

Dimension of the Hilbert scheme (1)

Note: Only discuss Hilbert Schemes of curves.

Recall: Tangent space to the Hilbert scheme of curves in \mathbb{P}^n at a point $u = \{C\}$ is given by

$$T_u(\text{Hilb}_n^{p(t)}) = H^0(C, -N_{C/\mathbb{P}^n})$$

where $-N_{C/\mathbb{P}^n} = \text{Hom}(T_C, \mathcal{O}_C)$.

In particular:

$$\dim_{u=\{C\}}(\text{Hilb}_n^{p(t)}) \leq h^0(C, -N_{C/\mathbb{P}^n}).$$

Naive guess:

$$\dim_{\{C\}}(\text{Hilb}_n^{p(t)}) = h^0(C, -N_{C/\mathbb{P}^n}) - \chi(-N_{C/\mathbb{P}^n})$$

THIS IS WRONG IN GENERAL

Actually:

$$\chi(\mathcal{N}_{C/\mathbb{P}^n}) \leq \dim_{\mathbb{C}} \left(\text{Hilb}_{\mathbb{C}}^{p(t)} \right) \leq h^0(C, \mathcal{N}_{C/\mathbb{P}^n})$$

↑
(we've already shown this)

By Prop. 5.12 (p. 33 of "Geom. of Alg. Curves, Vol. II")

Proof 5.12:

X - closed, local complete intersection subscheme of \mathbb{P}^n .
 $w_i = \{X\}$. Then the dimension of every irreducible component of $\text{Hilb}_{\mathbb{C}}^{p(t)}$ at w_i is at least

$$h^0(X, \mathcal{N}_{X/\mathbb{P}^n}) - h^1(X, \mathcal{N}_{X/\mathbb{P}^n}).$$

In particular

$$\dim_{\mathbb{C}} \left(\text{Hilb}_{\mathbb{C}}^{p(t)} \right) \geq \chi(\mathcal{N}_{C/\mathbb{P}^n})$$

$$\left(h^0(C, \mathcal{N}_{C/\mathbb{P}^n}) - h^1(C, \mathcal{N}_{C/\mathbb{P}^n}) \right).$$

Let's calculate $\chi(\mathcal{N}_{C/\mathbb{P}^n})$

Consider the sequence

$$0 \rightarrow T_C \rightarrow T_{\mathbb{P}^n} \otimes \mathcal{O}_C \rightarrow N_{C/\mathbb{P}^n} \rightarrow 0$$

Explanation? \swarrow apply contra. Funct. $\text{Hom}_{\mathcal{O}_C}(-, \mathcal{O}_C)$ to exact seq.

$$0 \rightarrow \mathcal{I}_{C/\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n} \otimes \mathcal{O}_C \rightarrow \mathcal{O}_{C/\mathbb{P}^n} \rightarrow 0$$

\Rightarrow

$$\begin{array}{ccccccc}
 0 & \rightarrow & \text{Hom}_{\mathcal{O}_C}(\mathcal{I}_{C/\mathbb{P}^n}, \mathcal{O}_C) & \rightarrow & \text{Hom}_{\mathcal{O}_C}(\mathcal{O}_{\mathbb{P}^n} \otimes \mathcal{O}_C, \mathcal{O}_C) & \rightarrow & \text{Hom}_{\mathcal{O}_C}(\mathcal{O}_{C/\mathbb{P}^n}, \mathcal{O}_C) \\
 & & \parallel & & \parallel & & \parallel \\
 & & T_C & & \text{Hom}_{\mathcal{O}_C}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_C) \otimes \mathcal{O}_C & & N_{C/\mathbb{P}^n} \\
 & & & & \parallel & & \parallel \\
 & & & & T_{\mathbb{P}^n} \otimes \mathcal{O}_C & & 0
 \end{array}$$

Returning to $0 \rightarrow T_C \rightarrow T_{\mathbb{P}^n} \otimes \mathcal{O}_C \rightarrow N_{C/\mathbb{P}^n} \rightarrow 0$

$$\Rightarrow \text{deg}(N_{C/\mathbb{P}^n}) = (n+1)d + 2g - 2$$

(why) $n+1$ - degree of tangent bundle

d = degree of C .

