Introduction to Tropical Geometry

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Historical sketch

- 1971 - Bergman: Logarithmic limit sets of algebraic varieties $\Longrightarrow$ Tropical fans
- 1980 - Viro: Patchworking of algebraic varieties $\Longrightarrow$ Tropical polynomials
- 1984 - Bieri, Groves: Valuation images of algebraic varieties $\Longrightarrow$ Affine tropical varieties
- 1990 - Berkovich: Skeleta of analytic varieties over non-Archimedeal fields $\Longrightarrow$ General tropical varieties
- 2000 - Kontsevich: Tropical curves, Kontsevich conjecture
- 2002 - Mikhalkin: Tropical enumerative geometry

Tropical objects via limit

Various tropical objects appear in the limit of one-parameter families of algebro-geometric objects

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \widehat{\mathcal{C}} \\ \downarrow & & \downarrow \\ \mathrm{D}^*_\eta & \hookrightarrow & \mathrm{D}_\eta \end{array} \quad \text{or} \quad \begin{array}{ccc} \mathcal{C} & \hookrightarrow & \widehat{\mathcal{C}} \\ \downarrow & & \downarrow \\ (0,\eta) & \hookrightarrow & [0,\eta) \end{array}$$

where $\mathrm{D}_\eta = \{|z| < \eta\}$, $\mathrm{D}^*_\eta = \mathrm{D}_\eta \setminus \{0\}$, and the central fiber $\widehat{\mathcal{C}}_0$ carries some algebraic/geometric/combinatoial structure called the **tropicalization** (or the **tropical limit**) of the family $\mathcal{C} \to \mathrm{D}^*_\eta$.

Example: *Tropical semifield* $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \max, +)$

Consider the family of maps

$$(\mathbb{R}_{>0}, +, \cdot) \to (\mathbb{R}, \oplus_t, \odot_t), \quad a \mapsto -\log_t a, \quad 0 < t < \eta,$$

$$u \oplus_t v = -\log_t(t^{-u} + t^{-v}), \quad u \odot_t v = -\log_t(t^{-u} t^{-v}).$$

Then

$$\lim_{t \to 0}(u \oplus_t v) = \max(u, v), \quad \lim_{t \to 0}(u \odot_t v) = u + v.$$

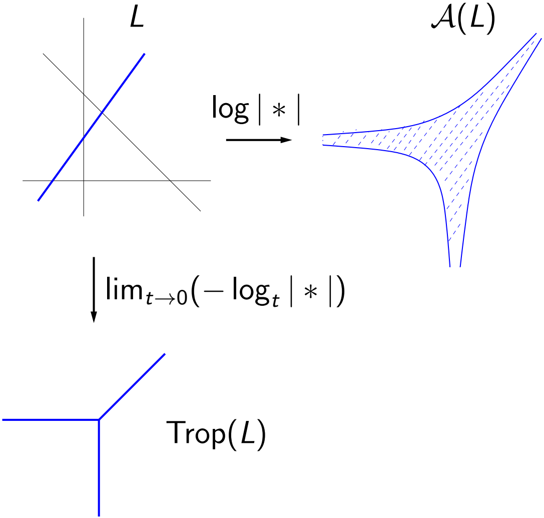
Example: *Complex and tropical amoeba of a line*

$$L \subset (\mathbb{C}^*)^2 \overset{(\log|z|, \log|w|)}{\Longrightarrow} \mathcal{A}(L) \subset \mathbb{R}^2$$

The complex amoeba $\mathcal{A}(L)$ admits a tropicalization:
build a family of maps, then go to the limit

$$L \subset (\mathbb{C}^*)^2 \overset{(-\log_t|z|, -\log_t|w|)}{\Longrightarrow} \mathcal{A}_t(L) \underset{t\to 0}{\Longrightarrow} \mathrm{Trop}(L)$$

where $\mathrm{Trop}(L)$ is the tropical line


L
$\mathcal{A}(L)$
$\log|*|$
$\lim_{t\to 0}(-\log_t|*|)$
$\text{Trop}(L)$

Tropical objects via non-Archimedean valuation

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, possessing a dense real non-Archimedean valuation

$$\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{-\infty\}$$

$$\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b), \quad \mathrm{val}(a+b) \begin{cases} = & \max(\mathrm{val}(a), \mathrm{val}(b)), \\ & \text{if } \mathrm{val}(a) \neq \mathrm{val}(b), \\ \leq & \max(\mathrm{val}(a), \mathrm{val}(b)), \\ & \text{if } \mathrm{val}(a) = \mathrm{val}(b) \end{cases}$$

