Introduction to Tropical-Topological(Tropological) Sigma Models

Tel Aviv University: Seminar in Real & Complex Geometry

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Talk Atmosphere

No string theory or quantum field theory knowledge is required.

Paraphased from mathoverflow: Philip Candelas once asked Michael Atiyah how to learn algebraic geometry.

Atiyah responded with “You can’t.”

Algebraic geometry is such a large subject that understanding it is a full-time occupation, string theory may very well be the same.

I am a physicist therefore I will occasionally fail to be a bit precise, please stop me to clarify! The goal of this seminar is to explain some physical results so that they may one day rigorously clarified.

Very high level perspective: String theory* is formally an infinite dimensional calculus** for maps from a punctured Riemann surface (worldsheet) Σ to a target space $\mathcal{M}$.

A punctured Riemann surface is interchangable with a compact complex curve with marked points.

The target space is a specific background that can radically change the form of stringy calculus and is generically dependent on geometric data like various additional sections (vector fields, metric tensors) and even gerbes. More complicated cases are "non-geometric".

Once we get to the "topological sigma model", we will focus on the case where the target space is a symplectic manifold (X, ω) with some ω-compatible almost complex structure J. For now, we keep $\mathcal{M}$ general.

Motivation: Nonequilibrium String Theory

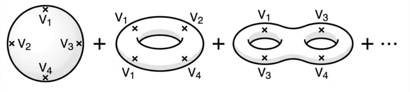


Figure 1: String Perturbation Theory

The stringy calculus computes “correlators of vertex operators” which in simple settings, admit a topological expansion in terms of the worldsheet topology (genus).

The vertex operators correspond to the punctures/marked points on the Riemann surface.

The problem is therefore reduced to figuring out how to “integrate over all” punctured Riemann surfaces at fixed genus and fixed puncture count.

Motivation: Nonequilibrium String Theory

The ”sum over all punctured Riemann surface” is interpreted as trying to count over all marked complex curves. This is a moduli space problem.

A detailed (physics) analysis gives us an idea of which curves to include. In particular, it tells us which degenerations we should be considering. It is partially answered by the Deligne-Mumford compactification.

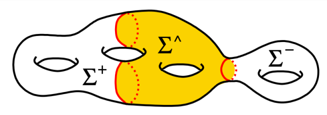
The DM compactification can identify the correct limiting stable curves for pinching degenerations but recent results in string theory have suggested that the story is more complicated. Physics tells us that we don't have Riemann surfaces but analytic continuations of them which are Lorentzian manifolds for backgrounds that are ”out of thermal equilibrium”.

Conclusion: The ”true” nonequilibrium formulation of string theory requires summing over Lorentzian geometries.

Motivation: Nonequilibrium String Theory

Summing over Lorentzian manifolds generically introduces a wealth of analytic problems i.e. divergences.

The Schwinger Keldysh formalism of nonequilibrium quantum mechanics suggest that the correct regularization involves Lorentzian surfaces that are forward moving and backwards moving combined in a (currently unknown) way to produce the correct physics.



Triple decomposition of Σ. Source: [2009.03940], [2]

There was an observation made by Petr Hořava and Christopher Mogni in [2008.11685], [3]: Schwinger Keldysh formalism suggests the worldsheet wedge region $\Sigma^{\wedge}$ is topologically 2-dimensional but geometrically (highly) anisotropic. What kind of geometry is this?

Motivation: Nonequilibrium Quantum Field Theory

How do we construct this wedge region explicitly?

The wedge region is topologically 2-dimensional but geometrically it is highly anisotropic i.e. the metric tensor is expected to become degenerate.

Similar objects in string theory (string networks from M-theory) show similar collapse $\rightarrow$ that are described by a balancing condition that we know from tropical geometry.

We want to see this at the level of the stringy calculus and hence from we turn over to Mikhalkin's correspondence theorem which effectively states that the Gromov Witten invariants of fixed degree *d* and genus *g* can be complicated by taking the "tropical limit and replacing the complex curve counting with corresponding tropical curves.

For the rest of the talk, we will try to construct this tropical stringy calculus from first principles and boil it down to a picture that can be made rigorous.

Real Algebraic Geometry/ Tropical Geometry in Physics

Tropical geometry has appeared in a wide variety of places.

Borinsky on tropicalized quantum field theory [2508.14263] [4].

Tourkine on tropical limits/infinite tension limits of string amplitudes. [1309.3551] [5]

Arkani-Hamed's push on tropical geometry as method to simplify scattering amplitudes [2309.15913] [6].

Eberhardt and Mizera in worldsheet unitarity cuts [2208.12233] [7].

