Dimension of the Hilbert scheme (1)

Note: Only discuss Hilbert schemes of curves.

Recall: Tangent space to the Hilbert scheme of curves in $\mathbb{P}^n$ at a point $h := [C]$ is given by

$$T_h\left(\mathrm{Hilb}_n^{P(t)}\right) = H^0(C, N_{C/\mathbb{P}^n})$$

where $N_{C/\mathbb{P}^n} = \mathcal{H}om(\mathcal{I}_C, \mathcal{O}_C)$.

In particular:

$$\dim_{h=[C]}\left(\mathrm{Hilb}_n^{P(t)}\right) \leq h^0(C, N_{C/\mathbb{P}^n}).$$

Naive guess:

$$\dim_{[C]}\left(\mathrm{Hilb}_n^{P(t)}\right) \overset{?}{=} h^0(C, N_{C/\mathbb{P}^n}) \overset{?}{=} \chi(N_{C/\mathbb{P}^n})$$

THIS IS UNTRUE IN GENERAL!

Actually:

$$\chi(\mathcal{N}_{C/\mathbb{P}^n}) \leq \dim_{[C]}(\mathrm{Hilb}_r^{p(t)}) \leq h^0(C, \mathcal{N}_{C/\mathbb{P}^n})$$

(we've already shown this)

By Prop. 5.12 (p. 33 of "Geom. of Alg. curves, Vol. II"

Prop 5.12:

X - closed, local complete intersection subscheme of $\mathbb{P}^n$. $h = [X]$. Then the dimension of every irreducible component of $\mathrm{Hilb}_r^{p(t)}$ at h is at least

$$h^0(X, \mathcal{N}_{X/\mathbb{P}^n}) - h^1(X, \mathcal{N}_{X/\mathbb{P}^n}).$$

In particular

$$\dim_{[C]}(\mathrm{Hilb}_r^{p(t)}) \geq \chi(\mathcal{N}_{C/\mathbb{P}^n})$$

$$= (h^0(C, \mathcal{N}_{C/\mathbb{P}^n}) - h^1(C, \mathcal{N}_{C/\mathbb{P}^n})).$$

Let's calculate $\chi(\mathcal{N}_{C/\mathbb{P}^n})$

Consider the sequence

$$0 \to T_C \to T_{\mathbb{P}^r} \otimes \mathcal{O}_C \to \mathcal{N}_{C/\mathbb{P}^r} \to 0$$

Explanation? Apply contravariant $\mathrm{Hom}_{\mathcal{O}_C}(-, \mathcal{O}_C)$ to exact seq.

$$0 \to I/I^2 \to \Omega_{X/k} \otimes \mathcal{O}_C \to \Omega_{C/k} \to 0$$

$\Rightarrow$

$$0 \to \mathrm{Hom}_{\mathcal{O}_C}(\Omega_{C/k}, \mathcal{O}_C) \to \mathrm{Hom}_{\mathcal{O}_C}(\Omega_{\mathbb{P}^r_k} \otimes \mathcal{O}_C, \mathcal{O}_C) \to \mathrm{Hom}_{\mathcal{O}_C}(I/I^2, \mathcal{O}_C) \to 0$$

$$\begin{array}{ccccc} \| & & \| & & \| \\ T_C & & \mathrm{Hom}_{\mathcal{O}_C}(\Omega_{\mathbb{P}^r_k}, \mathcal{O}_C) \otimes \mathcal{O}_C & & \mathcal{N}_{C/\mathbb{P}^r} \\ & & \| & & \\ & & T_{\mathbb{P}^r} \otimes \mathcal{O}_C & & \end{array}$$

Returning to $0 \to T_C \to T_{\mathbb{P}^r} \otimes \mathcal{O}_C \to \mathcal{N}_{C/\mathbb{P}^r} \to 0$

$\Rightarrow$

$$\deg(\mathcal{N}_{C/\mathbb{P}^r}) = (r+1)d + 2g - 2$$

(why?) $r+1$ - degree of tangent bundle

d = degree of C

$2g-2$ = degree of the canonical divisor

$= -\deg T_C = \deg K_C$

iv) By RRC:

$$\chi(N_{C/\mathbb{P}^r}) = \deg(N_{C/\mathbb{P}^r}) - (r-1)(g-1)$$

$$= (r+1)d + 2g-2 - (r-1)(g-1)$$

$$= \underbrace{(r+1)d - (r-3)(g-1)}_{\substack{\parallel \\ h_{d,g,r}}}.$$

Alternative proof of

$$\dim_{[C]}({}^{\prime}\mathcal{H}\cap \mathcal{G}^{\rho(t)}_{r}) \geq \chi(N_{\times/\mathbb{P}^r})$$

Consider family of line bundles of degree d & of genus g

$\Rightarrow$ the locus of those line bundle ~~with~~ w/ $r+1$ or more sections has codim. $(r+1)(g-d+r)$ in the nbhd of a line bundle w/ exactly $r+1$ sections.