Our main example: $\mathbb{K} = \bigcup_{m \geq 1} \mathbb{C}\{t^{1/m}\}$
the field of complex, locally convergent Puiseux series,

$$\mathrm{val}(a(t)) = -\min\{q \ : \ a_q \neq 0\}, \quad a(t) = \sum_{r \geq r_0} a_r t^r$$

Tropical amoeba as tropicalization of algebraic variety

Definition

Let $V \subset (\mathbb{K}^*)^n$ be an algebraic variety. Define its tropical amoeba (or tropicalization) by

$$\mathrm{Trop}(V) = \mathit{Closure}(\mathrm{Val}(V)) \subset \mathbb{R}^n$$

where $\mathrm{Val} : (\mathbb{K}^*)^n \to \mathbb{R}$ is the coordinate-wise valuation map.

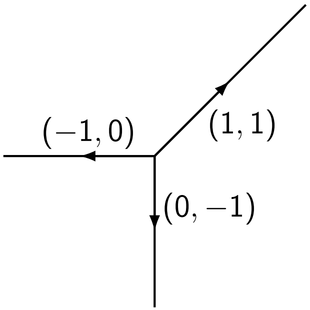
Theorem (d'aprés Bieri-Groves)

Let $V \subset (\mathbb{K}^*)^n$ be an algebraic variety of dimension $0 < r < n$. Then $\mathrm{Trop}(V)$ is a finite, rational, connected polyhedral complex of pure dimension r.
Moreover, its top-dimensional faces σ can be equipped with positive integral weights $\omega(\sigma)$ (uniquely determined by V) so that $\mathrm{Trop}(V)$ becomes balanced, i.e., at each face of dimension $r-1$ there holds a balancing condition.

Balancing condition: For each face δ denote by $\Lambda(\delta) \subset \mathbb{R}^n$ the linear space of dimension $\dim \delta$, parallel to δ. Pick a face $\tau \subset \mathrm{Trop}(V)$ of dimension $r-1$. For each face $\sigma \supset \tau$ of dimension r, pick a generator $\overline{a}_\tau(\sigma)$ of $(\Lambda(\sigma) \cap \mathbb{Z}^n)/(\Lambda(\tau) \cap \mathbb{Z}^n)$ directed inside σ. Then

$$\sum_{\sigma \supset \tau,\ \dim \sigma = r} \omega(\sigma) \cdot \overline{a}_\tau(\sigma) = 0 \in \mathbb{Z}^n/(\Lambda(\tau) \cap \mathbb{Z}^n).$$

Example: *Tropical line in the plane*



$$(-1,0)+(0,-1)+(1,1)=0$$

Affine tropical hypersurfaces

Let $F \in \mathbb{K}[z_1, ..., z_n]$, and $V = Z(F) \subset (\mathbb{K}^*)^n$ a hypersurface.
Define the tropicalization of the polynomial F as follows:

$$F(\underline{z}) = \sum_{\alpha \in \Delta \cap \mathbb{Z}^n} a_\alpha(t) \underline{z}^\alpha \quad \Longrightarrow \quad \mathrm{Trop}(F)(\underline{x}) = \max_{\alpha \in \Delta \cap \mathbb{Z}^n} (\langle \alpha, \underline{x} \rangle + \mathrm{val}(a_\alpha))$$

Theorem (Kapranov)

$\mathrm{Trop}(V)$ is the corner locus of the tropical polynomial $\mathrm{Trop}(F)$.

Example: $L = \{az + bw + c = 0\} \subset (\mathbb{K}^*)^2$

$$\mathrm{Trop}(L) = Corner(\max\{x + \mathrm{val}(a),\ y + \mathrm{val}(b),\ \mathrm{val}(c)\})$$

Define $\nu_F : \Delta \to \mathbb{R}$ to be the convex, piecewise linear function whose graph is the lower part of

$$ConvexHull\{(\alpha, -\mathrm{val}(a_\alpha)) \in \mathbb{R}^{n+1} \ : \ \alpha \in \Delta \cap \mathbb{Z}^n\}.$$

The linearity domains of ν_F define a subdivision Σ_F of the Newton polytope Δ into convex lattice polytopes.