$2g-2$ = degree of canonical divisor.

$= -\text{deg} T_C = \text{deg} K_C$

\Rightarrow By RRC:

$$\begin{aligned}\chi(N_{C_{10^r}}) &= \deg(N_{C_{10^r}}) - (r-1)(g-1) \\ &= (r+1)d + 2g - 2 - (r-1)(g-1) \\ &= \underbrace{(r+1)d - (r-3)(g-1)}_{\substack{!! \\ \text{h.d.g.r.}}}\end{aligned}$$

Alternative proof of

$$\dim_{\mathbb{C}} H^0(\mathbb{P}^1, \mathcal{O}(d)) \geq \chi(\mathcal{N}_{\mathbb{P}^1/\mathbb{P}^n})$$

Consider family of line bundles of degree d on
of genus g

\Rightarrow the locus of those line bundles ~~that~~ on
with $r+1$ or more sections has codim.

$(r+1)(g-d+r)$ in the world of a line
bundle with exactly $r+1$ sections.

\Rightarrow apply this to family of all line bundles
of deg. d on all curves of genus g .

\Rightarrow family of all linear series of deg = d on
genus = g has local dim. =

$$= 4(g-3) - (r+1)(g-d+r).$$

What does this tell us about linear series

define a map to a curve up to

the (r^2+r) -dim. family $\mathbb{P}^r \mathcal{H}(r+1, \mathcal{O}(1))$

of automorphisms of \mathbb{P}^r

$$\begin{aligned} \Rightarrow \dim_{\mathbb{C}} H^0 &\geq 4(g-3) - (r+1)(g-d+r) + (r^2+r) \\ &= (r+1)d - (r-3)(g-1). \end{aligned}$$

Alternative proof of

$$\dim_{\mathbb{C}} H^0(\mathbb{P}^1, \mathcal{O}(d)) \geq \chi(\mathcal{N}_{X/\mathbb{P}^1})$$

Consider family of line bundles of degree d of genus g

\Rightarrow the locus of those line bundles ~~with~~ with $r+1$ or more sections has codim.

$(r+1)(g-d+r)$ in the world of a line bundle with exactly $r+1$ sections.

\Rightarrow apply this to family of all line bundles of deg. d on all curves of genus g .

\Rightarrow family of all linear series of deg. $= d$ of genus $= g$ has local dim. =

$$= 4(g-3) - (r+1)(g-d+r).$$

Without loss of generality these linear series

determine a map to a curve up to

the (r^2+2r) -dim. family $\mathbb{P}^r \times \mathbb{P}^1$

of automorphisms of \mathbb{P}^1

$$\Rightarrow \dim_{\mathbb{C}} H^0 \geq 4(g-3) - (r+1)(g-d+r) + (r^2+2r)$$

$$= (r+1)d - (r-3)(g-1).$$

Important case: when $\dim(\mathbb{P}^n) = \mathcal{K}(N_{C/\mathbb{P}^n})$.

