}{=} \sum_{T \in \mathcal{T}(D^{tr}, g, \boldsymbol{x})} W(T)$$

is **invariant** *of the choice of the configuration $\boldsymbol{x}$.*

Corollary

The count of real curves $C \in |D|$ on a real toric surface X having any given genus $g \geq 0$ and equipped with Welschinger signs, is **invariant** *as long as the point constraint z is* **close to the tropical limit**, *i.e.,* $z = \{(t^{-a_i}(\alpha_i + O(t^{>0})), t^{-b_i}(\beta_i + O(t^{>0}))),\ i = 1, 2, ...\}$, *where* $\boldsymbol{x} = \{(a_i, b_i),\ i = 1, 2, ...\}$.

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Refinement (Block-Göttsche, 2013-16)

For any $\alpha \in \mathbb{R}$ and a formal parameter y let

$$[\alpha]_y^- = \frac{y^{\alpha/2} - y^{-\alpha/2}}{y^{1/2} - y^{-1/2}} \qquad \left(\text{note: } \lim_{y \to 1} [\alpha]_y^- = \alpha\right)$$

Theorem (Block-Göttsche, Itenberg-Mikhalkin)

$$BG_y(X, D, g) \overset{def}{=} \sum_{T \in \mathcal{T}(D^{tr}, g, \boldsymbol{x})} bg_y(T), \quad bg_y(T) = \prod_{V \in \mathrm{Vert}(T)} [\mu(T, V)]_y^-,$$

does not depend on the choice of a generic configuration $\boldsymbol{x} \subset \mathbb{R}^2$ of $-DK_X + g - 1$ points.

$BG_y(X, D, g)$ is a symmetric Laurent polynomial in y of degree $p_a(D) - g$, and it satisfies

$$BG_1(X, D, g) = GW_g(X, D), \quad BG_{-1}(X, D, g) = W_g^{trop}(X, D)$$

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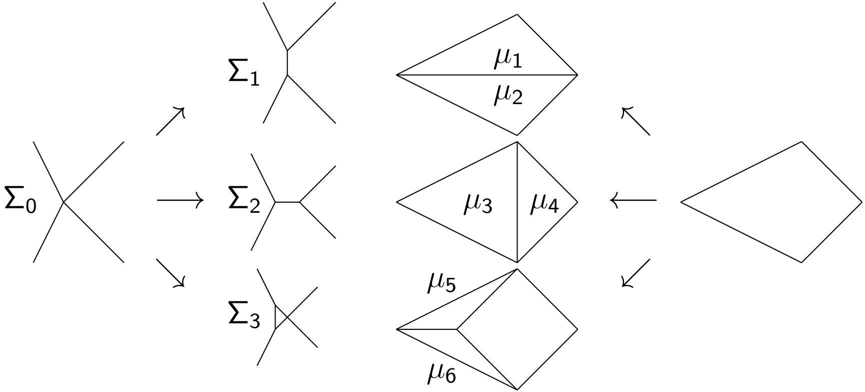
Proof of the invariance:

The main wall-crossing event is "4-valent vertex trifurcation"



(a) (b)

$$\mu_1\mu_2 = \mu_3\mu_4 + \mu_5\mu_6 \implies \left\{ \Sigma_1 \leftarrow \begin{array}{c}\text{4-valent}\\ \text{vertex}\end{array} \rightarrow \Sigma_2 + \Sigma_3 \right\}$$

In the refined setting:

$$
\begin{aligned}
&(y^{\mu_1/2} - y^{-\mu_1/2})(y^{\mu_2/2} - y^{-\mu_2/2}) \\
&\qquad = (y^{\mu_3/2} - y^{-\mu_3/2})(y^{\mu_4/2} - y^{-\mu_4/2}) \\
&\qquad\qquad + (y^{\mu_5/2} - y^{-\mu_5/2})(y^{\mu_6/2} - y^{-\mu_6/2})
\end{aligned}
$$

Follows from the *linear* relations

$$
\begin{cases}
\mu_1 + \mu_2 = \mu_3 + \mu_4, \\
\mu_1 - \mu_2 = \mu_6 - \mu_5, \\
\mu_3 - \mu_4 = \mu_5 + \mu_6
\end{cases}
$$

Questions:

(1) What is a conceptual algebraic-geometric meaning of BG_y?
Partial answers and conjectures: Block-Göttshe, Filippini-Stoppa, Bousseau, Mikhalkin, Mandel-Ruddat

(2) Are there WDVV type formulas for the genus zero refined invariants, which specialize to WDVV formulas for Gromov-Witten invariants (e.g., Kontsevich formula) as $y = 1$ and specialize to Solomon's formulas for open GW invariants as $y = -1$? Can one define "refined" quantum cohomology? - Completely open

Remark: There exists a refined Caposaro-Harris (aka, symplectic sum) formula which specializes to C-H type formulas for Gromov-Witten and (algebraic and tropical) Welschinger invariants as $y = 1$, resp., $y = -1$

(3) Are there other refined tropical invariants?
Some examples: Göttsche-Schroeter, Schroeter-Sh., Mandel-Ruddat, Itenberg-Sh., Blomme etc.