This list is not exhaustive. It my expectation that real algebraic geometry is going to become increasingly important as theoretical physics progresses.

Where are GWs in "string theory"?

In summary: Quantum field theory can be formally viewed as a calculus for infinite dimensional maps(fields).

String theory is an special example of a quantum field theory for maps from a punctured Riemann surface to a target space.

Topological string theory is a special example of a string theory where the calculus is independent of the target space complex structure. The computed "observables" are the Gromov Witten invariants.

Topological string theory is built upon a simpler topological quantum field theory known as the **topological sigma A model**, the observables of this theory give form/classes on some to be specified moduli space, not yet numbers.

Hence, we will begin at understanding topological sigma A models from the physicist's point of view which is grounded in formal probability theory.

Functional Integration

0-dimensional quantum field theory (qft) with 1 random variable:

$$\mathcal{Z} = \int_{\mathbb{R}} dx e^{-S(x)}, \quad S(x) = \frac{1}{2} a_2 x^2 + \sum_{n=3}^{\infty} a_n x^n. \tag{1}$$

Correlation functions/observables are defined as

$$\langle x^\kappa \rangle = \frac{1}{Z} \int_{\mathbb{R}} dx e^{-S(x)} x^\kappa. \tag{2}$$

Here $S(x)$ is known as a Euclidean action. We can extend this to obtain a 0D qft with N random variables

$$\mathcal{Z} = \int_{\mathbb{R}^N} dx_1 \cdots dx^N e^{-S[\vec{x}]}. \tag{3}$$

Observables are then

$$\left\langle x_i x_j x_k \ldots \right\rangle = \frac{1}{Z} \int dx_1 \cdots dx_N e^{-S(\vec{x})} x_i x_j x_k \ldots. \tag{4}$$

The Euclidean action for N random variables is roughly of the form

$$\begin{aligned} S(\vec{x}) &= \frac{1}{2}\vec{x}^{\top}A\vec{x} + W(\vec{x}) \\ &= \frac{1}{2}\sum_{i,j}^{N} x_i A_{ij} x_j + W\left(x_i\right). \end{aligned} \tag{5}$$

One might imagine that you can push this analogy to the continuum by making a formal replacement where $X_i \to X(t)$ for $t \in [0, T]$. This defines a 1D qft with effectively ∞ random variables (1 field)

$$\mathcal{Z} = \int \mathcal{D}x(t) e^{-S[x(t)]}, \quad S[x(t)] = \frac{1}{2}\int_{[0,T]^2} dt dt' x(t) A\left(t, t'\right) x\left(t'\right) + \int_{[0,T]} dt W\left[x(t)\right].$$

Physicists are generically motivated in the case where $A(t, t')$ is a local differential operator e.g.,

$$A\left(t, t'\right) = -\delta\left(t - t'\right)\frac{d^2}{dt^2}. \tag{6}$$

Direct calculus yields connections to heat kernels and other objects e.g.

$$\int_{x(0)=x}^{x(T)=y} \mathcal{D}x \exp\left(-\frac{1}{2}\int_0^T |\dot{x}(t)|^2 dt\right) = \frac{1}{(2\pi T)^{n/2}} \exp\left(-\frac{|x-y|^2}{2T}\right). \quad (7)$$

It turns out that these simple 1D qfts can be made rigorous and is understood from many different perspectives e.g., Wiener measures, rough paths theory and stochastic geometric analysis.

The high level problem is that we don't know a Lebesgue measure on the infinite dimensional space i.e. $\mathcal{D}x$ does not exist in the usual sense.

A solution is to interpret the kinetic term as part of the definition of a Wiener measure on the space of continuous paths $C_0([0, T])$.

2D Functional Integration

Some attempts have been made using the Gaussian Free field for 2d qfts and there have been rigorous constructions made e.g. Liouville quantum gravity.

Some additional recent attempts via stochastic quantization (many authors) and constructive QFT.

Quantum field theory (QFT) can be viewed as the calculus of these functional integrals.

We generically extend this to much more complicated maps like vector fields, connections, spinor fields and other geometric objects.

No common method to rigorously understand quantum field theories above 2D onwards.

Every dimension above seems to require increasingly more complicated methods.

Type A Topological Sigma Model (Relativistic)

Integration theory of maps (fields) $\Phi : \Sigma \to \mathcal{M}$ (Riemann surface to symplectic manifold).