Lemma

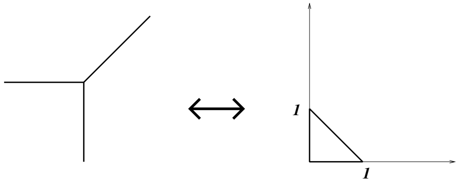
Trop(F) and ν_F are Legendre dual convex, piecewise linear functions.

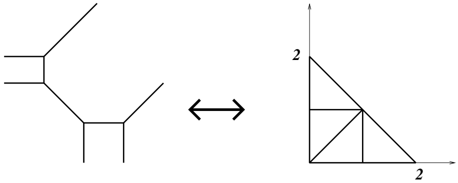
Corollary

There is a duality $\varphi : Cell(\mathbb{R}^n, \mathrm{Trop}(V)) \leftrightarrow Cell(\Sigma_F)$ such that

- $\dim \sigma + \dim \varphi(\sigma) = n$,
- $\sigma \perp \varphi(\sigma)$,
- $\tau \subset \sigma \Longleftrightarrow \varphi(\sigma) \subset \varphi(\tau)$.

Example: *Tropical line and tropical conic*





Define the weights of the top-dimensional faces of a tropical hypersurface by

$$\omega(\sigma) = LatticeLength(\varphi(\sigma))$$

Lemma

A tropical hypersurface equipped with the above weights is balanced.

Theorem

Every connected, rational polyhedral complex in $\mathbb{R}^n$ of pure dimension $n-1$, equipped with positive integral weights of the top-dimensional faces and satisfying the balancing condition, is a tropical hypersurface.

Warning: Not true in codimension > 1!

Intersection theory of affine tropical varieties

Let $T_1, T_2 \subset \mathbb{R}^n$ be tropical varieties, $\dim T_1 = r_1$, $\dim T_2 = r_2$, $r_1 + r_2 \geq n$. We would like to define the intersection "$T_1 \cap T_2$" as a tropical variety $T_1 T_2$ of pure dimension $r_1 + r_2 - n$.

(1) Pick a generic vector $\overline{c} \in \mathbb{R}^n \setminus \{0\}$ and denote $T_{2,\tau} = T_2 + \tau\overline{c}$, $0 < \tau \ll 1$. Then T_1 and $T_{2,\tau}$ intersect generically, an we can refine the cell structures so that $T_1 \cap T_{2,\tau}$ will consist of entire cells (of dimension $\leq r_1 + r_2 - n$). Each cell $\sigma \subset T_1 \cap T_{2,\tau}$ of dimension $r_1 + r_2 - n$ is an intersection $\sigma = \sigma_1 \cap \sigma_2$, $\sigma_1 \subset T_1$, $\sigma_2 \subset T_{2,\tau}$, $\dim \sigma_1 = r_1$, $\dim \sigma_2 = r_2$. Set

$$\omega(\sigma) = \omega(\sigma_1)\omega(\sigma_2)[\mathbb{Z}^n/\Lambda(\sigma) : (\Lambda(\sigma_1) + \Lambda(\sigma_2))/\Lambda(\sigma)]$$

(2) Let $\tau \to 0$ summing up the weight of $(r_1 + r_2 - n)$-cells that merge to one cell.

Theorem

The polyhedral complex $T_1 T_2 = \lim_{\tau \to 0}(T_1 \cap T_{2,\tau})$ is rational, weighted, balanced of pure dimension $r_1 + r_2 - n$. It does not depend on the choice of $\overline{c} \in \mathbb{R}^n \setminus \{0\}$ and is called the stable intersection of T_1 and T_2.

Example: Let $X_1, ..., X_n \subset \mathbb{R}^n$ be tropical hypersurfaces. Then $X_1 ... X_n$ is a finite set of weighted points, whose weight sum up to the normalized mixed volume of the Newton polygons $\Delta(X_1), ..., \Delta(X_n)$ (Bernstein-Koushnirenko theorem).