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Complex formula

$$GW_g(d,\alpha,\beta)=\sum_{\beta_k>0} k\cdot GW_g(d,\alpha+\overline{e}_k,\beta-\overline{e}_k)$$
$$+\sum\frac{1}{|\mathrm{Aut}|}\binom{\alpha}{\alpha^{(1)},...,\alpha^{(m)}}\frac{(n-1)!}{n_1!...n_m!}$$
$$\times\prod_{i=1}^m\left(\binom{\beta^{(i)}}{\beta^{(i)}-\gamma^{(i)}}\prod_{\gamma_k^{(i)}>0}k^{\gamma_k^{(i)}}\cdot GW_{g_i}(d_i,\alpha^{(i)},\beta^{(i)})\right)$$

where $GW_g(d,\alpha,\beta)$ is relative GW invariant,

$GW_g(d)=GW_g(d,0,d\overline{e}_1)$ is absolute GW invariant, and

$$\sum_{i=1}^m d_i=d-1$$

Real formula

$$W_g(d,\alpha,\beta)=\sum_{k\ \text{odd},\beta_k>0}W_g(d,\alpha+\overline{e}_k,\beta-\overline{e}_k)$$

$$+\sum\frac{1}{|\mathrm{Aut}|}\binom{\alpha}{\alpha^{(1)},...,\alpha^{(m)}}\frac{(n-1)!}{n_1!...n_m!}$$

$$\times\prod\left(\binom{\beta^{(i)}}{\beta^{(i)}-\gamma^{(i)}}W_{g_i}(d_i,\alpha^{(i)},\beta^{(i)})\right)$$

where $W_g(d,\alpha,\beta)$ is relative W invariant in a neighborhood of the tropical limit, $W_0(d)=W_0(d,0,d\overline{e}_1)$ is genuine W invariant, and

$$\sum_{i=1}^{m}d_i=d-1$$

Complex tropical formula

$$GW_g^{trop}(d,\alpha,\beta)=\sum_{\beta_k>0}k\cdot GW_g^{trop}(d,\alpha+\overline{e}_k,\beta-\overline{e}_k)$$

$$+\sum\frac{1}{|\mathrm{Aut}|}\binom{\alpha}{\alpha^{(1)},...,\alpha^{(m)}}\frac{(n-1)!}{n_1!...n_m!}$$

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where

$$GW_g^{trop}(d,\alpha,\beta)=\sum_T\left(\prod_{V\in\mathrm{Vert}(T)}\mu(T,V)\cdot\prod_{\alpha_k>0}k^{-\alpha_k}\right)$$

$$\sum_{i=1}^m d_i=d-1$$

Refined formula

$$BG_y(d,g,\alpha,\beta) = \sum_{\beta_k>0} [k]_y^- BG_y(d,g,\alpha+\overline{e}_k,\beta-\overline{e}_k)$$

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$$\times \prod \left(\binom{\beta^{(i)}}{\beta^{(i)}-\gamma^{(i)}} \prod_{\gamma_k^{(i)}>0} ([k]_y^-)^{\gamma_k^{(i)}} \cdot BG_y(d_i,g_i,\alpha^{(i)},\beta^{(i)}) \right)$$

where

$$BG_y(d,g,\alpha,\beta) = \sum_T \prod_{k\geq 1} ([k]_y^-)^{-\alpha_k} \prod_{V\in \mathrm{Vert}(T)} [\mu(T,V)]_y^-$$

$$\sum_{i=1}^m d_i = d-1$$

Block-Göttsche invariants from the wall-crossing (Filippini-Stoppa)

The wall-crossing in the tropical vertex theory consists in the computation of the commutators

$$[\theta_{(a_1,b_1),f_1}, \theta_{(a_2,b_2),f_2}]$$

of automorphisms $\theta_{(a,b),f}$ of the $\mathbb{C}[[t]]$-algebra

$$A = \mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$$

where

$$f = 1 + tx^a y^b g(x^a y^b, t), \quad g \in \mathbb{C}[z][[t]], \quad (a, b) \in \mathbb{Z}^2 \setminus \{0\},$$

$$\theta_{(a,b),f}(x) = xf^{-b}, \quad \theta_{(a,b),f}(y) = yf^a.$$

Remark: $\theta^*_{(a,b),f}\omega = \omega$, where $\omega = \frac{dx \wedge dy}{xy}$

Theorem (Gross-Pandharipande-Siebert)

$$\left[\theta_{(0,1),(1+ty)^{-\ell_2}},\theta_{(1,0),(1+tx)^{\ell_1}}\right]=\prod^{\rightarrow}\theta_{(a,b),f_{(a,b)}},$$

$$\log f_{(a,b)}=\sum kc^k_{a,b}[\ell_1,\ell_2](tx)^{ak}(ty)^{kb},$$

$$c^k_{a,b}[\ell_1,\ell_2]=\sum_{|\mathrm{P}_a|=ka,\#\mathrm{P}_a=\ell_1}\ \sum_{|\mathrm{P}_b|=kb,\#\mathrm{P}_b=\ell_2}N_{a,b}[\mathrm{P}_a,\mathrm{P}_b],$$

$$N_{a,b}[\mathrm{P}_a,\mathrm{P}_b]=\sum_{\mathrm{S}_a\ge\mathrm{P}_a,\ \mathrm{S}_b\ge\mathrm{P}_b)}\frac{\Phi(\mathrm{P}_a,\mathrm{P}_b,\mathrm{S}_a,\mathrm{S}_b)}{\prod\mathrm{S}_a\prod\mathrm{S}_b}N^{trop}_{a,b}(\mathrm{S}_a,\mathrm{S}_b),$$