The A-model comes equipped with a nilpotent, Grassmann odd operator $\hat{Q}$ on the space of fields known as the *BRST differential.*

We restrict to symplectic geometry target spaces $(\mathcal{M}, \omega, J)$ with a compatible almost complex structure $J^i_{\ j}$ defined as:

$$J^i_{\ j} J^j_{\ k} = -\delta^i_k \tag{8}$$

Hence, a theory of psuedoholomorphic maps $\Phi : (\Sigma, \varepsilon) \to (\mathcal{M}, J)$.

Known that the topological A model only depends on the target space symplectic form ω.

Notation

For completeness, the action takes the form

$$L = 2t \int d^2z \left(\frac{1}{2} g_{IJ}(\Phi) \partial_z \phi^I \partial_{\bar{z}} \phi^J + \frac{i}{2} g_{IJ} \psi_-^I D_z \psi_-^J \right. \\ \left. + \frac{i}{2} g_{IJ} \psi_+^I D_{\bar{z}} \psi_+^J + \frac{1}{4} R_{IJKL} \psi_+^I \psi_+^J \psi_-^K \psi_-^L \right). \tag{9}$$

where by using local complex coordinates on and $\mathcal{M}$, the fields ϕ^I can be locally described by functions. If K and $\bar{K}$ are canonical and anti canonical line bundles over Σ, then we can construct the square roots of these.

ψ^I are then sections of the tangent bundle of X pulled back to the worldsheet i.e. $K^{1/2} \otimes \Phi^*(TX)$

$$D_{\bar{z}} \psi_+^I = \frac{\partial}{\partial \bar{z}} \psi_+^I + \frac{\partial \phi^J}{\partial \bar{z}} \Gamma_{JK}^I \psi_+^K, \tag{10}$$

BRST Differential

The BRST differential can be formally understood in terms of equivariant cohomology.

It is defined by the fact that we require our maps to satisfy a topological invariance generated by a vector field ϵ^i.

$$\delta_\epsilon \Phi^i = \epsilon^i \tag{11}$$

The standard BRST procedure then replaces the topological symmetry generated by ϵ^i with a Grassmann odd vector field c^i known as a ghost field which enforces the topological symmetry.

Localization of the functional integral is then via the fixed points of the BRST differential. This can be concretely understood in finite-dimensions.

A finite-dimensional analogy

Let M be a compact manifold with a G-action generated by a vector field v. In the Cartan model, the equivariant differential is defined as

$$d_G = d - \iota_v, \quad d_G^2 = 0. \tag{12}$$

Take an equivariantly closed form α and one asks how to compute

$$\mathcal{Z} = \int_M \alpha. \tag{13}$$

We deform α by an equivariant d_G exact deformation and obtain

$$\alpha_t = \alpha e^{-t d_G \Psi}, \quad \mathcal{Z}(t) = \int_M \alpha_t. \tag{14}$$

A finite-dimensional analogy

Then, a short calculation and Stokes' theorem shows

$$\frac{d\mathcal{Z}}{dt} = 0. \tag{15}$$

Hence, we can compute $\mathcal{Z}$ by taking either asymptotic limits

$$\lim_{t\to 0} \mathcal{Z}(t) = \lim_{t\to\infty} \mathcal{Z}(t). \tag{16}$$

The $t \to \infty$ limit is a sum over fixed points and the transverse fluctuations which can be captured by the equivariant Euler class of the transverse directions to the fixed point locus.

Infinite Dimensional Localization

Localization in the context of field theory does this at the functional integral level. We are usually interested in "observables/correlation functions" which take the schematic form

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \mathcal{O}_1(\Phi) \cdots \mathcal{O}_n(\Phi) e^{-S[\Phi]} \tag{17}$$

Here $\mathcal{O}_j$ is a functional of the fields Φ. This generically reduces the functional integral down to finite-dimensional objects which can be made rigorous.

The correlation functions in a topological field theory usually corresponds to topological invariants of the target space.

For A model, it will be the GW invariants.

Localization for Sigma A Model

Fixed point of BRST differential is the localization locus i.e., $\rightarrow$ $\delta\Phi = 0$ for all Φ.

Using local complex coordinates on Σ and $\mathcal{M}^D$, the fields are $\phi^i, \bar{\phi}^i, \psi^i_\pm, \bar{\psi}^i_\pm$.