+ Apply this to family of all line bundles of deg. d on all curves of genus g.

$\Rightarrow$ family of all linear series of deg $= d$ & genus $= g$ has local dim. $=$

$= 4g-3-(r+1)(g-d+r)$.

Without base pts these linear series determine a map to a curve up to the (r^2+2r)-dim. family $PGL(r+1,\mathbb{C})$ of autom's of $\mathbb{P}^r$

$\Rightarrow \dim_{[C]}\mathcal{H} \geq 4g-3-(r+1)(g-d+r)+(r^2+2r)$

$= (r+1)d-(r-3)(g-1)$.

Alternative proof of

$$\dim_{[C]}(\mathrm{Hilb}_r^{P(t)}) \geq \chi(N_{X/\mathbb{P}^r})$$

Consider family of line bundles of degree d & of genus g

$\Rightarrow$ the locus of those line bundle ~~that~~ w/ $r+1$ or more sections has codim. $(r+1)(g-d+r)$ in the nbhd of a line bundle w/ exactly $r+1$ sections.

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Without base pts these linear series determine a map to a curve up to the (r^2+2r)-dim. family $PGL(r+1, \mathbb{C})$ of autom's of $\mathbb{P}^r$

$\Rightarrow \dim_{[C]} H \geq 4g-3-(r+1)(g-d+r)+(r^2+2r)$

$= (r+1)d-(r-3)(g-1)$.

Import case: when $\dim \mathrm{Hilb}^{P(t)}_{-} = \chi(N_{C/\mathbb{P}^r})$.

Proof. ?

C – smooth, nondegen curve of deg. $= d$ & genus $= g$ (?)

$\mathcal{O}_C(1)$ – nonspecial (i.e.

Consider: ...

Want to show. $H^1(C, N_{C/\mathbb{P}^r}) = 0$.

Consider:

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & T_C \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] & \mathcal{O}_C^{r+1}(1) \arrow[r] & T_{\mathbb{P}^r}\otimes\mathcal{O}_C \arrow[r] \arrow[d] & 0 \\
 & & & N_{C/\mathbb{P}^r} \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

restrict to C, the Euler seq. Recall Euler seq.! for the tangent bundle of projective space.

Apply cohom. $H^1(C, -)$

$$\begin{tikzcd}
H^1(C, \mathcal{O}_C(1))^{r+1} \arrow[r] & H^1(C, T_{\mathbb{P}^r}\otimes\mathcal{O}_C) \arrow[r] \arrow[d] & 0 \\
 & H^1(C, N_{C/\mathbb{P}^r}) \arrow[d] & \\
 & 0 &
\end{tikzcd}$$

$\Rightarrow$ $0 =$

as $\mathcal{O}_C(1)$ is nonspecial

$\Rightarrow H^1(C, N_{C/\mathbb{P}^r}) = 0$.

Severi Varieties

Will see your summer 2010 For me: tutorial is say more.

Def. In the space of plane curves of dimension d, define:

- $V_{d,g}$:= locus of reduced, irr. curves of degree d of genus g.

- $U_{d,g}$:= $V_{d,g} \cap$ {curves having only nodes as singularities} (not ", but okay?)

- $\overline{V_{d,g}}$:= closure of $V_{d,g}$ in $\mathbb{P}^N$. ($3d+g-1$ = $(r+1)d-(r-3)(g-1)|_{r=2}$)

THM (Zariski–Harris)

1) $U_{d,g}$ is smooth of dim. $3d+g-1 = h_{d,g,2}$

2) $U_{d,g}$ is dense in $V_{d,g}$

3) $V_{d,g}$ is irr.

We will show that $U_{d,g}$ is smooth of dimension $h_{d,g,2}$.

simple ... $C: x^2 - y^2 = 0 \xrightarrow{\text{aff}} 1 - xy = 0$ $a_{11} = -1$, $a_{10} = a_{01} = 0$

$$\Sigma = \{ (C, p_1, \dots, p_\delta) \mid p_i \in C_{sing} \} \subseteq \mathbb{P}^N \times (\mathbb{P}^2)^\delta$$

$\boxed{N = \frac{d(d+3)}{2} \text{ plane curves of degree } d \text{ is } \mathbb{P}^N}$

Write down equations.