$N^{trop}_{a,b}(\mathrm{S}_a,\mathrm{S}_b)$ counts plane rational tropical curves with $\#\mathrm{S}_a$ ends directed by $(0,-1)$, lying on fixed lines and having weight distribution S_a, and with $\#\mathrm{S}_b$ ends directed by $(-1,0)$, lying on fixed lines and having weight distribution S_b, and with one end of weight k directed by (b,a).

Comments:

(1) In the right-hand side of the first formula, the vectors (a, b) range all primitive integral vectors in the positive quadrant ordered by an increasing slope.

(2) $N_{a,b}(\mathrm{P}_a, \mathrm{P}_b)$ is a logarithmic GW invariant of the toric surface $X_{a,b}$ associated with the triangle $\mathrm{conv}\{(0,0),(a,0),(0,b)\}$. It counts complex rational curves on the blow up $\widetilde{X}_{a,b} \to X_{a,b}$ at ℓ_1 (resp., ℓ_2) points on the toric divisor D_a (resp., D_b) associated with the segment $[(0,0),(a,0)]$ (resp., $[(0,0),(0,b)]$). The curves are constrained by the conditions to touch the incline toric divisor at one point with multiplicity k and to realize the divisor class $kL - \sum p_i E_i - \sum p'_j E'_j$ with $p_i, p'_j \geq 0$ dictated by the ordered partitions $\mathrm{P}_a, \mathrm{P}_b$.

(3) $N_{a,b}(\mathrm{P}_a, \mathrm{P}_b)$ can be expressed via relative GW invariants $N_{a,b}(\mathrm{S}_a, \mathrm{S}_b)$ counting complex rational curves on $X_{a,b}$ with tangency conditions on D_a and D_b dictated by refined partitions $\mathrm{S}_a \geq \mathrm{P}_a$, $\mathrm{S}_b \geq \mathrm{P}_b$, respectively. In turn, $N_{a,b}(\mathrm{S}_a, \mathrm{S}_b) = N^{trop}_{a,b}(\mathrm{S}_a, \mathrm{S}_b)$ by Mikhalkin's correspondence.

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q-deformations from Donaldson-Thomas theory

$$\theta_1 = \theta_{(1,0),(1+tx)^{\ell_1}}, \quad \theta_2 = \theta_{(0,1),(1+ty)^{\ell_2}}$$

admit a q deformation acting on the $\mathbb{C}[[t]]$-algebra $\mathbb{C}[\widehat{x}, \widehat{x}^{-1}, \widehat{y}, \widehat{y}^{-1}][[t]]$ generated by quantum variables $\widehat{x}, \widehat{y}$ satisfying

$$\widehat{x}\widehat{y} = q\widehat{y}\widehat{x}$$

Namely,

$$\begin{cases} \widehat{\theta}_1(\widehat{x}) = \widehat{x}, \\ \widehat{\theta}_1(\widehat{y}) = \widehat{y}(1 + q^{1/2}t\widehat{x})^{\ell_1} \end{cases} \quad \begin{cases} \widehat{\theta}_2(\widehat{x}) = \widehat{x}(1 + q^{1/2}t\widehat{y})^{\ell_2}, \\ \widehat{\theta}_2(\widehat{y}) = \widehat{y} \end{cases}$$

Theorem (Filippini-Stoppa)

The commutator

$$\widehat{\theta}_2^{-1}\widehat{\theta}_1\widehat{\theta}_2\widehat{\theta}_1^{-1}$$

admits a similar development, where in the right-hand side, the numerical tropical invariants $N^{trop}_{(a,b)}(S_a, S_b)$ should be replaced with the q-deformed tropical invariants $\widehat{N}^{trop}_{(a,b)}(S_a, S_b)$ that coincide with the corresponding relative Block-Göttsche invariants $BG_q(X_{a,b}, S_a, S_b)$.