Remark: One can define the rational equivalence of affine tropical varieties in $\mathbb{R}^n$ and show that it commutes with the intersection. For example, shifts of the same tropical variety are rationally equivalent.

Let $T \subset \mathbb{R}^n$ be a tropical variety of dimension $r > 0$, $f : T \to \mathbb{R}$ a continuous piecewise linear function with integral gradients. We shall define $\mathrm{div}(f) \subset T$ as a tropical cycle of dimension $r - 1$, i.e., a finite rational weighted balanced polyhedral complex with weights of the top-dimensional faces in $\mathbb{Z}$.
We can suppose that the subdivision into linearity domains of f is inscribed into the cellular structure of T. Then to each $(r-1)$-dimensional face $\tau \subset T$, we assign the weight

$$\omega(\tau) = \sum_{\tau\subset\sigma,\ \dim\sigma=r} \omega(\sigma) f_\sigma(\overline{a}_\tau(\sigma)) - f_\tau\left(\sum_{\tau\subset\sigma,\ \dim\sigma-r} \omega(\sigma)\overline{a}_\tau(\sigma)\right)$$

Theorem

$\mathrm{div}(f) \overset{\text{def}}{=} \bigcup_{\omega(\tau)\neq 0} \omega(\tau)\cdot\tau$ is an $(r-1)$-dimensional tropical cycle.

Tropical maps

(1) An integral-affine map $f : \mathbb{R}^n \to \mathbb{R}^m$ is called tropical map. That is, $f(\underline{x}) = A\underline{x} + \underline{y}_0$, where $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$.
(2) Let $T \subset \mathbb{R}^n$ be a tropical variety of dimension r, and $f : \mathbb{R}^n \to \mathbb{R}^m$ a tropical map such that $\dim f(T) = r$.
Define the push-forward f_*T as follows: for each r-dimensional cell $\sigma \subset T$ such that $\dim f(\sigma) = r$, set

$$f_*\sigma = \omega(f_*\sigma) \cdot f(\sigma), \quad \omega(f_*\sigma) = \omega(\sigma)[\Lambda(f(\sigma)) : f_*\Lambda(\sigma)]$$

Lemma

$$f_*T \overset{\text{def}}{=} \sum_{\dim \sigma = \dim f(\sigma) = r} f_*\sigma$$

is a tropical variety, called the push-forward of T by f.

Example: *Degree of the projection*

$T \subset \mathbb{R}^n$ a tropical variety of dimension r, $f : \mathbb{R}^n \to \mathbb{R}^r$ the projection. Then $f_* T = d \cdot \mathbb{R}^r$, where d is called the degree of $f\big|_T$.

Lemma

Let $q \in \mathbb{R}^r$ be a generic point, $(f\big|_T)^{-1}(q) = \{p_1, ..., p_k\}$, where

$$p_i \in \sigma_i, \quad \dim \sigma_i = r, \quad i = 1, ..., k.$$

Then

$$\deg(f\big|_T) = \sum_{i=1}^{k} \Big(\omega(\sigma_i) \cdot \big| \det D(f\big|_{\sigma_i}) \big| \Big).$$

Modification as the tropical blowing-up

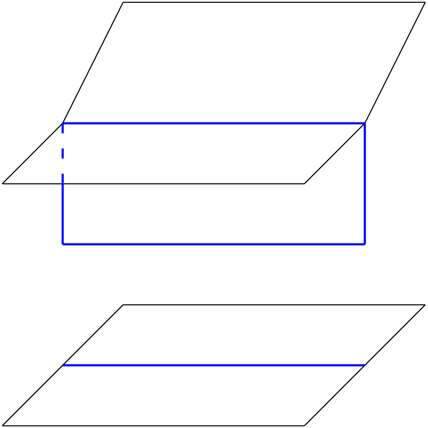
Let $T \subset \mathbb{R}^n$ be a tropical variety of dimension r, $f : \mathbb{R}^n \to \mathbb{R}^m$ a tropical map.
The map $f : T \to f_* T$ is called a modification if it is one-to-one over open top-dimensional cells of $f_* T$, and $[\Lambda(f(\sigma)) : f_* \Lambda(\sigma)] = 1$ as long as $\dim \sigma = \dim f(\sigma) = r$.

