a talk about
Moduli of Curves
Advanced Seminar in Geometry and Topology 1
0366-4856-01
Prof. E. Shustin

Mendy Shoval
ID 0-2745444-6

December 7, 2011

Chapter 1

The Hilbert Scheme

We have seen that the Hilbert polynomial of a projective variety X of dimension s and degree d in $\mathbb{P}^r$ has leading term $h_X(m) = \frac{dm^s}{s!} + \cdots$.

Exercise 1.13(2) A subscheme $X \subset \mathbb{P}^r$ is a hypersurface of degree d if and only if

$$h_X(m) = \binom{r+m}{r} - \binom{r+m-d}{r}.$$

Proof. Suppose X is a hypersurface of degree d, given by the equation $F_d = 0$ where F_d is a homogeneous form of degree d in the variables $X_0, \dots, X_r$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^r}(m-d) \xrightarrow{\cdot F_d} \mathcal{O}_{\mathbb{P}^r}(m) \longrightarrow \mathcal{O}_X(m) \longrightarrow 0$$

for all $m \in \mathbb{Z}$. It is known that $H^1(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)) = 0$, $\forall m \gg 0$, hence the resulting long exact sequence on cohomology gives

$$0 \longrightarrow H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m-d)) \xrightarrow{\cdot F_d} H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)) \longrightarrow H^0(X, \mathcal{O}_X(m)) \longrightarrow 0$$

for all $m \gg 0$. We therefore get that

$$h_X(m) = h^0(X, \mathcal{O}_X(m)) = h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)) - h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m-d)) = \binom{r+m}{r} - \binom{r+m-d}{r},$$

for all $m \gg 0$, as needed.

Conversely, suppose $h_X(m)$ has the required form, then expanding the binomial coefficients we see that

$$h_X(m) = \frac{dm^{r-1}}{(r-1)!} + \cdots .$$

By the discussion above we see that $\dim X = r-1$, but it may be reducible with lower dimension components. Suppose $X = Y \cup Z$ where Y is the union of components of X of dimension $r-1$ and Z is the union of components of X of lower dimension. Let $\mathcal{J}_{Y/X}$ be the ideal sheaf of Y on X and let $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}_Y$. Then $(Z, \mathcal{O}_Z)$ is a scheme structure on Z such that $h_X(m) = h_Y(m) + h_Z(m)$ for all $m \in \mathbb{Z}$. Moreover, since $\dim Z < r-1$ we have $\deg h_Z < r-1$, so the leading term of h_Y is also $\frac{dm^{r-1}}{(r-1)!}$. Since Y is now really a hypersurface in $\mathbb{P}^r$ we get, by the form of its Hilbert polynomial, that $\deg Y = d$. Now by the first direction we deduce that $h_Y(m) = \binom{r+m}{r} - \binom{r+m-d}{r} = h_X(m)$ for all m. This implies that $h_Z = 0$, so Z is actually empty, and we have $X = Y$, a hypersurface of degree d.

Exercise 1.13(3) A subscheme $X \subset \mathbb{P}^r$ is a linear space of dimension s if and only if

$$h_X(m) = \binom{s+m}{s}.$$

Proof. If X is linear then $X \cong \mathbb{P}^s$, so the result follows from the fact that

$$h_{\mathbb{P}^s}(m) = \dim \mathbb{C}[X_0, \dots, X_s]_m = \binom{s+m}{s}.$$

Conversely, if $h_X(m) = \binom{s+m}{s}$ then its leading term is just $\frac{m^s}{s!}$, hence $\dim X = s$ and $\deg X = 1$. But a projective variety has degree 1 if and only if it is a linear space, so we are done.

Exercise 1.14 The Hilbert scheme of lines in $\mathbb{P}^3$ is the Grassmannian $\mathcal{G} = \mathbb{G}(2,4)$ (written projectively as $\mathbb{G}(1,3)$).