Quantum invariants (Mikhalkin, 2017)

Lemma (Mikhalkin)

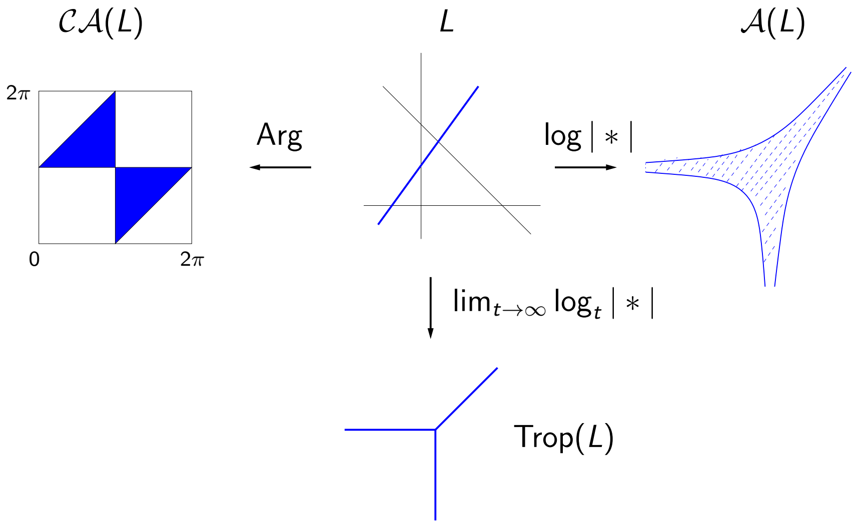
Let $C = \{F(z,w)=0\} \subset (\mathbb{C}^*)^2$ be a dividing real algebraic curve, i.e., $C \setminus C(\mathbb{R}) = C_+ \cup C_-$ - two conjugate connected components. Suppose that $\mathrm{Closure}(C) \subset \mathbb{P}^2$ intersects each coordinate axis only in real points. Then

$$\int_{C_+} \frac{dx \wedge dy}{xy} = \varkappa \frac{\pi^2}{2}, \quad \text{where } \varkappa \in \mathbb{Z} \quad \text{and } x = |z|,\ y = |w|.$$

$QI(C, C_+) \overset{\text{def}}{=} \varkappa$ is called the **quantum index** of (C, C_+).

Hint: The integral is the signed area of the amoeba $\mathcal{A}(C_+)$, which turns to be equal to the signed area of the coamoeba $\mathcal{CA}(C_+)$, while the coamoeba is a polygonal object with vertices in $(\pi\mathbb{Z})^2$

Example


$\mathcal{CA}(L)$
L
$\mathcal{A}(L)$
2π
0
2π
Arg
$\log|*|$
$\lim_{t\to\infty}\log_t|*|$
$\mathrm{Trop}(L)$

Theorem (Mikhalkin)

The number $W_0^{\varkappa}(\mathbb{P}^2, d)^*$ *of real plane rational oriented curves of degree* d*, passing through a fixed configuration of* $3d-1$ *real points on the coordinate axes or another configuration obtained by replacement of any point by the opposite one, having quantum index* $\varkappa$*, and counted with Welschinger signs is an* **invariant** *of the point constraint. Moreover,*

$$\sum_{\varkappa}(-y)^{\varkappa} W_0^{\varkappa}(\mathbb{P}^2, d)^* = BG_y^*(\mathbb{P}^2, d, 0) \cdot (y^{1/2} - y^{-1/2})^{3d-2}$$

Remarks:

(1) $BG_y^*(\mathbb{P}^2, d, 0)$ is the relative rational Block-Göttsche invariant satisfying relative constraints but without multiplicative contribution of the constraints.

(2) $(y^{1/2} - y^{-1/2})^{3d-2}$ is the common denominator of all weights $bg_y(T)$ of the corresponding tropical curves T.

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Quantization in arbitrary genus

Theorem (Itenberg-Sh.)

For any $d > 0$, and $g \geq 0$, the number $W^+_{g,\varkappa}(\mathbb{P}^2, 2d)$ of real plane dividing oriented curves

(1) having degree $2d$, genus g, and $QI = 4\varkappa$,

(2) passing through any fixed configuration of $3d + g - 1$ generic real points in a neighborhood of the tropical limit,

(3) quadratically tangent to the coordinate axes and having all but finitely many real points in the positive quadrant,

(4) counted with modified Welschinger signs,

is an **invariant** of the point constraint. Moreover,

$$\sum_k (-y)^{\varkappa/4} W^+_{g,\varkappa}(\mathbb{P}^2, d) = BG_y(\mathbb{P}^2, d, g) \cdot (y^{1/2} - y^{-1/2})^{3d+g-2}$$

Remarks: (1) Enumeration of oriented dividing curves perfectly fits the setting of symplectic geometry (and string theory) in which the counted objects are Riemann surfaces with boundaries on Lagrangian submanifolds. In the case of real algebraic varieties these surfaces with boundaries are just the halves of real dividing algebraic curves.
(2) It is interesting that the enumeration of real dividing curves determines the count of complex curves. This confirms the fact that the numbers of real and complex curves in the considered problems are asymptotically equivalent on the logarithmic scale.