Example: *Modification of* $\mathbb{R}^n$
Let $F \in \mathbb{K}[z_1, ..., z_n]$, $T = \mathrm{Trop}(w - F(\underline{z}) = 0) \subset \mathbb{R}^{n+1}$ the graph of the tropical polynomial $\mathrm{Trop}(F)$. Then

$$f : T \to \mathbb{R}^n, \quad f(\underline{z}, w) = \underline{z},$$

is a modification.

Example: *Modification of the plane along a line*

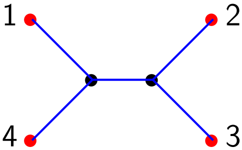


Abstract marked tropical curves

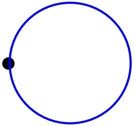
An abstract tropical curve with $n \geq 0$ markings is a finite, connected, metric graph having $n' \geq n$ univalent vertices among which n vertices are marked, and such that the edges containing a univalent vertex are isometric to $[0, \infty]$ (where ∞ is the univalent vertex) while the other edges are isometric to finite closed intervals.
The genus of an abstract tropical curve T is $g(T) = b_1(T)$.
Abstract marked tropical curves are considered up to the following equivalence:

- a bivalent vertex which is the intersection of two edges can be removed,
- A leaf ended at an unmarked univalent vertex can be contracted (a kind of modification),
- isometry of graphs.

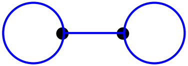
Examples:



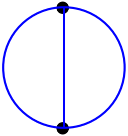
$g = 0, \; n = 4$



$g = 1, \; n = 0$



$g = 2, \; n = 0$



$g = 2, \; n = 0$

Moduli spaces of stable abstract marked tropical curves

$\mathcal{M}^{trop}_{g,n}$ denotes the moduli space of stable abstract marked tropical curves of genus g with n markings.

$$\dim \mathcal{M}^{trop}_{g,n} = \begin{cases} n-3, & g=0,\ n\ge 3, \\ n+1, & g=1,\ n\ge 0, \\ 3g-3+n, & g\ge 2,\ n\ge 0 \end{cases}$$

- the parameters are the lengths of finite edges.

Theorem (Mikhalkin)

The map $\Phi : \mathcal{M}^{trop}_{0,n} \to \mathbb{R}^{n(n-1)(n-2)(n-3)}$, $n \ge 3$,

$$\Phi(T) = \{\mathrm{SignedLength}(\overline{p_i p_j} \cap \overline{p_k p_l}) \ : \ i,j,k,l, \in \{1,...,n\}\}$$

defines an embedding of $\mathcal{M}^{trop}_{0,n}$ onto an affine tropical variety.

Plane marked tropical curves

Let $\overline{\Gamma}$ be an abstract tropical curve, $\overline{\Gamma}^0_\infty$ the set of its univalent vertices, $\Gamma = \overline{\Gamma} \setminus \overline{\Gamma}^0_\infty$, and Γ^1_∞ the set of unbounded edges of Γ (called **ends**). Let $\boldsymbol{p} = (p_1, ..., p_n)$ be a sequence of n distinct points of Γ (markings).

A plane marked tropical curve is a map $h : (\Gamma, \boldsymbol{p}) \to \mathbb{R}^2$ such that

- h is integral-affine on each edge of Γ, and is non-constant on each end;
- at each vertex $V \in \Gamma^0$ there holds a balancing condition

$$\sum_{E \in \Gamma^1,\ V \in E} D\left(h\big|_E\right)(\overline{e}_V(E)) = 0,$$

where $\overline{e}_V(E)$ is the unit tangent vector to E emanating from V.

Let $\xi = [h : (\Gamma, \boldsymbol{p}) \to \mathbb{R}^2]$ be the isomorphism class of a plane marked tropical curve $h : (\Gamma, \boldsymbol{p}) \to \mathbb{R}^2$..
The genus of ξ is $g(\Gamma) = b_1(\Gamma)$.
The (tropical) degree of ξ is the multiset

$$\Delta = \Delta(\xi) = \{D\left(h\big|_E\right)(\overline{e}(E)) \ : \ E \in \Gamma^1_\infty\},$$

where $\overline{e}(E)$ is the unit tangent vector oriented towards infinity. Note that the degree is **balanced**, i.e.,

$$\sum_{\overline{a} \in \Delta} \overline{a} = 0.$$

For simplicity, in what follows we will assume that

$$\Delta = \{d \times (-1, 0),\ d \times (0, -1),\ d \times (1, 1)\}$$

that corresponds to plane algebraic curves of degree d.