Proof. Note that a closed subscheme of $\mathbb{P}^3$ is a line if and only if $h_X(m) = \binom{m+1}{1} = m+1$, by the previous exercise, so we are talking about $\mathcal{H}_{3,m+1}$. First, recall that $\mathcal{G}$ has a universal bundle $\mathcal{U}$ of rank 2, whose projectivization $\mathbb{P}(\mathcal{U})$ is the $\mathbb{P}^1$-bundle whose fibre over any point $\Lambda \in \mathcal{G}$ is the line $\mathbb{P}(\Lambda) \subset \mathbb{P}^3$. Now let $\mathcal{X} \hookrightarrow \mathbb{P}^3 \times B$ be a closed subscheme such that $\varphi : \mathcal{X} \to B$ is a family of lines in $\mathbb{P}^3$ over B:

$$\begin{tikzcd}
\mathcal{X} \arrow[r, hook] \arrow[dr, "\varphi"'] & \mathbb{P}^3 \times B \arrow[r] \arrow[d, "\pi"'] & \mathbb{P}^3 \\
 & B &
\end{tikzcd}$$

Then $\mathcal{X}$ is given locally as the span of two 4-vectors

$$\begin{pmatrix} a_{11}(b) & \cdots & a_{14}(b) \\ a_{21}(b) & \cdots & a_{24}(b) \end{pmatrix} = \begin{pmatrix} \underline{a}_1(b) \\ \underline{a}_2(b) \end{pmatrix}, b \in B.$$

We may form, then, the morphism $B \to \mathcal{G}$ by mapping $b \mapsto \underline{a}_1(b) \wedge \underline{a}_2(b) \in \Lambda^2 \mathbb{A}^4_B$. This morphism also has the property that it is unique such that the family $\mathcal{X} \to B$ is the pullback of $\mathcal{U} \to \mathcal{G}$, hence $\mathbb{G}(2,4)$ is a fine moduli space for this problem. By uniqueness of a fine moduli space, it is isomorphic to the Hilbert scheme $\mathcal{H}_{3,m+1}$.

Exercise 1.15 The Hilbert scheme of plane curves of degree d is $\mathbb{P}^N$, where $N = \frac{d(d+3)}{2}$, and a curve (which is always given by a homogeneous polynomial of degree d) is sent to its coefficients.

Proof. 1. The incidence correspondence

$$\mathcal{T} = \left\{ (f, P) \in \mathbb{P}^N \times \mathbb{P}^2 \mid f(P) = 0 \right\}$$

is flat over $\mathbb{P}^N$. Indeed, $\mathcal{T}$ is a projective scheme over the base $\mathbb{P}^N$ which is reduced, and all the fibres over the closed points of $\mathbb{P}^N$ have the same Hilbert polynomial $h_X(m) = \binom{m+2}{2} - \binom{m+2-d}{2}$. By a well-known criterion for flatness, this gives the result. Actually, the map $\mathcal{T} \to \mathbb{P}^N$ will turn out to be the universal family. To show this we proceed as follows. Let $\varphi : \mathcal{X} \to B$ be a flat family of plane curves, i.e., there is a closed immersion

$$\begin{tikzcd}
\mathcal{X} \arrow[r, hook] \arrow[dr, "\varphi"'] & \mathbb{P}^2 \times B \arrow[r] \arrow[d, "\pi_B"'] & \mathbb{P}^2 \\
& B &
\end{tikzcd}$$

whose fibres are curves of degree d in $\mathbb{P}^2$. Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{X}$ in $\mathbb{P}^2 \times B$.