BRST transformation laws (with Grassmann odd parameter α) are:

$$\delta\phi^i = i\alpha\psi^i_+, \quad \delta\bar{\phi}^i = i\tilde{\alpha}\bar{\psi}^i_-, \tag{18}$$

$$\delta\psi^i_+ = 0, \quad \delta\bar{\psi}^i_- = 0, \tag{19}$$

$$\delta\bar{\psi}^i_+ = -\alpha\partial\bar{\phi}^i - i\tilde{\alpha}\bar{\psi}^j_-\bar{\Gamma}^i_{jm}\bar{\psi}^m_+, \tag{20}$$

$$\delta\psi^i_- = -\tilde{\alpha}\bar{\partial}\phi^i - i\alpha\psi^j_+\Gamma^i_{jm}\psi^m_-. \tag{21}$$

In particular, localization requires:

$$\partial\bar{\phi}^i = \bar{\partial}\phi^i = 0, \tag{22}$$

$$\psi^i_+ = \bar{\psi}^i_- = 0. \tag{23}$$

Tropical limit of localization - Viro hyperfields

Pick local complex coordinates $z, \bar{z}$ on Σ and $Z, \bar{Z}\,(\phi, \bar{\phi})$ on 1-complex dimensional $\mathcal{M}$. The even localization equations are just the Cauchy-Riemann equations:

$$\bar{\partial} Z = 0. \tag{24}$$

We use Viro's subtropical deformation [8] (effectively the Maslov dequantization) of complex numbers:

$$S_\hbar(z) = \begin{cases} |z|^{1/\hbar} \frac{z}{|z|}, & \text{if } z \neq 0; \\ 0 & \text{if } z = 0. \end{cases} \tag{25}$$

This can be parametrized via polar coordinates as

$$z = e^{r+i\theta} \to S_\hbar(z) = e^{r/\hbar + i\theta}. \tag{26}$$

For the addition operation to be associative, it requires multi-valuedness.

Do we need to somehow define a multi-valued functional integral?

For real numbers $a, b \in \mathbb{R}$ and non-negative $\hbar \in \mathbb{R}$ consider:

$$a \odot b = \lim_{\hbar \to 0} \hbar \log \left\{ e^{(a+b)/\hbar} \right\} = a + b, \tag{27}$$

$$a \oplus b = \lim_{\hbar \to 0} \hbar \log \left\{ e^{a/\hbar} + e^{b/\hbar} \right\} = \begin{cases} a, & \text{if } a \geq b; \\ b, & \text{if } b > a. \end{cases} \tag{28}$$

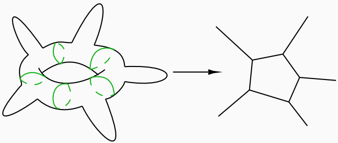
This defines the tropical semifield $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ with arithmetic operations $a \oplus b = \max(a, b)$ and $a \odot b = a + b$. This is known as Litvinov-Maslov dequantization [0507014] [9].

Tropicalization of manifolds

For a complex manifold M with equipped with complex coordinates Z^I.

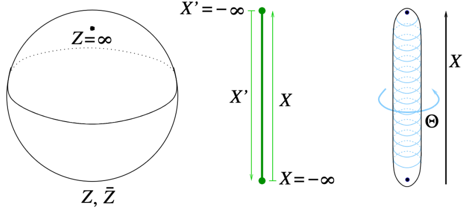
The log map: $(Z^1, \ldots, Z^n) \rightarrow (\hbar \log |Z^1|, \ldots, \hbar \log |Z^n|)$ forgets the phase and the resulting image is called an amoeba.

The limit $\hbar \rightarrow 0$ leads to tropical i.e. piecewise linear coordinates X^I.



A complex surface degenerating into its underlying tropical geometry.

Tropicalization of $\mathbb{CP}^1$



(a) (b) (c)

The tropicalization of $\mathbb{CP}^1$. **a)** The original complex manifold with two coordinates system covering it. **b)** The standard tropicalization $\mathbb{TP}^1$, again with coordinate systems indicated. **c)** The representation of $\mathbb{TP}^1$ with the coordinate Θ preserved. We call this the covering space perspective.

Tropical limit of the localization equation

It turns out that we can get away with the standard formulation of the functional integral in terms of complex numbers as long as we allow a gauge symmetry.

Following Viro's subtropical deformation, the target space local coordinates become:

$$Z = e^{X/\hbar + i\Theta}. \tag{29}$$

In the new variables, the localization equations is now:

$$\bar{\partial} Z = \frac{Z}{2\bar{z}} \left(\partial_r X - \partial_\theta \Theta + \frac{i}{\hbar} (\partial_\theta X + \hbar^2 \partial_r \Theta) \right) = 0. \tag{30}$$

Formally taking $\hbar \to 0$ leads to a tropicalized version of Cauchy-Riemann equations:

$$\partial_r X - \partial_\theta \Theta = 0, \quad \partial_\theta X = 0. \tag{31}$$

Solution for tropical localization equation

From the definition $\theta \sim \theta + 2\pi$ and $\Theta \sim \Theta + 2\pi$.