Proof,

C - smooth, irreducible curve of $\deg. = d$ & genus $= g$

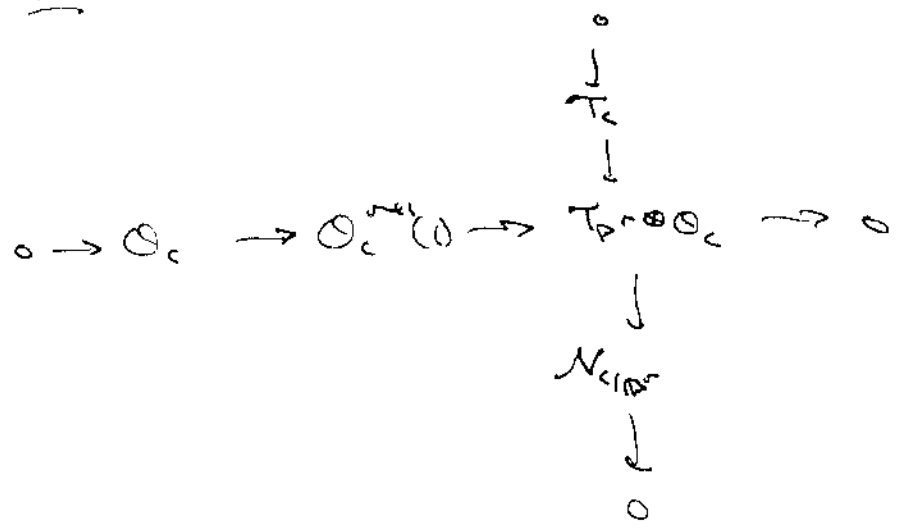
$\mathcal{O}_C(1)$ - nonspecial (i.e.)

(?)

Consider:

Want to show: $H^1(C, N_{C/\mathbb{P}^n}) = 0$.

Consider:



restrict to C , the Euler seq.
Recall Euler seq.!

for the tangent bundle of projective space.

Apply covar. $H^1(C, -)$

$$\Rightarrow H^1(C, \mathcal{O}_C^{\oplus n+1}) \rightarrow H^1(C, T_{\mathbb{P}^n} \otimes \mathcal{O}_C) \rightarrow 0$$

$$\begin{array}{ccc}
 \mathcal{O}_C & \rightarrow & 0 \\
 \uparrow & & \downarrow \\
 H^1(C, \mathcal{O}_C) & \rightarrow & H^1(C, N_{C/\mathbb{P}^n}) \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

$\Rightarrow H^1(C, N_{C/\mathbb{P}^n}) = 0$.

Severs Varieties

with some assumptions: For $r \geq 2$: r lines \rightarrow say more.

Def. In the space of plane curves of dimension d , define:

$V_{d,g} :=$ locus of reduced, irr. curves of degree d & genus g .

$U_{d,g} := V_{d,g} \rightarrow$ { curves having only nodes as singularities
(no \tilde{c} , but "jökay?")

$\overline{V}_{d,g} :=$ closure of $V_{d,g}$ in \mathbb{P}^N . ($3d + g - 1$)

THM (Zariski-Harris)

$$(r+1)d - (r-3)(g-1) \Big|_{r=2}$$

① $U_{d,g}$ is smooth & dim. $3d + g - 1 = h_{d,g,2}$

② $U_{d,g}$ is dense in $V_{d,g}$ i.e. $U_{d,g} = V_{d,g}$

③ $V_{d,g}$ is irr. $(\overline{V}_{d,g} = V_{d,g})$

We will show that $U_{d,g}$ is smooth & dimension $h_{d,g,2}$.

[Faded handwritten notes and diagrams, possibly showing a sequence of points or curves]

$C: x^2 + y^2 = c \xrightarrow{c=0} x^2 + y^2 = 0$

$a_{11} = \dots = a_{1, n-1} = 0$
 $a_{10} = a_{01} = 0$

$\Sigma = \{ (C, (F_1, \dots, F_S)) \mid P, (C, S, N) \} \subseteq (P^N) \times (P^S)^C$

Write down equations.

$$N = \frac{d(d+3)}{2}$$
 plus number of degrees of freedom is P^d .

For each coord. $(x, y) \in P^2$

a_{ij} - coeffs. in eq. of C

(x, y) - coords. of P^2

Then Σ is given by

$F_\alpha(a_{ij}, x, y) = \sum a_{ij} x^i y^j = 0$

$\frac{\partial F_\alpha}{\partial x} = G_\alpha(a_{ij}, x, y) = \sum i a_{ij} x^{i-1} y^j = 0$

$\frac{\partial F_\alpha}{\partial y} = H_\alpha(a_{ij}, x, y) = \sum j a_{ij} x^i y^{j-1} = 0$

Σ is singular!

