Residues and Duality for Schemes and Stacks

Amnon Yekutieli

Department of Mathematics
Ben Gurion University

Notes available at http://www.math.bgu.ac.il/~amyekut/lectures

Written: 24 Oct 2012

Outline

1. Rigid Dualizing Complexes over Rings

2. Rigid Residue Complexes over Rings

3. Rigid Residue Complexes over Schemes

4. Residues and Duality for Proper Maps of Schemes

5. Finite Type DM Stacks

Some of the work discussed here was done with James Zhang several years ago.

1. Rigid Dualizing Complexes over Rings

All rings and algebras in this talk are commutative.

We fix a base ring $\mathbb{K}$, which is finite dimensional, regular and noetherian (e.g. a field or $\mathbb{Z}$).

Let A be an *essentially finite type* $\mathbb{K}$-algebra. Recall that this means A is a localization of a finite type $\mathbb{K}$-algebra. So A is noetherian.

We denote by $\mathsf{C}(\mathsf{Mod}\, A)$ the category of complexes of A-modules, and by $\mathsf{D}(\mathsf{Mod}\, A)$ its derived category.

There is a functor

$$Q : \mathsf{C}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A)$$

which is the identity on objects. The morphisms in $\mathsf{D}(\mathsf{Mod}\, A)$ are all of the form $Q(\phi) \circ Q(\psi)^{-1}$, where ψ is a quasi-isomorphism.

Inside $\mathsf{D}(\mathsf{Mod}\, A)$ there is the full subcategory $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ of bounded complexes with finitely generated cohomology modules.

We defined a functor

$$\mathrm{Sq}_{A/\mathbb{K}} : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A)$$

called the *squaring*.

It is a quadratic functor: if $\phi : M \to N$ is a morphism in $\mathsf{D}(\mathsf{Mod}\, A)$, and $a \in A$, then

$$\mathrm{Sq}_{A/\mathbb{K}}(a\phi) = a^2 \, \mathrm{Sq}_{A/\mathbb{K}}(\phi).$$

If A is flat over $\mathbb{K}$ then there is an easy formula for the squaring:

$$\mathrm{Sq}_{A/\mathbb{K}}(M) = \mathrm{RHom}_{A \otimes_{\mathbb{K}} A}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M).$$

But in general we have to use DG algebras to define $\mathrm{Sq}_{A/\mathbb{K}}(M)$.

A *rigidifying isomorphism* for M is an isomorphism

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{A/\mathbb{K}}(M)$$

in $\mathsf{D}(\mathsf{Mod}\, A)$.

If $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$, then the pair (M, ρ) is called a *rigid complex over* A *relative to* $\mathbb{K}$.

Suppose (N, σ) is another rigid complex. A *rigid morphism*

$$\phi : (M, \rho) \to (N, \sigma)$$

is a morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, A)$, such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho} & \mathrm{Sq}_{A/\mathbb{K}}(M) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \mathrm{Sq}_{A/\mathbb{K}}(\phi)} \\ N & \xrightarrow{\sigma} & \mathrm{Sq}_{A/\mathbb{K}}(N) \end{array}$$

is commutative.

We denote by $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$ the category of rigid complexes, and rigid morphisms between them.

Here is the important property of rigidity: let (M, ρ) be a rigid complex, such that canonical morphism $A \to \mathrm{RHom}_A(M, M)$ is an isomorphism. Then the only automorphism of (M, ρ) in $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$ is the identity.

The idea of rigid dualizing complex goes back to M. Van den Bergh's paper [VdB] from 1997. More progress (especially the passage from base field to base ring) was done in the papers "YZ" in the references.

2. Rigid Residue Complexes over Rings

The next definition is from [RD].

A complex $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is called *dualizing* if it has finite injective dimension, and the canonical morphism $A \to \mathrm{RHom}_A(M, M)$ is an isomorphism.

Grothendieck proved that for a dualizing complex M, the functor

$$\mathrm{RHom}_A(-, M)$$

is a duality (i.e. contravariant equivalence) of $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$

A *rigid dualizing complex over* A is a rigid complex (M, ρ) such that M is dualizing.

We know that any essentially finite type $\mathbb{K}$-algebra A has a rigid dualizing complex (M, ρ).

Moreover, any two rigid dualizing complexes over A are uniquely isomorphic in $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$.