Global solutions to localization equations on a sleeve $I \times S^1$ are

$$X(r, \theta) = x_0 + nr, \tag{32}$$

$$\Theta(r, \theta) = \Theta_0(r) + n\theta, \tag{33}$$

with $n \in \mathbb{Z}$ and $\Theta_0(r)$ is an arbitrary projectable function of r only.

Repeating the argument across many local patches gives that the solution for X is given by piece-wise linear functions.

Symmetries of tropicalized target space

"Tropical manifolds" exhibit only discrete symmetries.

Since tropical coordinates are (locally) given by piecewise linear function, the most general transformation is given by

$$\tilde{X}^I = n^{IJ} X^J + a^I, \tag{34}$$

with n^{IJ} being a matrix of integer coefficients.

Therefore, the symmetry group $\mathcal{G}$ has to be a subgroup of:

$$\mathcal{G} \subset GL(n, \mathbb{Z}). \tag{35}$$

The important consequence is that this effectively allows us to consider only free field theories.

In the relativistic topological sigma model, the localization equation in covariant form is

$$\hat{\varepsilon}_{\alpha}{}^{\beta}\partial_{\beta}Y^{i} - \hat{J}_{j}{}^{i}\partial_{\alpha}Y^{j} = 0, \tag{36}$$

where $\hat{\varepsilon}_{\alpha}{}^{\beta}$ and $\hat{J}_{j}{}^{i}$ are complex structure on Σ and $\mathcal{M}$ respectively.

In the tropical limit ε, J become nilpotent endomorphisms:

$$\varepsilon^2 = 0, \quad J^2 = 0. \tag{37}$$

We call these Jordan structures and using the adapted coordinates:

$$\varepsilon_{\alpha}{}^{\beta} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_{j}{}^{i} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{38}$$

Covariant form of localization equation

The tropical limit of the localization equations then become

$$E_{\alpha}{}^{i} \equiv \varepsilon_{\alpha}{}^{\beta} \partial_{\beta} Y^{i} - J_{j}{}^{i} \partial_{\alpha} Y^{j}. \tag{39}$$

In adapted coordinates (r, θ) and (X, Θ) we recover the coordinate result:

$$E_{r}{}^{X} = -E_{\theta}{}^{\Theta} = \partial_{\theta} X, \quad E_{r}{}^{\Theta} = \partial_{\theta} \Theta - \partial_{r} X, \quad E_{\theta}{}^{X} = 0. \tag{40}$$

Induced foliation on manifold

In the relativistic case, the worldsheet complex structure decomposes the tangent space as a direct sum

$$V \equiv T\Sigma = T\Sigma^{(1.0)} \oplus T\Sigma^{(0,1)} \tag{41}$$

In the non-relativistic tropical limit, the Jordan structure $\varepsilon^2 = 0$ will induce a natural 2-step filtration on the tangent space.

Instead of direct sum, we now have a flag structure

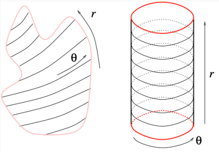
$$F^0V \equiv 0 \subset F^1V \subset F^2V \equiv V. \tag{42}$$

Induced foliation on manifold

The filtration due to Jordan structure induces an integrable distribution of the tangent bundle of Σ through the Frobenius theorem (up to singular points)

However, we begin by solving the tropological sigma model equations on a patch so we stay away from singular points.

The foliation is an equivalence relation on Σ with leaves of foliation being equivalence classes.



Examples of non-singular foliations. Source: [1]

Symmetries of the Jordan structure

In the adapted basis, one can show that the infinitesimal transformations that preserve Jordan structure are:

$$\delta r = f(r), \tag{43}$$

$$\delta\theta = F(r) + \theta\partial_r f(r), \tag{44}$$

where $f(r)$ and $F(r)$ are real-valued projectable functions.

One can show that you can arrive at the same set of symmetries by considering the tropical limit of the metric $g_{\alpha\beta}$ and its inverse $h^{\alpha\beta}$; in this limit, they now satisfy a mutual invisibility condition:

$$g_{\alpha\beta}h^{\beta\gamma} = 0, \tag{45}$$

we find that symmetries of Jordan structure arrive as the intersection of a conformal rescaling of g and h (with opposite weight).