Moduli space of plane marked tropical curves

Denote by $\mathcal{M}_{g,n}^{trop}(\Delta, \mathbb{R}^2)$ the moduli space of plane n-marked tropical curves of genus g and degree Δ. There are finitely many combinatorial types of plane n-marked tropical curves of genus g and degree Δ. Each combinatorial type is parameterized either by a point, or by an open convex polyhedron.
From the enumerative point of view it is natural to choose
$n = 3d + g - 1$.
The evaluation map

$$\mathrm{Ev} : \mathcal{M}_{g,n}^{trop}(\Delta, \mathbb{R}^2) \to \mathbb{R}^{2n}, \quad \mathrm{Ev}(h : (\Gamma, \boldsymbol{p}) \to \mathbb{R}^2) = h(\boldsymbol{p})$$

is a tropical map, i.e., it is integral-affine in the parameters of each cell of $\mathcal{M}_{g,n}^{trop}(\Delta, \mathbb{R}^2)$.

Lemma (Mikhalkin)

(1) $\dim \mathcal{M}^{trop}_{g,n}(\Delta, \mathbb{R}^2) = 6d + 2g - 2 = 2n$.
(2) Each $2n$-dimensional cell Σ of $\mathcal{M}^{trop}_{g,n}(\Delta, \mathbb{R}^2)$ such that $\dim \mathrm{Ev}(\Sigma) = 2n$ parameterizes elements $[h : (\Gamma, \boldsymbol{p}) \to \mathbb{R}^2]$ such that

- Γ is trivalent,
- $\boldsymbol{p} \cap \Gamma^0 = \emptyset$,
- no edge of Γ is contracted,
- any component of $\Gamma \setminus \boldsymbol{p}$ is a tree containing exactly one end.

Such $2n$-cells are called enumeratively essential.

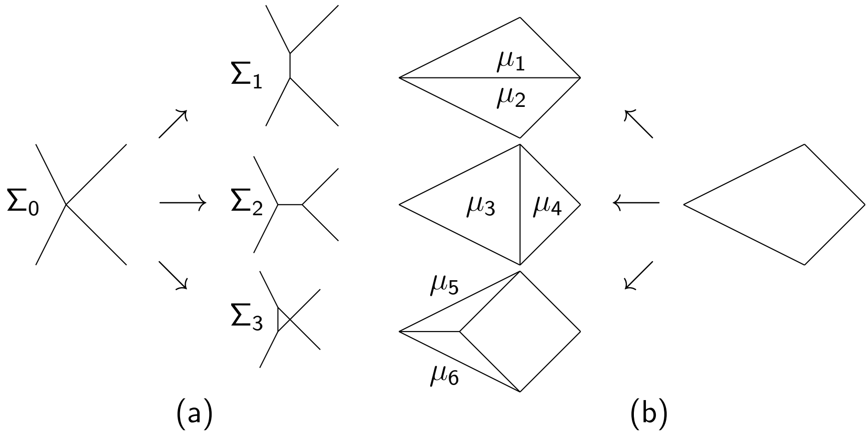
$(2n-1)$-cells in the boundary of enumeratively essential $2n$-cell (d'aprés Gathmann-Markwig):

(1) Either exactly one vertex of Γ is four-valent,

(2) or $\boldsymbol{p} \cap \Gamma^0$ is one point,

(3) or (in case $g > 0$) the image of a four-leg cycle collapses to a segment.