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Refined broccoli invariants (Göttsche-Schroeter)

For any $a \in \mathbb{R}$ and a formal parameter y let

$$[a]_y^+ = \frac{y^{a/2} + y^{-a/2}}{y^{1/2} + y^{-1/2}}$$

and define the **Göttsche-Schroeter weight** of a trivalent rational plane tropical curve T with marked points on edges and at vertices

$$gs_y(T) = \prod_{V \text{ unmarked}} [\mu(T,V)]_y^- \cdot \prod_{V \text{ marked}} [\mu(T,V)]_y^+$$

Theorem (Göttsche-Schroeter, Gathmann-Markwig-Schroeter, Markwig-Rau)

(1) If $n_0 + 2n_1 = -DK_X - 1$ (i.e., $g = 0$), then the expression

$$GS_y(X, D, 0, (n_0, n_1)) := \sum_{T \in \mathcal{T}(D^{tr}, 0, \boldsymbol{x})} gs_y(T)$$

does not depend on the choice of a generic configuration $\boldsymbol{x} \subset \mathbb{R}^2$ of $n_0 + n_1$ points, provided that n_0 points lie on edges of T and n_1 points are trivalent vertices of T.

(2) If X is toric del Pezzo, then $GS_y(X, D, 0, (n_0, n_1))$ is a symmetric Laurent polynomial in y of degree $p_a(D)$, and it satisfies

$$GS_1(X, D, 0, (n_0, n_1)) = \langle \tau_0(2)^{n_0} \tau_1(2)^{n_1} \rangle_{X,D}^{g=0}$$

$$GS_{-1}(X, D, 0, (n_0, n_1)) = W_0(X, D, (n_0, n_1))$$

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Remark: Here the mixed Welschinger invariant

$$W_0(X, D, (n_0, n_1))$$

counts real rational curves $C \in |D|$ in the surface X passing through n_0 real points and n_1 pairs of complex conjugate points (in general position) and equipped with Welschinger signs.

The mystery of refined broccoli invariants:

(1) GS_1 is the descendant Gromov-Witten invariant, i.e., the integral over the moduli space $\mathcal{M}_{0,n}(\mathbb{P}^2, d)$ of the product of the evaluation pull-backs of the point classes and of the ψ-classes (Chern classes of the tautological bundles).

(2) The enumerative meaning of GS_1 (Graber-Kock-Pandharipande): it counts complex plane rational curves of degree d passing through $n_0 + n_1$ points in the plane and having prescribed tangents at n_1 points.

(3) GS_{-1} equals the mixed Weslchinger invariant which counts signed real plane rational curves of degree d passing through n_0 real points and n_1 pairs of complex conjugate points.

(4) The number of real plane rational curves of degree d passing through $n_0 + n_1$ real points and having prescribed tangents at n_1 points **is not an invariant**, and **is not equal to** GS_{-1}. Furthermore, there is **no** any correspondence statement over $\mathbb{R}$ for tropical broccoli curves.

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Extension to higher descendant invariants

Let X be a toric surface, $D \in \mathrm{Pic}(X)$ ample, $\overline{n} = (n_0, n_1, ...)$ such that $\sum_k (k+1)n_k = DK_X - 1$. Denote by $\mathcal{T}_0(X, D, \overline{n}, \boldsymbol{x})$ the set of plane rational curves T of degree D, matching point constraint $\boldsymbol{x}$, and having n_k marked vertices of valency $k+2$, $k \geq 0$, while other vertices are trivalent. For a marked vertex $V \in \mathrm{Vert}(T)$ of valency $k+2$ put

$$rb(T, V) = \begin{cases} 1, & k = 0, \\ [\mu(T, V)]_y^+, & k = 1, \\ \sum \prod [\mu(v)]_y^+, & k > 1, \end{cases}$$

where the sum ranges over all deformations of V into a trivalent tree and the product ranges over all (trivalent) vertices in such a tree. Put

$$RB_y(T) = \prod_{V \text{ unmarked}} [\mu(T, V)]_y^- \cdot \prod_{V \text{ marked}} rb_y(T, V)$$

Theorem (Blechman-Sh., Markwig-Rau)

The expression

$$RB_y(X,D,0,\overline{n}) = \sum_{T\in\mathcal{T}_0(X,D,\overline{n},\boldsymbol{x})} RB_y(T)$$

does not depend on the choice of a generic constraint $\boldsymbol{x}$.
If X *is toric del Pezzo, then*

$$RB_1(X,D,0) = \prod_{k\ge 2}\left(\frac{(k+2)!(k+1)!}{3\cdot 2^{k+2}}\right)^{n_k}\cdot\left\langle\prod_{k\ge 0}\tau_k(2)^{n_k}\right\rangle_{X,D}^{g=0}$$

Remark:

$$RB_y(X,D,\overline{n}) \underset{y\to -1}{\longrightarrow} \infty \quad \text{if} \quad \sum_{k\ge 2} n_k > 0$$

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Refined broccoli invariants for $g = 1$

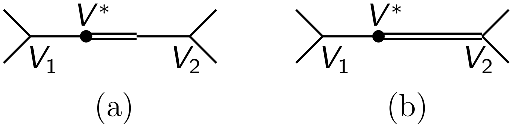
Two types of plane elliptic tropical curves with marked trivalent vertices:

(1) the image of the cycle is not contained in a line

$$RB_y(T) = \prod_{V \text{ unmarked}} [\mu(T, V)]_y^- \cdot \prod_{V \text{ marked}} rb_y(T, V)$$