Symmetries of localization equations

The localization equations in adapted coordinates suggest an additional symmetry

$$\partial_r X - \partial_\theta \Theta = 0, \quad \partial_\theta X = 0. \tag{46}$$

They are invariant under:

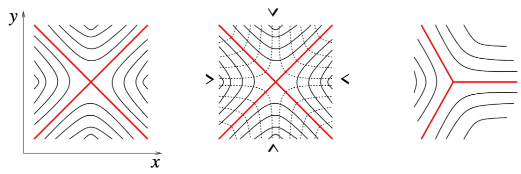
$$\delta X = \alpha_1(r), \tag{47}$$

$$\delta \Theta = \alpha_0(r) + \theta \partial_r \alpha_1(r). \tag{48}$$

In the process of canonical quantization, when we treat r as time, α is interpreted as gauge symmetry imposing constraints on momenta.

Admissible singularities

To formulate theory on higher genus Σ, we have to consider singular foliations.



Examples of singular junctions of sleeves. Source [1].

- Introduce new local coordinates so that localization equations can be extended.
- Construct global solutions with an appropriate gluing.

Junctions of sleeves

Recall the global solution we found on $I \times S^1$:

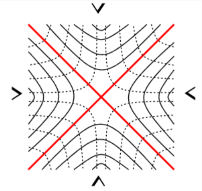
$$X = nr, \quad \Theta = \Theta_0(r) + n\theta, \tag{49}$$

α gauge symmetry is used to adjust r and Θ_0 to glue solutions on the overlap region.

In the case of four sleeves meeting at a junction. The total winding number at the singular point can be shown to satisfy

$$n_\wedge + n_> + n_< + n_\vee = 0. \tag{50}$$

This is precisely the balancing condition at a vertex that appears in tropical geometry. Punctures can be treated similarly.



Junction of 4 sleeves. Source: [1].

BRST construction of Tropological Sigma Model

We introduce Grassmann odd vector fields known as ghost fields (ψ, Ψ). The grading is known as ghost number degree. We assign ghost degree 1 to both ψ and Ψ

We then group fields into components of a field vector that transform under the BRST symmetry known as BRST multiplet

In the adapted coordinates, the BRST multiplets satisfy:

$$[Q, X] = \psi, \quad \{Q, \psi\} = 0, \tag{51}$$

$$[Q, \Theta] = \Psi, \quad \{Q, \Psi\} = 0. \tag{52}$$

Covariant action

In the BRST construction, the action is still a grade 0 object and hence we introduce a grade (-1,0) BRST multiplet known as the antighost-auxiliary multiplet (χ, B).

Under the BRST differential, they satisfy

$$\{Q, \chi^{\alpha}{}_{i}\} = \mathcal{B}^{\alpha}{}_{i}, \tag{53}$$

$$[Q, \mathcal{B}^{\alpha}{}_{i}] = 0. \tag{54}$$

Then a BRST representative for the action can be constructed as

$$\int_{\Sigma} d^2\sigma \mathcal{B}^{\alpha}{}_{i} E_{\alpha}{}^{i}, \tag{55}$$

modulo quadratic terms in $\mathcal{B}^{\alpha}{}_{i}$ which can be integrated out.

Covariant action

The action is invariant under an additional gauge symmetry:

$$\delta \mathcal{B}^{\alpha}{}_{i} = f^{\alpha}_{+i}(\sigma), \quad \delta Y^{i} = 0, \tag{56}$$

with $Y^i = (X, \Theta)$ and gauge parameter satisfying self dual equation $\varepsilon_{\beta}{}^{\alpha} f^{\beta}_{+i}(\sigma) = J_i{}^j f^{\alpha}_{+j}$.

One can check that a fully covariant (on worldsheet) gauge fixing that preserves worldsheet conformal invariance is

$$\mathcal{B}^{\alpha}{}_{X} = 0. \tag{57}$$

In the adapted coordinates the leftover components are denoted as:

$$\mathcal{B}^{r}{}_{\Theta} = B, \quad \mathcal{B}^{\theta}{}_{\Theta} = -\beta. \tag{58}$$

Covariant action

One can show that there is a term consistent with all these symmetries that one can add to the action in the adapted coordinates which is

$$-\frac{1}{2}\int dtd\theta B^2, \tag{59}$$

which can be used to integrate out B. However, no such term can be added for β.

The antighosts follow the same pattern so we denote remaining components as:

$$\chi^{\alpha}{}_{X} = 0, \quad \chi^{r}{}_{\Theta} = \mathfrak{X}, \quad \chi^{\theta}{}_{\Theta} = -\chi. \tag{60}$$

Action for cohomological field theories

It is easy to see that this action may be written as

$$S = \frac{1}{e} \int d^2\sigma \{Q, V\}, \tag{61}$$

with e a coupling constant and V a ghost degree -1 term chosen so that we get non-degenerate kinetic term. One can then show that this action satisfies is BRST invariant $\hat{Q}S = 0$.