Example: The star of a $(2n-1)$-cell of type (1)



$$\mu_1\mu_2 = \mu_3\mu_4 + \mu_5\mu_6$$

Theorem (Gathmann-Markwig)

(1) The star of each $(2n-1)$-cell as above admits an embedding onto an affine tropical variety in some $\mathbb{R}^N$.
(2) There is a well-defined push-forward $\mathrm{Ev}_*\mathcal{M}^{trop}_{g,n}(\Delta,\mathbb{R}^2)$ and $\deg\left(\mathrm{Ev}:\mathcal{M}^{trop}_{g,n}(\Delta,\mathbb{R}^2)\right)$ such that, for a generic point $\boldsymbol{q}\in\mathbb{R}^{2n}$ with $\mathrm{Ev}^{-1}(\boldsymbol{q})=\{\xi_1,...,\xi_s\}$,

$$\deg\mathrm{Ev}=\sum_{i=1}^{s}\prod_{V\in\Gamma_i^0}\mu(V)$$

where

$$\xi_i=[h_i:(\Gamma_i,\boldsymbol{p}_i)\to\mathbb{R}^2],\quad \mu(V)=\left|D(h_i)(\overline{e}_1)\wedge D(h_i)(\overline{e}_2)\right|$$

Mikhalkin's correspondence theorem

Theorem (Mikhalkin)

Let $\boldsymbol{w}$ be sequence of $n = 3d + g - 1$ points in $(\mathbb{K}^*)^2$ such that $\mathrm{Val}(\boldsymbol{w}) = \boldsymbol{q}$ is a generic point of $\mathbb{R}^{2n}$.
Then, for each $\xi = [h : (\Gamma, \boldsymbol{p}) \to \mathbb{R}^2] \in \mathcal{M}^{trop}_{g,n}(\Delta, \mathbb{R}^2)$ such that $\mathrm{Ev}(\xi) = h(\boldsymbol{p}) = \boldsymbol{q}$, there exist exactly

$$\prod_{V \in \Gamma^0} \mu(V)$$

irreducible algebraic curves $C \subset \mathbb{P}^2_{\mathbb{K}}$ of genus g and degree d such that

$$C \supset \boldsymbol{w} \quad \text{and} \quad \mathrm{Trop}(C) = h_* \Gamma$$

Sketch of the proof

Let $[\boldsymbol{n} : (\widehat{C}, \boldsymbol{p}) \to \mathbb{P}^2_{\mathbb{K}}] \in \mathcal{M}_{g,n}(d, \mathbb{P}^2_{\mathbb{K}})$, $\boldsymbol{n}(\boldsymbol{p}) = \boldsymbol{w}$. Then

$$\begin{array}{ccccccc} (\widehat{C}^*, \boldsymbol{p}) & \to & (\mathbb{P}^2_{\mathbb{C}} \times \mathrm{D}^*_\eta, \boldsymbol{w}) & & (\widehat{C}, \boldsymbol{p}) & \to & (\boldsymbol{X}, \boldsymbol{w}) \\ \downarrow & & \downarrow & \stackrel{t \mapsto t^M}{\Longrightarrow} & \downarrow & & \downarrow \\ \mathrm{D}^*_\eta & = & \mathrm{D}^*_\eta & & \mathrm{D}_\eta & = & \mathrm{D}_\eta \end{array}$$

where $\boldsymbol{X}_0$ is a certain complex surface, a flat limit of $\mathbb{P}^2_{\mathbb{C}}$, and $\boldsymbol{n}_0 : \widehat{C}_0 \to \boldsymbol{X}_0$ is a map of a connected nodal complex curve $\widehat{C}_0$ of arithmetic genus g to $\boldsymbol{X}_0$.

The first approximation to $\boldsymbol{n}_0 : \widehat{C}_0 \to \boldsymbol{X}_0$

Let $C = \boldsymbol{n}(\widehat{C}) \cap (\mathbb{K}^*)^2$ be given by a polynomial $F \in \mathbb{K}[x, y]$. We can write

$$F(\underline{z}) = \sum_{(i,j)\in\Theta_d\cap\mathbb{Z}^2} a_{ij}(t)x^i y^j = \sum_{(i,j)\in\Theta_d\cap\mathbb{Z}^2} t^{\nu(i,j)}(a^0_{ij} + O(t))x^i y^j$$

where $\Theta_d = \mathrm{conv}\{(0,0),(d,0),(0,d)\}$ is the Newton triangle, $\nu : \theta_d \to \mathbb{R}$ a convex, piecewise linear function Legendre dual to $\mathrm{Trop}(F)$. Then we define a flat family of surfaces

$$\boldsymbol{X}' = \mathrm{Tor}(OG(\nu)) \to \mathrm{D},$$

$$OG(\nu) = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \ : \ (\lambda_1, \lambda_2) \in \Theta_d,\ \lambda_3 \geq \nu(\lambda_1, \lambda_2)\}$$