(2) the image of the cycle is contained in a line (flat cycle), and its trivalent vertex V^* is marked

$$RB_y(T) = rb_y^*(T, V^*) \cdot \prod_{V \text{ unmarked}} [\mu(T, V)]_y^- \cdot \prod_{V \neq V^* \text{ marked}} rb_y(T, V)$$



(1) case (a)

$$rb_y(T,V^*)=\frac{2}{y^{1/2}+y^{-1/2}}\cdot\frac{[m_1]_y^-[m_2]_y^-}{[m]_y^-}$$

(2) case (b)

$$rb^*(T,V^*)=\frac{[\frac{m_1}{m}\mu(T,V_2)]_y^-[\frac{m_2}{m}\mu(T,V_2)]_y^-}{[\mu(T,V_2)]_y^-}-\frac{[m_1]_y^-[m_2]_y^-}{[m]_y^-}$$

Theorem (Schroeter-Sh., Sh.-Sinichkin)

Let $2n_0 + n_1 = -DK_X$. The expression

$$RB_y(X, D, 1, (n_0, n_1)) = \sum_{T \in \mathcal{T}_1(X, D, (n_0, n_1), \boldsymbol{x})} RB_y(T)$$

does not depend on the choice of a generic constraint $\boldsymbol{x}$.
At $y = 1$, $RB_1(X, D, 1, (n_0, n_1))$ equals the number of complex elliptic curves on X in the linear system $|D|$, passing through $n_0 + n_1$ points so that at n_1 points they have prescribed tangents.

Note: Extension to higher genera appears to be problematic.

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Kontsevich formula

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} d_1^2 d_2 \left(d_2 \binom{3d-4}{3d_1-2} - d_1 \binom{3d-4}{3d_1-1} \right)$$

$$N_d = GW_0(\mathbb{P}^2, d), \quad N_0 = 0,\ N_1 = N_2 = 1,\ N_3 = 12,\ \dots$$

WDVV for rational descendant GW invariants of the plane

(1) String equation

$$\left\langle \tau_0(0)\prod_{k\in N}\tau_{r_k}(2)\right\rangle_d = \sum_{\substack{k\in N\\ r_k>0}} \left\langle \tau_{r_k-1}(2)\prod_{k\neq k'\in N}\tau_{r_{k'}}(2)\right\rangle_d$$

(2) Divisor equation

$$\left\langle \tau_0(1)\prod_{k\in N}\tau_{r_k}(2)\right\rangle_d = d\left\langle \prod_{k\in N}\tau_{r_k}(2)\right\rangle_d$$

(3) WDVV equation

$$\left\langle \tau_0(1)\prod_{k\in N}\tau_{r_k}(2)\right\rangle_d + \sum D\left\langle \tau_0(e)\tau_0(0)\prod_{k\in N_1}\tau_{r_k}(2)\right\rangle_{d_1}$$

$$\times\left\langle \tau_0(f)\tau_{r_1}(2)\tau_{r_2}(2)\prod_{k\in N_2}\tau_{r_k}(2)\right\rangle_{d_2}$$

$$=\sum D\left\langle \tau_0(e)\tau_{r_{r_1}}(2)\prod_{k\in N_1}\tau_{r_k}(2)\right\rangle_{d_1}\cdot\left\langle \tau_0(f)\tau_0(1)\tau_{r_2}(2)\prod_{k\in N_2}\tau_{r_k}(2)\right\rangle_{d_2}$$

where $d_1+d_2=d$, $e+f=2$, $N_1\coprod N_2=N\setminus\{1,2\}$, $r_1>0$,

$$D=\binom{d}{d_1}^{-3}$$

First Solomon's formula

$$W_{d,m} = \sum_{2d'+d''=2} (-1)^{d'} 2^{3d'-2} N_{d'} W_{d'',m-3d'} (d')^2 d''$$

$$\times \left(d'' \binom{m-2}{3d'-2} - 2d' \binom{m-2}{3d'-1} \right)$$

$$+ \sum_{\substack{d_1+d_2=d \\ m_1+m_2=m-1}} W_{d_1,m_1} W_{d_2,m_2} d_1 \binom{m-2}{m_1}$$

$$\times \left(d_2 \binom{3d-2m-1}{3d_1-2m_1-2} - d_1 \binom{3d-2m-1}{3d_1-2m_1-1} \right)$$

Second Solomon's formula

$$
\begin{aligned}
W_{d,m} = & (-1)^{d/2-1} 2^{d/2-4} d^2 N_{d/2} \delta_m^{3d/2-1} \\
& - \sum_{2d'+d''=d} (-1)^{d'} 2^{3d'-1} N_{d'} W_{d'',m-3d'} (d')^3 d'' \binom{m-1}{3d'-1} \\
& \quad + \sum_{\substack{d_1+d_2=d \\ m_1+m_2=m-1}} W_{d_1,m_1} W_{d_2,m_2} d_1 \binom{m-1}{m_1} \\
& \quad \times \left(d_2 \binom{3d-2m-1}{3d_1-2m_1-2} - d_1 \binom{3d-2m-1}{3d_1-2m_1-1} \right)
\end{aligned}
$$