Physical observables satisfy $[Q, \mathcal{O}] = 0$. Any variation of metric can be be written as $\{Q, \delta V\}$. Hence, any "interesting" observables are topologically invariant in the sense that they do not depend on the metric (Jordan structure) up to a BRST cohomology class.

Action for tropological sigma model

Explicitly, the action is now

$$S = \frac{1}{e}\int drd\theta\{Q, \mathfrak{X}(\partial_\theta\Theta - \partial_r X - \frac{1}{2}B) + \chi\partial_\theta X\} \tag{62}$$
$$= \frac{1}{e}\int drd\theta\{B(\partial_\theta\Theta - \partial_r X) + \beta\partial_\theta X - \frac{1}{2}B^2 - \mathfrak{X}(\partial_\theta\Psi - \partial_r\phi) - \chi\partial_\theta\psi\}.$$

Integrating out the B field, the action becomes:

$$S = \frac{1}{e}\int drd\theta\Big\{\frac{1}{2}(\partial_\theta\Theta - \partial_r X)^2 + \beta\partial_\theta X - \mathfrak{X}(\partial_\theta\Psi - \partial_r\phi) - \chi\partial_\theta\psi\Big\}, \tag{63}$$

the bosonic part of the action is minimized by the tropical localization equations.

Physical observables satisfy BRST invariance: $[Q, \mathcal{O}] = 0$ up to BRST exact terms.

In fact, we are able to identify the correct observables by mapping the target space deRham cohomology to the BRST cohomology i.e.,

$$W = W_{i_1,\ldots,i_p} dX^{i_1} \wedge \cdots \wedge dX^{i_p} \in \Omega(\mathcal{M}), \tag{64}$$

we associate the operator

$$\mathcal{O}_W^{(p,0)} = W_{i_1,\ldots,i_p} \psi^{i_1} \ldots \psi^{i_p} \in H_Q(\mathcal{F}) \tag{65}$$

where $\mathcal{F}$ is the space of fields. This operator has ghost number p and deRham degree 0.

Observables

Since $\{Q, \psi\} = 0$ we find

$$\{Q, \mathcal{O}_W^{(p,0)}\} = -\partial_{i_0} W_{i_1,\ldots,i_p} \psi^{i_0} \ldots \psi^{i_p} = -\mathcal{O}_{dW}^{(p+1,0)}, \tag{66}$$

which implies that $\mathcal{O}_W^{(0)}$

- is a physical observable if W is closed,
- is a BRST commutator only if W is exact.

One can show that you can construct a hierarchy of observables via the descent equations

$$dO_W^{(p,0)}(x) = \left\{Q, O_W^{(p-1,1)}(x)\right\}. \tag{67}$$

We view $\mathbb{TP}^1$ as a foliated manifold diffeomorphic to $\mathbb{CP}^1$.

The real cohomology of TP^1 is two dimensional spanned by zero form - 1 and two form - $[\omega_T] = \frac{1}{2\pi} dX \wedge d\Theta$.

In the tropical limit, the notion of finite-dim integration measure changes. We recover the correct instanton number by using the definition

$$\int_{TP^1}^{\oplus} \omega_T = \frac{1}{2\pi} \int^{\oplus} dX \int_0^{2\pi} d\Theta = 1, \tag{68}$$

where $\int_R^{\oplus} f(X) dX = \max(f(X), X \in R)$. Alternatively, we can use the distribution-valued two form $[\omega_T] = \frac{\delta(X)}{2\pi} dX \wedge d\Theta$.

Tropical CP^1 model

We construct correlation functions using the $\mathbb{TP}^1$ cohomology.

The correlation function vanishes unless ghost zero modes are saturated. Similar to the relativistic case, we find that

$$\langle \mathcal{O}_\omega(P_1)\mathcal{O}_\omega(P_2)\dots\mathcal{O}_\omega(P_n)\rangle = \begin{cases} 0, & \text{if} \quad n = 2k, \\ \lambda^k, & \text{if} \quad n = 2k+1, \end{cases} \tag{69}$$

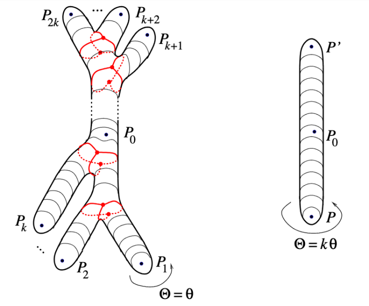
where k is the instanton number given by the pullback $\Phi^\star(\omega_T)$. This can be realized on genus zero Σ with a fixed, generic Jordan structure.

