joint works w/ Lang, Barash, He, Christ

Severi varieties on toric surfaces

① Toric surfaces

$\Delta \subset \mathbb{R}^2$ a lattice polygon $\rightsquigarrow$ polarized surface $(X_\Delta, \mathcal{O}(\Delta))$

① $\mathbb{G}_m^2 \hookrightarrow X_\Delta$ open & dense. The action of $\mathbb{G}_m^2$ extends to X_Δ

② A natural order preserving 1-1 corresp. between the faces of Δ & orbits in X_Δ

③ $H^0(X_\Delta, \mathcal{O}(\Delta)) \simeq \bigoplus_{m \in \Delta \cap \mathbb{Z}^2} K x^m$

A construction: $\Phi_\Delta : \mathbb{G}_m^2 \longrightarrow \mathbb{P}^{|\Delta \cap \mathbb{Z}^2|}$

$p \mapsto \{x^m(p)\}$

$X_\Delta = \overline{\Phi_\Delta(\mathbb{G}_m^2)}$ $\qquad \mathcal{O}(\Delta) = \mathcal{O}(1)|_{X_\Delta}$

Facts: $\partial_i \Delta \subset \Delta$ facet $\qquad \mathcal{O}_{\partial_i\Delta}$ orbit $\qquad D_i := \overline{\mathcal{O}_{\partial_i \Delta}}$

① $K_{X_\Delta} = -\sum_i D_i$ $\qquad$ ② $\mathrm{div}(x^m) = \sum_i (m, n_i) D_i$

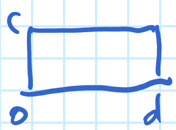
n_i: primitive inner normal to $\partial_i \Delta$

③ X_Δ is normal $\qquad$ ④ $[C] \in |\mathcal{O}(\Delta)|$ general $\quad p_a(C) = |\Delta^\circ \cap \mathbb{Z}^2|$

Examples: (I)

$\rightsquigarrow (X_\Delta, \mathcal{O}(\Delta)) \cong (\mathbb{P}^2, \mathcal{O}(d))$

(II)



$\rightsquigarrow (X_\Delta, \mathcal{O}(\Delta)) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d,c))$

② Severi varieties

$(X_\Delta, \mathcal{O}(\Delta))$ a polarized toric surface

$$V^{irr}_{g,\Delta} = \{ [C]\in|\mathcal{O}(\Delta)| : \begin{array}{l} C \text{ reduced (integral)} \\ g(\underset{\text{normalization}}{C^\nu})=g,\ C\cap X_\Delta^{sing}=\emptyset \end{array} \}$$

Severi variety

Facts: ① If $V_{g,\Delta}\neq\emptyset$ then it is equidimensional of dimension $-K_{X_\Delta}\cdot\mathcal{O}(\Delta)+g-1 = |\partial\Delta\cap\mathbb{Z}^2|+g-1$

② a general curve $[C]\in V_{g,\Delta}$ intersects $\cup D_i$ transversally and away of the 0-dim. orbits

③ If char $=0$ then a general $[C]\in V_{g,\Delta}$ is a nodal curve and intersects any given curve G transversally

Q: What if char >0. A: ③ may fail

Example: Let $M_0 := \langle \mathbb{Z}^2\cap\partial\Delta\rangle \subseteq \mathbb{Z}^2$

$(X_{\Delta_{M_0}}, \mathcal{O}(\Delta_{M_0}))$ – polarized toric surface associated to Δ but w.r.t. lattice $M_0\subset\mathbb{R}^2$

$$X_{\Delta_{M_0}} \xrightarrow{\pi} X_\Delta \cong X_{\Delta_{M_0}}/\mu_n$$

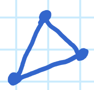
$n=[\mathbb{Z}^2:M_0]$

$\mu_n<\mathbb{G}_m$ roots of unity

$$\pi_*|\mathcal{O}(\Delta_{M_0})| = |\mathcal{O}(\Delta)|$$

$$\pi_*(V^{(irr)}_{g,\Delta_{M_0}}) \subseteq V^{(irr)}_{g,\Delta}$$ union of irred. comp.

Assume $n=\operatorname{char} K$ μ_n

Simplest case:  char $=3$ $g=0$

③ Severi problem

Is $V^{irr}_{g,\Delta}$ irreducible? If not, can one classify its irred. components?

Thm (Harris '86): $V^{irr}_{g,d}/\mathbb{C}$ are either empty or irred.

$(X_\Delta, \mathcal{O}(\Delta)) = (\mathbb{P}^2, \mathcal{O}(d))$

Approach: difficult ① $V^{irr}_{0,d} \subseteq \overline{V}$ $\forall$ $V \subseteq V^{irr}_{g,d}$ irred. comp.

rel. easy $\Big\{$ ② describe locally $\overline{V^{irr}_{g,d}}$ along $V^{irr}_{0,d}$

③ Show that monodromy acts transitively on branches of $\overline{V^{irr}_{g,d}}$ along $V^{irr}_{0,d}$

Prop.: $\forall \Delta$ $\forall$ char $V^{irr}_{0,\Delta}$ is irred

Reason: It is easy to construct a dominant map

$$\mathbb{G}_m^N \dashrightarrow V^{irr}_{0,\Delta}$$

Thm (Lang, T. '23): $g \geq 1$. Then the number of irred. comp. of $V^{irr}_{g,\Delta}$ is at least

$$\#\{M_0 \subseteq M \subseteq \mathbb{Z}^2 \mid |\Delta^\circ \cap M| \geq g\}$$

$$M_0 = \langle \partial\Delta \cap \mathbb{Z}^2 \rangle$$

Q: Is it the right answer?

A (Lang, T. '23): No in general

Thm (Barash, T '25): In genus 1 Yes

$$\#\text{ irred comp. of } V^{irr}_{1,\Delta} = \#\{M_0 \subseteq M \subseteq \mathbb{Z}^2 \mid |\Delta^\circ \cap M| \geq 1\}$$

Thm (Christ, He, T, 23–25): Let $\Delta \subset \mathbb{R}^2$ be an admissible(*) lattice polygon, and $0 \le g \le |\Delta^\circ \cap \mathbb{Z}^2|$. Then $V^{irr}_{g,\Delta}$ is non-empty and irreducible over any alg. closed field. Furthermore, a general curve $[C] \subset V^{irr}_{g,\Delta}$ is nodal if Δ is very admissible(**)

Remark: All classical polarized surfaces $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, Hirzebruch surfaces and their toric blow ups (once) (w/ a unique exception) are very admissible.

(very) admissible = h-transverse w/ a horizontal side admitting a convex subdiv. into $\Delta \& \square$ dual to a mult-one tropical curve of genus g. unimodular

Corollary: Hurwitz schemes $H_{g,d}$ are irred. in any character.