Appendix B: Tropical vertex

$$\mathcal{A} = \mathrm{Aut}_{\mathbb{C}[[t]]}(\mathbb{C}^* \times \mathbb{C}^* \times \mathrm{Spec}\mathbb{C}[[t]])$$
$$\simeq \mathrm{Aut}_{\mathbb{C}[[t]]}(\mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]),$$
$$f = 1 + tx^a y^b g(x^a y^b, t), \quad g \in \mathbb{C}[z][[t]], \quad (a, b) \in \mathbb{Z}^2 \setminus \{0\},$$
$$\theta_{(a,b),f}(x) = xf^{-b}, \quad \theta_{(a,b),f}(y) = yf^a,$$
$$\theta^{-1}_{(a,b),f} = \theta_{(a,b),f^{-1}},$$
$$\mathbb{V} = \left[\overline{\langle \theta_{(a,b),f} \ : \ (a,b), \ f \rangle}\right]_{(t)} \subset \mathcal{A},$$
$$\mathbb{V} \subset \{\theta \in \mathcal{A} \ : \ \theta^* \omega = \omega\}, \quad \omega = \frac{dx \wedge dy}{xy}.$$

$$X_{a,b} = \mathrm{Tor}(\mathrm{conv}\{(0,0),(a,0),(0,b)\}), \quad (a,b) \in \mathbb{N}^2,$$

$$p_1 + ... + p_{l_a} = ka, \quad p'_1 + ... + p'_{l_b} = kb, \quad p_i, p'_j \geq 0,$$

$$\nu : X_{a,b}(\mathrm{P}) = \mathrm{Bl}_{l_a+l_b}(X_{a,b}) \to X_{a,b},$$

$$\beta_k(\mathrm{P}) = \nu^*(\beta_k) - \sum p_i E_i - \sum p'_j E'_j,$$

$N_{a,b}(\mathrm{P})$ is the GW invariant counting rational curves in $X^o_{a,b}(\mathbb{P})$ of class $\beta_k(\mathbb{P})$, having the unique intersection point with D^o_{out}

$$\left[\theta_{(0,1),(1+ty)^{-\ell_2}}, \theta_{(1,0),(1+tx)^{\ell_1}}\right] = \vec{\prod}\theta_{(a,b),f_{(a,b)}},$$

$$\log f_{(a,b)} = \sum kc^k_{a,b}[\ell_1,\ell_2](tx)^{ak}(ty)^{kb},$$

$$c^k_{a,b}[\ell_1,\ell_2] = \sum_{|\mathrm{P}_a|=ka, \#\mathrm{P}_a=\ell_1} \sum_{|\mathrm{P}_b|=kb, \#\mathrm{P}_b=\ell_2} N_{a,b}[\mathrm{P}], \quad \mathrm{P}=(\mathrm{P}_a,\mathrm{P}_b),$$

$$N_{a,b}[\mathrm{P}] = \sum_{\overline{w}\geq(\mathrm{P}_a,\mathrm{P}_b)} \frac{\Phi(P,\overline{w})}{\prod\overline{w}} N^{trop}_{a,b}(\overline{w}),$$

$N^{trop}_{a,b}(\overline{w})$ counts plane rational tropical curves with $\#\overline{w}_a$ ends directed by $(-1,0)$, lying on fixed lines and having weight distribution $\overline{w}_a$, and with $\#\overline{w}_b$ ends directed by $(0,-1)$, lying on fixed lines and having weight distribution $\overline{w}_b$, and with one end of weight k directed by (a,b)

Donaldson-Thomas invariants

Given a compact moduli space of sheaves on a Calabi–Yau threefold, its Donaldson–Thomas invariant is the virtual number of its points, i.e., the integral of the cohomology class 1 against the virtual, zero-dimensional fundamental class.

Theorem (Bousseau)

For any $d > 0$ and $g \geq 0$, the following equality holds:

$$\sum_{g' \geq g} N_{g'}^{d,g} u^{2g'-2+3d} = BG_y(\mathbb{P}^2, d, g) \cdot ((-i)(y^{1/2} - y^{-1/2}))^{2g-2+3d},$$

where $y = e^{iu}$ and $N_{g'}^{d,g}$ are log Gromov-Witten invariants.

(1) Log GW invariants are integrals over $\mathcal{M}_{g',n}(\mathbb{P}^2, d)$ of the product of pull-backs of point classes with one λ-class (suitable Chern class of the Hodge bundle).

(2) The Hodge bundle $\mathbb{E} \to B$ of the family $\mathcal{C} \to B$ of smooth or nodal curves of arithmetic genus g' is the rank g' vector bundle with fibers $H^0(\mathcal{C}_b, \omega(\mathcal{C}_b))$.
The λ-classes $\lambda_j(\mathcal{C}, B)$, $j = 0, ..., g'$, are the Chern classes of the Hodge bundle $\mathbb{E}$.

(3) A particular idea of the proof:
Both sides reduce to enumeration of the same collection of plane tropical curves, while the weights are products of contributions of vertices. Then one compares the weights of the vertices.