$\forall \alpha, \alpha \in \Sigma$

\rightarrow Expect accordingly: $\dim(\Sigma) = \underbrace{N + 2S}_{\text{number of coordinates of elements in } \Sigma} - \underbrace{3S}_{\text{3S equations.}}$

number of coordinates of elements in Σ .

3S equations.

Q: Maybe these 3S equations are dependent.

∴ In a neighborhood of a nodal curve C of nodes

(11)

$f_1 \rightarrow f_2$, the variety Σ maps 1:1 to V_d, g .

To do this, we show that $d\phi_1^{-1} \circ d\phi_2^{-1}$ map is injection at C .

Ex: 1 nodal pt, i.e. $\delta=1, p=(0,0)$ - u.c. l.c.s.

$(C, p=(0,0)) \in \Sigma_{\delta=1} = \Sigma_{\delta=1}$

$F = a_{10}x + a_{20}x^2 + a_{02}y^2 = 0$

$G = a_{11}x + 2a_{20}x = 0$

$H = a_{11}x + 2a_{02}y$

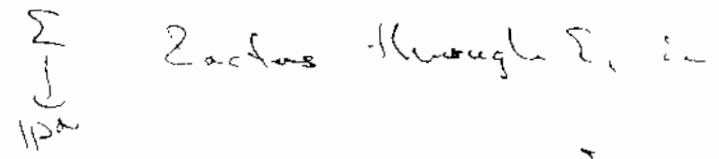
Matrix of partials of F, G, H w.r.t. x, y, a_{00} is

	F	G	H
$\frac{\partial}{\partial x}$	0	a_{20}	a_{11}
$\frac{\partial}{\partial y}$	0	a_{11}	$2a_{02}$
$\frac{\partial}{\partial a_{00}}$	1	0	0

The discr: $(2a_{20}a_{02} - a_{11}^2) \neq 0$ when C has an ordinary node at p .

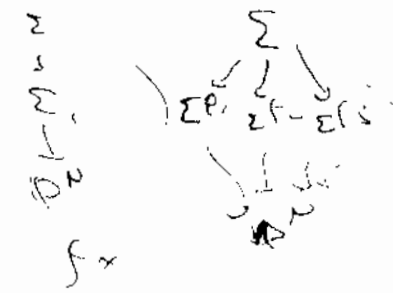
∴ At $(C, p) \in \Sigma_{\delta=1}$ is smooth of codim. 3 in $(\mathbb{P}^1 \times \mathbb{P}^2) \in \Sigma_{\mathbb{P}^2}$ is an immersion (why?) at (C, p) - apply $d\phi_1^{-1} \circ d\phi_2^{-1}$ isomorphically to the polynomials $\delta_{\text{deg } 2}$ w/ $a_{00}=0$, i.e. vanishing at p .

In general, \mathbb{C} -curve intersecting Σ nodes first as singularities, then



Σ diff. covs.

(\mathbb{P}^1 guess as before)



~~just desc.~~ just factors on a different

each time.

(in an analytic neighborhood of Σ)

$$\Rightarrow V_{d,g} = \bigcup (\text{images of analytic neighborhoods of pts } (c, p) \in \Sigma)$$

$$\Rightarrow T_{\mathbb{C}^3} V_{d,g} = \left\{ \text{poles of deg. } d \text{ s.t. } F(f, z) = 0, \text{ visited} \right\}$$

↓
dim-subspace

Fact: Each p_i imposes independent conditions on curves of any degree $m \geq d-3$.

$$\Rightarrow \dim(T_{\mathbb{C}^3} V_{d,g}) = N - \delta$$

$\Rightarrow V_{d,g}$ is smooth of this dim.

Note: $N - \delta = \frac{d(d+3)}{2} - \left(\frac{(d-1)(d-2)}{2} - g \right)$

$$= 3d + g - 1$$

$$= h_{d,g, 2}$$