If $A = K$ is a field, then its rigid dualizing complex M must be isomorphic to $K[d]$ for an integer d. We define

$$\dim_{\mathbb{K}}(K) := d.$$

If $\mathbb{K}$ is a field then

$$\dim_{\mathbb{K}}(K) = \mathrm{tr.deg}_{\mathbb{K}}(K),$$

but in general it could be negative.

For a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ we define

$$\dim_{\mathbb{K}}(\mathfrak{p}) := \dim_{\mathbb{K}}(\boldsymbol{k}(\mathfrak{p})),$$

where $\boldsymbol{k}(\mathfrak{p})$ is the residue field.

The resulting function

$$\dim_{\mathbb{K}} : \operatorname{Spec} A \to \mathbb{Z}$$

is a dimension function (it has the expected property for specialization of primes).

For any $\mathfrak{p} \in \operatorname{Spec} A$ we denote by $J(\mathfrak{p})$ the injective hull of the A-module $\boldsymbol{k}(\mathfrak{p})$. This is an indecomposable injective module.

A *rigid residue complex* over A (relative to $\mathbb{K}$) is a rigid dualizing complex $(\mathcal{K}, \rho)$, such that for every i there is an isomorphism of A-modules

$$\mathcal{K}^{-i} \cong \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ \dim_{\mathbb{K}}(\mathfrak{p}) = i}} J(\mathfrak{p}).$$

A morphism $\phi : (\mathcal{K}, \rho) \to (\mathcal{K}', \rho')$ between rigid residue complexes is a homomorphism of complexes $\phi : \mathcal{K} \to \mathcal{K}'$ in $\mathsf{C}(\mathsf{Mod}\, A)$, such that $Q(\phi) : (\mathcal{K}, \rho) \to (\mathcal{K}', \rho')$ is a morphism in $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$.

We denote by $\mathsf{C}(\mathsf{Mod}\, A)_{\mathrm{res}/\mathbb{K}}$ the category of rigid residue complexes.

The algebra A has a rigid residue complex $(\mathcal{K}_A, \rho_A)$. It is unique up to a unique isomorphism in $\mathsf{C}(\mathsf{Mod}\, A)_{\mathrm{res}/\mathbb{K}}$. So we call it *the rigid residue complex* of A.

Let me mention two important functorial properties of rigid residue complexes.

Suppose $A \to B$ is an *étale homomorphism* of $\mathbb{K}$-algebras. Consider the rigid residue complexes $(\mathcal{K}_A, \rho_A)$ and $(\mathcal{K}_B, \rho_B)$ of A and B respectively.

There is a unique *rigid localization homomorphism*

$$\mathrm{q}_{B/A} : \mathcal{K}_A \to \mathcal{K}_B.$$

The induced homomorphism of complexes

$$B \otimes_A \mathcal{K}_A \to \mathcal{K}_B$$

is bijective.

If $B \to C$ is another étale homomorphism, then

$$\mathrm{q}_{C/A} = \mathrm{q}_{C/B} \circ \mathrm{q}_{B/A}\,.$$

In this way rigid residue complexes form a quasi-coherent sheaf on the étale topology of $\operatorname{Spec} A$. This will be important for us.

Now let $A \to B$ any homomorphism between essentially finite type $\mathbb{K}$-algebras.

There is a homomorphism of graded A-modules

$$\mathrm{Tr}_{B/A} : \mathcal{K}_B \to \mathcal{K}_A$$

called the *rigid trace homomorphism.*

It is functorial: if $B \to C$ is another homomorphism, then

$$\mathrm{Tr}_{C/A} = \mathrm{Tr}_{B/A} \circ \mathrm{Tr}_{C/B}\,.$$

When $A \to B$ is a *finite* homomorphism, then $\mathrm{Tr}_{B/A}$ is a homomorphism of complexes.

The rigid traces and the rigid localizations commute with each other.

3. Rigid Residue Complexes over Schemes

Now we look at a finite type $\mathbb{K}$-scheme X. If $U \subset X$ is an affine open set, then $A := \Gamma(U, \mathcal{O}_X)$ is a finite type $\mathbb{K}$-algebra.

Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. For any affine open set U, $\Gamma(U, \mathcal{M})$ is a $\Gamma(U, \mathcal{O}_X)$-module.

