Examination in Basic Algebraic Topology

(course 0366490301, Fall semester 2011-12)

Lecturer: Prof. E. Shustin
Assignment: Solve 8 problems from the following list

Problem 1. A homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow\varphi & & \downarrow\psi \\ X' & \xrightarrow{f'} & Y' \end{array} \quad \text{with} \quad \psi\circ f \overset{h}{\sim} f'\circ\psi',$$

in which φ and ψ are homotopy equivalences, is called a homotopy equivalence of the maps $f : X \to Y$ and $f' : X' \to Y'$.

(1) Prove that any map $f : X \to Y$ is homotopy equivalent to the Borsuk pair inclusion

$$\mathrm{in} : X \to \mathrm{Cyl}(f) := (X \times I) \amalg Y/((x, 0) \sim f(x)), \quad \mathrm{in}(x) = (x, 1) \ .$$

(2) Prove that any map $f : X \to Y$ is homotopy equivalent to the Hurewicz fibration $F : X_1 \to Y$, where

$$X_1 = \{(x, s) \in X \times C(I, Y) \ : \ s(0) = f(x)\}, \quad F(x, s) = s(1) \ .$$

Problem 2. Classify the 3-fold coverings of $S^1 \vee S^1$ and find all normal coverings.

Problem 3. Find the fundamental group of the complement of the two circles $\{x^2 + y^2 = 1, \ z = 1\}$ and $\{x^2 + y^2 = 1, \ z = -1\}$ in $\mathbb{R}^3$.

Problem 4. Let X be a connected CW space with a continuous associative operation $(x, y) \mapsto x \circ y$ such that

$$x \circ x = x, \quad x \circ e = e \circ x$$

for any $x \in X$ and certain $e \in X$. Prove that X is contractible.

Problem 5. Prove that the join $X * Y := X \times Y \times I/\sim$ where

$$(x, y', 0) \sim (x, y'', 0) \text{ for all } x \in X, \ y', y'' \in Y \ ,$$

$$(x', y, 1) \sim (x'', y, 1) \text{ for all } x', x'' \in X, \ y \in Y \ ,$$

of simply connected spaces X and Y is simply connected too.

Problem 6. (1) Prove that, for a field $K = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ and any topological space X,

$$H_q(X; K) \simeq H_q(X) \otimes K, \quad H^q(X; K) \simeq \mathrm{Hom}(H_q(X), K), \quad q \geq 0 \ .$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

(2) Prove that, for any field K and any topological space X,

$$H^q(X;K) \simeq \mathrm{Hom}(H_q(X;K),K), \quad q \geq 0 \ .$$

Problem 7. (1) Prove that $\pi_*(S^2) \simeq \pi_*(S^3 \times \mathbb{C}P^\infty)$, but $H_*(S^2) \not\simeq H_*(S^3 \times \mathbb{C}P^\infty)$.

(2) Prove that if $1 < m < n$, then $\pi_*(S^m \times \mathbb{R}P^n) \simeq \pi_*(\mathbb{R}P^m \times S^n)$, but $H_*(S^m \times \mathbb{R}P^n) \not\simeq H_*(\mathbb{R}P^m \times S^n)$.

Problem 8. (1) Prove that $H_*(S^2 \vee S^1 \vee S^1) \simeq H_*(S^1 \times S^1)$, but $\pi_*(S^2 \vee S^1 \vee S^1) \not\simeq \pi_*(S^1 \times S^1)$.

(2) Prove that the natural projection $f : S^1 \times S^1 \to S^1 \times S^1/(S^1 \vee S^1) = S^2$ induces the zero homomorphism $f_* : \pi_*(S^1 \times S^1) \to \pi_*(S^2)$ and a non-zero homomorphism $f_* : \widetilde{H}_*(S^1 \times S^1) \to \widetilde{H}_*(S^2)$.

Problem 9. Prove that a loop $f : S^1 \to X$ defines an element of $\mathrm{Ker}\{\pi_1(X) \to H_1(X)\}$ if and only if f extends up to a map $F : M_g \to X$, where M_g is a sphere with $g \geq 0$ handles and with a cut out disc so that $\partial M_g = S^1$ and $F\big|_{S^1} = f$. Show that the minimal genus g in the above extension is equal to the minimal number of commutators in the representation

$$[f] = a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} \in \pi_1(X) \ .$$

Problem 10. Show that in any H'-space X the $\cup$-product is trivial, i.e., $c_1 \cup c_2 = 0$ for any $c_1 \in H^n(X,G)$, $c_2 \in H^m(X,G)$ as $n,m \geq 1$.

Hint. An H'-space X possesses a comultiplication $\mu : X \to X \vee X$ and a coinverse map $\nu : X \to X$ such that the maps $(\mathrm{Id}_X \vee \nu) \circ \mu : X \to X$ and $(\nu \vee \mathrm{Id}_X) \circ \mu : X \to X$ are homotopic to Id_X.

Good luck!