For aff. coords (x, y) in $\mathbb{P}^2$

a_{ij} - coeff.s in eq. of C

(x_α, y_α) - coord.s of p_α.

Then Σ is given by

$$F_\alpha(a_{ij}, x_\alpha, y_\alpha) = \sum a_{ij} x_\alpha^i y_\alpha^j = 0 \quad \text{— } \underline{p_\alpha \text{ is on } C}.$$

$$\frac{\partial F_\alpha}{\partial x_\alpha} = G_\alpha(a_{ij}, x_\alpha, y_\alpha) = \sum i a_{ij} x_\alpha^{i-1} y_\alpha^j = 0$$

$$\frac{\partial F_\alpha}{\partial y_\alpha} = H_\alpha(a_{ij}, x_\alpha, y_\alpha) = \sum j \cdot a_{ij} \cdot x_\alpha^i \cdot y_\alpha^{j-1} = 0$$

p_α is singular! $\qquad \forall\, 1 \le \alpha \le \delta$.

$\longrightarrow$ Expect naively: $\dim(\Sigma) = \underbrace{N + 2\delta}_{} - 3\delta = N - \delta$

number of coordinates of elements in Σ.

3δ equations.

Q: Maybe these 3δ equations are dependent.

∴ In nbhd of nodal curve C w/ nodes $p_1, \dots, p_\delta$, the variety Σ maps 1:1 to $V_{d,g}$.

To do this, we show that diff. of this map is injective at C.

Ex. 1 nodal pt, i.e. $\delta = 1$, $p = (0,0)$ near C.

$(C, p = (0,0)) \in \Sigma_1 = \Sigma_{\delta=1}$.

$F = a_{00} + a_{11}xy + a_{20}x^2 + a_{02}y^2 = 0$

$G = a_{11}y + 2a_{20}x = 0$

$H = a_{11}x + 2a_{02}y$

Matrix of partials of F, G, H w.r.t. x, y, a_{00} is

$$\begin{array}{c|ccc} & F & G & H \\ \hline \frac{\partial}{\partial x} & 0 & a_{20} & a_{11} \\ \frac{\partial}{\partial y} & 0 & a_{11} & a_{02} \\ \frac{\partial}{\partial a_{00}} & 1 & 0 & 0 \end{array}$$

The det: $(a_{20}a_{02} - a_{11}^2) \neq 0$ when C has an ordinary node at p.

(!) $\longrightarrow$ At $(C, p) \in \Sigma_1$ is $\Sigma_{\delta=1}$ is smooth of codim. 3 in $\mathbb{P}^N \times \mathbb{P}^2$ & $\begin{array}{c}\Sigma \\ \downarrow \psi \\ \mathbb{P}^N\end{array}$ is an immersion (with ψ) at (C, p) mapping isomorphically to the polynomials of deg. 2 w/ $a_{00} = 0$, i.e. vanishing at p.

In general, (i.e. curve w/ exactly δ nodes for its singularities), the $\begin{matrix}\Sigma \\ \downarrow \\ \mathbb{P}^N\end{matrix}$ factors through Σ_1 in

δ diff. ways. (I guess as before $\searrow$ $\begin{matrix}\Sigma \\ \downarrow \\ \Sigma_1 \\ \downarrow \\ \mathbb{P}^N\end{matrix}$)

$$\begin{tikzcd}[column sep=small] & \Sigma \arrow[dl] \arrow[d] \arrow[dr] & \\ \Sigma^{p_1} \arrow[dr] & \Sigma^{p_2} \cdots \arrow[d] & \Sigma^{p_\delta} \arrow[dl] \\ & \mathbb{P}^N & \end{tikzcd}$$

~~just dist.~~ just focus on a different fix each time.

(in an analytic nbhd of C) (?)

$\Rightarrow V_{d,g} = \bigcap$ (images of analytic nbhds of pts $(C, p_i) \in \Sigma_1$)

$\Rightarrow T_{[C]} V_{d,g} = \{$polys F of deg. d s.t. $F(p_\alpha) = 0,\ \forall 1 \le \alpha \le \delta\}$

linear subspace

FACT: Each pt imposes independent conditions on curves of any degree $m \ge d-3$.

$$\Rightarrow \dim(T_{[C]} V_{d,g}) = N - \delta$$

$\Rightarrow V_{d,g}$ is smooth of this dim.

Note: $N - \delta = \dfrac{d(d+3)}{2} - \left(\dfrac{(d-1)(d-2)}{2} - g\right)$

$= 3d + g - 1$

$= h_{d,g,2}$.