If $V \subset U$ is another affine open set, then

$$\Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X)$$

is an *étale ring homomorphism.*

And there is a homomorphism

$$\Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$$

of $\Gamma(U, \mathcal{O}_X)$-modules.

A *rigid residue complex on* X is a complex $\mathcal{K}$ of quasi-coherent $\mathcal{O}_X$-modules, together with a rigidifying isomorphism ρ_U for the complex $\Gamma(U, \mathcal{K})$, for every affine open set U.

There are two conditions:

(i) The pair $\big(\Gamma(U, \mathcal{K}), \rho_U\big)$ is a rigid residue complex over $\Gamma(U, \mathcal{O}_X)$ relative to $\mathbb{K}$.

(ii) For an inclusion $V \subset U$ of affine open sets, the canonical homomorphism

$$\Gamma(U, \mathcal{K}) \to \Gamma(V, \mathcal{K})$$

is the unique rigid localization homomorphism between these rigid residue complexes.

We denote by $\boldsymbol{\rho} := \{\rho_U\}$ the collection of rigidifying isomorphisms, and call it a *rigid structure.*

Suppose $(\mathcal{K}, \boldsymbol{\rho})$ and $(\mathcal{K}', \boldsymbol{\rho}')$ are two rigid residue complexes on X.

A morphism of rigid residue complexes $\phi : (\mathcal{K}, \boldsymbol{\rho}) \to \mathcal{K}', \boldsymbol{\rho}')$ is a homomorphism $\phi : \mathcal{K} \to \mathcal{K}'$ of complexes of $\mathcal{O}_X$-modules, such that for every affine open set U, with $A := \Gamma(U, \mathcal{O}_X)$, the induced homomorphism $\Gamma(U, \phi)$ is a morphism in $\mathsf{C}(\mathsf{Mod}\, A)_{\mathrm{res}/\mathbb{K}}$.

We denote the category of rigid residue complexes by $\mathsf{C}(\mathsf{QCoh}\, X)_{\mathrm{res}/\mathbb{K}}$.

Every finite type $\mathbb{K}$-scheme X has a rigid residue complex $(\mathcal{K}_X, \boldsymbol{\rho}_X)$; and it is unique up to a unique isomorphism in $\mathsf{C}(\mathsf{QCoh}\, X)_{\mathrm{res}/\mathbb{K}}$.

Suppose $f : X \to Y$ is any map between finite type $\mathbb{K}$-schemes.

The complex $f_*(\mathcal{K}_X)$ is a bounded complex of quasi-coherent $\mathcal{O}_Y$-modules.

The rigid traces for rings that we talked about before induce a homomorphism of graded quasi-coherent $\mathcal{O}_Y$-modules

$$\mathrm{Tr}_f : f_*(\mathcal{K}_X) \to \mathcal{K}_Y, \tag{3.0}$$

that we also call the *rigid trace homomorphism.*

It is functorial: if $g : Y \to Z$ is another map, then

$$\mathrm{Tr}_{g \circ f} = \mathrm{Tr}_g \circ \mathrm{Tr}_f \,.$$

It is not hard to see that if f is a finite map of schemes, then Tr_f is a homomorphism of complexes.

4. Residues and Duality for Proper Maps of Schemes

Theorem 4.1. (Residue Theorem) *Let $f : X \to Y$ be a proper map between finite type $\mathbb{K}$-schemes. Then*

$$\mathrm{Tr}_f : f_*(\mathcal{K}_X) \to \mathcal{K}_Y$$

is a homomorphism of complexes.

The idea of the proof (imitating [RD]) is to reduce to the case when $Y = \operatorname{Spec} A$, A is a local artinian ring, and $X = \mathbf{P}^1_A$ (the projective line).

In this case we show that the complex of A-modules $\Gamma(X, \mathcal{K}_X)$ has an induced rigidifying isomorphism. We use this, plus a trick, to prove that

$$\mathrm{Tr}_f : \Gamma(X, \mathcal{K}_X) \to \mathcal{K}_Y$$

is a homomorphism of complexes.

Theorem 4.2. (Duality Theorem) *Let $f : X \to Y$ be a proper map between finite type $\mathbb{K}$-schemes. Then for any $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, X)$ the morphism*

$$\mathrm{R}f_*\big(\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}_X)\big) \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\mathrm{R}f_*(\mathcal{M}), \mathcal{K}_Y\big)$$

in $\mathsf{D}(\mathsf{Mod}\, Y)$, that is induced by

$$\mathrm{Tr}_f : f_*(\mathcal{K}_X) \to \mathcal{K}_Y,$$

is an isomorphism.