Correlation functions at genus zero

To get an unique instanton solution without moduli, we map:

- k points $(P_1, \ldots, P_k)$ to $X = -\infty$
- k points $(P_{k+1}, \ldots, P_{2k})$ to $X = \infty$
- single point P_0 to $X = 0$

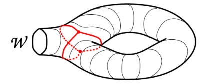
Such map will have desired value of instanton number: k given by the pullback of ω.



Instanton contribution to $2k + 1$ correlation function. Source: [1].

We can construct handle operators $\mathcal{W}$, to increase the genus by 1.

In the case of tropological models this can be only represented by a torus with one finite sleeve attached and generic foliation structure.



Handle operator which raises the genus by one. Source: [1]

The balancing condition imposes that the total winding number at the juncture is $n = 0$ and it can be shown that this implies the handle operator can be mapped to a point.

Correlation functions at higher genus

Therefore, $\mathcal{W}$ is represented by an insertion of a local operator:

$$\mathcal{W} = 2\mathcal{O}_\omega. \tag{70}$$

In particular, the partition function can be calculated from the torus one point correlation function evaluated at all loop orders i.e.,

$$\langle\langle\mathcal{O}_1\rangle\rangle = \sum_{g=0}^{\infty} g_s^{2g-2}\langle\mathcal{O}_1\rangle_g = \sum_{g=0}^{\infty} g_s^{2g-2}\langle\mathcal{O}_1\mathcal{W}^{g-1}\rangle_0 = \frac{2}{1-4g_s^4\lambda}. \tag{71}$$

This is precisely the same result as the relativistic case for higher genus correlation functions matching the well known result of the $\mathbb{CP}^1$ model.

Further Results

Analytic continuation to real time

We set a real worldsheet time to run along the leaves of the foliation. Then we compactify r on a circle with periodicity $r \sim r + 2\pi$. If we expand the generators of the worldsheet conformaal symmetries in terms of Fourier modes

$$f(r) = \sum_{\mathbb{Z}} L_m e^{imr}, \quad F(r) = \sum_{\mathbb{Z}} J_m e^{imr}. \tag{72}$$

The generators result in the BMS_3 algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c_L}{12} m(m^2-1)\delta_{m+n,0}, \\
[J_m, J_n] &= 0, \\
[L_m, J_n] &= (m-n)J_{m+n} + \frac{c_J}{12} m(m^2-1)\delta_{m+n,0}.
\end{aligned} \tag{73}$$

This also appears in the study of asymptotic symmetries of gravity, flat space holography and other nonrelativistic limits of string theory.

One can investigate the boundary conditions for the tropological sigma models and find an new boundary conformal algebra [2412.12337], [10]

$$[\tilde{L}_m, \tilde{L}_n] = \frac{i}{2}\left((m-n)\tilde{L}_{m+n} - (m+n)\tilde{L}_{m-n}\right),$$
$$[J_m, J_n] = 0,$$
$$[\tilde{L}_m, J_n] = -\frac{i}{2}\left((m-n)J_{m+n} - (m+n)J_{m-n}\right) + \frac{c_J}{12}m(m^2-1)(\delta_{m+n,0} + \delta_{-m+n,0})$$

The canonical quantization of the boundary states result in a Hamiltonian that produces an infinite number of free particles with increasingly mass which is precisely the behavior we expect from string amplitudes. Mathematicians [11] have begun studying this algebra in [2508.21603] showing a connection to Carrollian physics.

Coupling to Tropological Gravity

A natural question is what sort of dynamical worldsheet gravity is appropriate for topological sigma models?

Can one create a quantum field theory of dynamical Jordan structures/foliations?

A construction was discussed in an unreleased paper where we instead replace the kinematics of Jordan structures with einbeins e^a_α and a Galilean connection $\omega^a_{\alpha b}$.

The construction yields some direct connections to Novikov Morse theory and is suggestive of a deeper connection to Ricci flow.

Future Directions

From here, many directions (and some preliminary answers) to study:

Is there a natural notion of tropological B model?

What occurs in higher dimensional target spaces? Filtered geometries and Nil-equivariance [2507.23072], [12].

Can we develop an anisotropic conformal field theory for the observables we've written down?

Are there other tropical limits of topological field theory that might yield something? [2503.15856], [13].

What is the connection to other nonrelativistic limit of string theory? e.g. Newton Cartan, Galilean, ambitwistor and Carollian strings? [2311.10565], [14].

..many more, limited time.

Thank you! Any questions?

Long form questions via andresfranco@berkeley.edu.

I’d be very happy to answer!

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