We have

- $\boldsymbol{X}_0' = \bigcup_{i=1}^N \mathrm{Tor}(\theta_i)$, where θ_i's are linearity domains of ν.
- The embedded plane tropical curve defined by $\mathrm{Trop}(F)$ is $h'_*\Gamma'$, where Γ' is an abstract trivalent tropical curve of genus g.
- The polygons θ_i are triangles and parallelograms.
- The family of curves $C = \boldsymbol{n}(\widehat{C}) \to \mathrm{D}_\eta^*$ flatly extends to the central point with $C_0' = \bigcup_{k=1}^N C^{(k)}$,

$$C^{(k)} = \left\{ \sum_{(i,j)\in\theta_k\cap\mathbb{Z}^2} a_{ij}^0 x^i y^j = 0 \right\} \subset \mathrm{Tor}(\theta_k), \quad k = 1, ..., N,$$

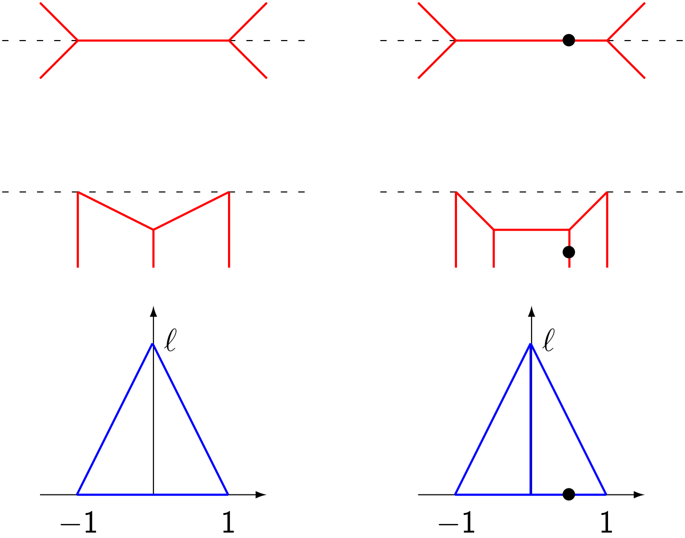
if θ_k is a triangle, then $C^{(k)}$ is a rational curve touching each toric divisor at one point, if θ_k is a parallelogram, then $C^{(k)}$ is the union of two multiple rational curves given by powers of binomial,

- the family of maps $\left[\boldsymbol{n}:(\widehat{C},\boldsymbol{p})\to(\mathbb{P}^2_{\mathbb{C}}\times\mathrm{D}^*_\eta)\right]\to\mathrm{D}^*_\eta$ flatly extends to the central point with the fiber $\boldsymbol{n}'_0:\widehat{C}'_0\to\boldsymbol{X}'_0$, where $\widehat{C}'_0$ is a connected union of rational curves of arithmetic genus g; for example, over the curve $C_k\subset\mathrm{Tor}(\theta_k)$, θ_k a parallelogram, we have two disjoint components of $\widehat{C}'_0$ isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ which multiply cover the components of C_k with ramification at the intersection points with toric divisors; the incidence graph of $\widehat{C}'_0$ is $\widetilde{\Gamma}'$, the graph obtained from Γ' by contracting ends and inserting binodal vertices - a pair of binodal vertices over each self-intersection point of $h'_*\Gamma'$.

Warning: For a given tropical curve $h:\Gamma\to\mathbb{R}^2$ of degree d and genus g passing through $\boldsymbol{q}$, the number of ways to recover the central fiber $\boldsymbol{n}'_0:(\widehat{C}'_0,\boldsymbol{p}_0)\to\boldsymbol{X}'_0$ matching the points $\boldsymbol{w}_0\subset\boldsymbol{X}'_0$ equals

$$\prod_{V\in(\Gamma')^0}\mu(V)\cdot\left(\prod_{E\in(\Gamma')^1}LL(D(h'\big|_E))\right)^{-1}\cdot\left(\prod_{E\in(\Gamma')^1,\ E\cap\boldsymbol{p}\neq\emptyset}LL(D(h'\big|_E))\right)^{-1}$$

Correction via modifications



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