The proof of Theorem 4.2 imitates the proof of the corresponding theorem in [RD], once we have the Residue Theorem 4.1 at hand.

The proofs of Theorems 4.1 and 4.2 are sketched in the incomplete preprint [YZ1]. Complete proofs will be available in [Ye2].

One advantage of our approach – using rigiditiy – is that it is much cleaner and shorter than the original approach in [RD]. This is because we can avoid extremely complicated diagram chasing (that was not actually done in [RD], but rather in follow-up work by Lipman, Conrad and others).

Another advantage, as we shall see next, is that the rigidity approach promises to give a useful duality theory for stacks.

5. Finite Type DM Stacks

Unfortunately I do not have time to give background on stacks. For those who do not know about stacks, it is useful to think of a Deligne-Mumford stack $\mathfrak{X}$ as a scheme, with an extra structure: the points of $\mathfrak{X}$ are clumped into finite groupoids.

Here are some good references on algebraic stacks: [LMB, SP, Ol].

We will only consider noetherian finite type DM $\mathbb{K}$-stacks.

Given a quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$, and an étale map $U \to \mathfrak{X}$ from a scheme U, we denote by $\mathcal{M}|_U$ the corresponding quasi-coherent $\mathcal{O}_U$-module (in the Zariski topology of U).

The definition of a rigid residue complex on a stack $\mathfrak{X}$ relative to $\mathbb{K}$ is very similar to the scheme definition:

A rigid residue complex on $\mathfrak{X}$ is a complex of quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules $\mathcal{K}_{\mathfrak{X}}$, together with a rigid structure $\boldsymbol{\rho}_{\mathfrak{X}} = \{\rho_U\}$.

However here the indexing is by étale maps $U \to \mathfrak{X}$ from affine schemes, and ρ_U is a rigidifying isomorphism for the complex $\Gamma(U, \mathcal{K}_{\mathfrak{X}}|_U)$ over the $\mathbb{K}$-algebra $\Gamma(U, \mathcal{O}_U)$.

The conditions are:

(i) The pair $\big(\Gamma(U, \mathcal{K}_{\mathfrak{X}}|_U), \rho_U\big)$ is a rigid residue complex over the ring $\Gamma(U, \mathcal{O}_X)$ relative to $\mathbb{K}$.

(ii) For an étale map $V \to U$ of affine schemes, the canonical homomorphism

$$\Gamma(U, \mathcal{K}_{\mathfrak{X}}|_U) \to \Gamma(V, \mathcal{K}_{\mathfrak{X}}|_V)$$

is the unique rigid localization homomorphism.

Theorem 5.1. ([Ye2]) *Let $\mathfrak{X}$ be a finite type DM stack over $\mathbb{K}$. Then $\mathfrak{X}$ has a rigid residue complex $(\mathcal{K}_{\mathfrak{X}}, \boldsymbol{\rho}_{\mathfrak{X}})$. It is unique up to a unique isomorphism in $\mathsf{C}(\mathsf{QCoh}\,\mathfrak{X})_{\mathrm{res}/\mathbb{K}}$.*

The proof is a standard étale descent argument, using the fact that for affine schemes the rigid residue complexes are quasi-coherent sheaves in the étale topology.

Similarly we have:

Theorem 5.2. ([Ye2]) *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map between finite type DM $\mathbb{K}$-stacks. Then there is a homomorphism of graded quasi-coherent $\mathcal{O}_{\mathfrak{Y}}$-modules*

$$\mathrm{Tr}_f : f_*(\mathcal{K}_{\mathfrak{X}}) \to \mathcal{K}_{\mathfrak{Y}}$$

called the rigid trace.

It satisfies, and is uniquely characterized by, these properties:

(i) *Functoriality:* $\mathrm{Tr}_{g \circ f} = \mathrm{Tr}_g \circ \mathrm{Tr}_f$.

(ii) *If $\mathfrak{X}$ and $\mathfrak{Y}$ are schemes, then Tr_f is the rigid trace from equation* (3.0).

The obvious question now is: do the Residue Theorem and the Duality Theorem hold for a proper maps $f : \mathfrak{X} \to \mathfrak{Y}$ between stacks?