Appendix D: Göttsche-Shende refinement

Göttsche and Shende considered the following equation (Macdonald, Gopakumar-Katz-Klemm-Vafa, Göttsche-Shende):

$$\sum_{n=0}^{\infty} q^{n+1-g}\chi(\mathcal{C}^{[n]}) = \sum_{i=0}^{\infty} n^i_{\mathcal{C}/B}\left(\frac{q}{(1-q)^2}\right)^{i+1-g}$$

where $\mathcal{C}^{[n]}$ is the relative Hilbert scheme of the family $\mathcal{C}$ of curves of arithmetic genus g over the base B.

Under certain conditions (e.g., B is a generic $\mathbb{P}^g$ in a very ample linear system on a smooth algebraic surface, i sufficiently small) the coefficients $n^i_{\mathcal{C}/B}$ count complex i-nodal curves in the family $\mathcal{C} \to B$.

Idea of the refinement: make replacement

$$\chi(X)=\sum_{a,b}(-1)^{a+b}h^{a,b}(X) \implies \chi_{-y}(X)=\sum_{a,b}(-1)^a(-y)^b h^{a,b}(X)$$

and obtain

$$\sum_{n=0}^{\infty} q^{n+1-g}\chi_{-y}(\mathcal{C}^{[n]})=\sum_{i=0}^{\infty} N^i_{\mathcal{C}/B}(y)\left(\frac{q}{(1-q)(1-yq)}\right)^{i+1-g}$$

Observation:
$N^i_{\mathcal{C}/B}(1)=n^i_{\mathcal{C}/B}$ counts complex curves
$N^i_{\mathcal{C}/B}(-1)$ counts real i-nodal curves with appropriate signs (under additional restrictions)

A general guess: The count of complex curves is related to the Euler characteristics of appropriate complex moduli spaces, while the count of real curves corresponds to the Euler characteristics of the real part of these moduli spaces

However, $\chi_{-1}(X) = \chi(X)$, while $\chi_1(X) = \sigma(X)$ - the signature

Question: *For which complex algebraic varieties X possessing a real structure, one has $\chi(X(\mathbb{R})) = \sigma(X) = \chi_1(X)$?*

Known: (i) $\chi(X(\mathbb{R})) \equiv (-1)^{\dim X/2}\sigma(X) \mod 4$ for all smooth X of even dimension (Kharlamov)

(ii) $\chi(X(\mathbb{R})) \equiv \sigma(X) \mod 16$ for a smooth M-variety X, i.e., satisfying $b_*(X, \mathbb{Z}/2) = b_*(X(\mathbb{R}), \mathbb{Z}/2)$ (Rokhlin)

(iii) $\chi(X(\mathbb{R})) = \sigma(X)$ for projective complete intersections which are close to the tropical limit (Bertrand-Bihan)

(iv) $\chi(X(\mathbb{R})) = \sigma(X)$ for a few specific $\mathcal{C}^{[n]}$ (Göttsche-Shende)

Question: *Which moduli spaces associated with real enumerative problems satisfy the relation $\chi(X(\mathbb{R})) = \sigma(X)$?*

Appendix E: Plane tropical curves

An **abstract tropical curve** is a finite connected metric graph Γ without one- and bi-valent vertices, whose edges are isometric either to compact intervals, or to the ray $[0, \infty)$.

A **plane tropical curve** is a pair (Γ, h), where $h : \Gamma \to \mathbb{R}^2$ is a continuous map, affine integral along each edge that satisfies the **balancing condition** at each vertex $V \in \Gamma^0$:

$$\sum_{V \in E,\ E \in \Gamma^1} \overline{a}(V, E) = 0 ,$$

where $\overline{a}_V(E) = Dh(\overline{e}(V, E))$ with $\overline{e}(V, E)$ being the unit tangent vector to the edge E at its endpoint V.

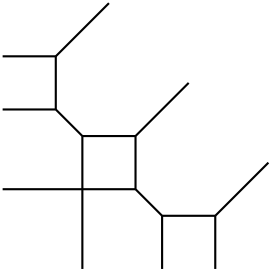
The **genus** of (Γ, h) is $g(\Gamma, h) = b_1(\Gamma)$ (e.g., rational means that Γ is a tree).

Degree $\Delta = \Delta(\Gamma, g_\Gamma, h)$ of a plane tropical curve is a multiset of vectors $\overline{a}(V, E)$, where E ranges over all infinite edges of Γ. Vectors $\overline{a}(V, E)$ rotated by $\frac{\pi}{2}$ clockwise sum up to zero and form a convex **Newton polygon** $\Delta^\perp$.

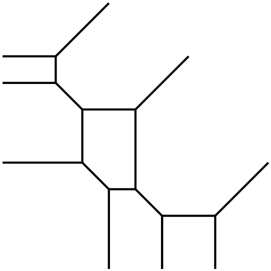
An n-**marked plane tropical curve** is a triple $(\Gamma, h, \boldsymbol{p})$, where $\boldsymbol{p} = (p_1, ..., p_n)$ a sequence of n distinct points of Γ.

Example: *Plane tropical curves*

rational cubic



elliptic cubic



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