The answer I can give is not so clear cut.

We know by the Keel-Mori Theorem that a separated finite type DM stack $\mathfrak{X}$ has a *coarse moduli space* $\pi : \mathfrak{X} \to X$. The map π is proper and quasi-finite, and X is, in general, an *algebraic space*.

Let us call $\mathfrak{X}$ a *coarsely schematic stack* if its coarse moduli space X is a scheme.

This appears to be a rather mild restriction: most DM stacks that come up in examples are of this kind.

A map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *coarsely schematic map* if for some surjective étale map $V \to \mathfrak{Y}$ from an affine scheme V, the stack

$$\mathfrak{X}' := \mathfrak{X} \times_{\mathfrak{Y}} V$$

is coarsely schematic.

Conjecture 5.3. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a proper coarsely schematic map between finite type DM $\mathbb{K}$-stacks. Then

$$\mathrm{Tr}_f : f_*(\mathcal{K}_{\mathfrak{X}}) \to \mathcal{K}_{\mathfrak{Y}}$$

is a homomorphism of complexes of $\mathcal{O}_{\mathfrak{Y}}$-modules.

It is not expected that duality will hold in this generality. In fact, there are easy counter examples. The problem is *finite group theory in positive characteristic*!

For a DM stack $\mathfrak{X}$ and a field K, there is a groupoid $\mathfrak{X}(K)$ where the automorphism groups of objects are finite.

Following [AOV], the stack $\mathfrak{X}$ is called *tame* if for every algebraically closed field K, the automorphism groups in $\mathfrak{X}(K)$ have orders prime to the characteristic of K.

A map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *tame map* if for some surjective étale map $V \to \mathfrak{Y}$ from an affine scheme V, the stack $\mathfrak{X}' := \mathfrak{X} \times_{\mathfrak{Y}} V$ is tame.

Conjecture 5.4. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a proper, tame, coarsely schematic map between finite type DM $\mathbb{K}$-stacks. Then Tr_f induces duality (as in Theorem 4.2).

I believe I have an idea how to prove these conjectures.

It is likely that the "coarsely schematic" could be removed; but I don't know how.

- END -

References

[AOV] D. Abramovich, M. Olsson and A. Vistoli, Tame stacks in positive characteristic, *Ann. Inst. Fourier* **58**, 4 (2008), 1057-1091.

[LMB] G. Laumon and L. Moret-Bailly, "Champs Algébriques", Springer, 2000.

[Na] S. Nayak, Compactification for essentially finite-type maps, Advances Math. **222** (2009), 527-546.

[Ni] F. Nironi, Grothendieck duality for deligne-mumford stacks, eprint arXiv:0811.1955v2.

[Ol] M. Olsson, "An Introduction to Algebraic Spaces and Stacks", book in preparation.

[RD] R. Hartshorne, "Residues and Duality," Lecture Notes in Math. **20**, Springer-Verlag, Berlin, 1966.

[SP] The Stacks Project, J.A. de Jong (Editor), `http://math.columbia.edu/algebraic_geometry/stacks-git`

[VdB] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered ring, J. Algebra **195** (1997), no. 2, 662-679.

[Ye1] A. Yekutieli, Rigid Dualizing Complexes via Differential Graded Algebras (Survey), in "Triangulated Categories", LMS Lecture Note Series **375**, 2010.

[Ye2] A. Yekutieli, Rigidity, residues, and Grothendieck duality for schemes and DM stacks, in preparation.

[YZ1] A. Yekutieli and J.J. Zhang, Rigid Dualizing Complexes on Schemes, Eprint math.AG/0405570 at http://arxiv.org.

[YZ2] A. Yekutieli and J.J. Zhang, Residue complexes over noncommutative rings, J. Algebra **259** (2003) no. 2, 451-493.

[YZ4] A. Yekutieli and J.J. Zhang, Dualizing complexes and perverse sheaves on noncommutative ringed schemes, Selecta Math. **12** (2006), 137-177.

[YZ5] A. Yekutieli and J.J. Zhang, Rigid Complexes via DG Algebras, Trans. AMS **360** no. 6 (2008), 3211-3248.

[YZ6] A. Yekutieli and J.J. Zhang, Rigid Dualizing Complexes over Commutative Rings, Algebras and Representation Theory **12**, Number 1 (2009), 19-52