2. $\mathcal{I}$ is flat over B. Indeed, consider the twisted exact sequence

$$0 \longrightarrow \mathcal{I}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^2\times B}(m) \longrightarrow \mathcal{O}_{\mathcal{X}}(m) \longrightarrow 0 \qquad (m \in \mathbb{Z}).$$

Applying the functor $(\pi_B)_*$ we get the long exact sequence

$$0 \longrightarrow (\pi_B)_*(\mathcal{I}(m)) \longrightarrow (\pi_B)_*(\mathcal{O}_{\mathbb{P}^2\times B}(m)) \longrightarrow (\pi_B)_*(\mathcal{O}_{\mathcal{X}}(m)) \longrightarrow (R^1\pi_B)_*(\mathcal{I}(m)) \longrightarrow \cdots$$

where $(R^1\pi_B)_*$ is the right-derived direct image functor. $\mathcal{I}$ is coherent, so for $m \gg 0$ we have $(R^1\pi_B)_*(\mathcal{I}(m)) = 0$.

Now, it is known that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^2 \times B$ is flat over B if and only if $(\pi_B)_*(\mathcal{F}(m))$ is locally free on B for $m \gg 0$. This applies, in particular, to $\mathcal{O}_{\mathcal{X}}$, since φ is flat. We get that $(\pi_B)_*(\mathcal{O}_{\mathcal{X}}(m))$ is locally free for m large enough, and $(\pi_B)_*(\mathcal{O}_{\mathbb{P}^2\times B}(m))$ is always locally free of rank $\binom{m+2}{2}$. We see that $(\pi_B)_*(\mathcal{I}(m))$ is locally free for $m \gg 0$, hence by the same criterion $\mathcal{I}$ is flat over B.

3. Since $\mathcal{I}$ is flat we have that $\mathcal{I}(d)$ is locally free on $\mathbb{P}^2 \times B$. Actually it is locally given by $F_d(x : y : z; b) = 0$ where F_d is a form of degree d in the homogeneous coordinates x, y, z on $\mathbb{P}^b2$ and b is a set of local coordinates on B (not all the coefficients of F_d vanish simultaneously for any $b \in B$). We get that $(\pi_B)_*(\mathcal{I}(d))$ is locally free of rank 1, i.e., a line bundle on B. Its global sections correspond to F_d at every $b \in B$, and so yield a morphism $\chi : B \to \mathbb{P}^N$, given by $b \mapsto [$ coefficients of $F_d(b)]$.

4. Finally, the construction above shows that the original map $\varphi : \mathcal{X} \to B$ is the pull-back via χ of the universal family $\pi : \mathcal{T} \to \mathbb{P}^N$. If $\chi_1 : B \to \mathbb{P}^N$ is any other morphism with the same property then $\chi_1^*(\mathcal{O}(1))$ is isomorphic to $(\pi_B)_*(\mathcal{I}(d))$, hence the induced map is the same and we have $\chi_1 = \chi$. This finished the proof.

Exercise 1.16 Let $C \subset \mathbb{P}^3$ be the union of a plane quartic and a non-coplanar line meeting it at one point. Then C is not the flat specialization of a smooth curve of degree 5. Same question for a union of a plane quartic and a non-coplanar conic meeting it in 2 points.

Solution: The Hilbert polynomial of a plane quartic C_4 is, as we've already seen,

$$h_{C_4}(m) = \binom{m+2}{2} - \binom{m-2}{2} = 4m - 2$$

and the Hilbert polynomial of the line L is $h_L(m) = m + 1$. It follows that the Hilbert polynomial of $C = C_4 \cup L$ is

$$h_C(m) = (4m - 2) + (m + 1) - 1 = 5m - 2.$$

Note that the genus of a smooth space quintic C_5 is given by $g = p_a(C_5) = 1 - h_{C_5}(0)$. If C is a flat specialization of a family of such quintics then we must have the same Hilbert polynomial

$h_{C_5}(m) = h_C(m) = 5m - 2$, which implies that $g = 3$. But a smooth plane cubic has genus 6, whereas a nondegenerate smooth quintic in $\mathbb{P}^3$ has genus at most 2. This contradiction gives the result.

Now for the second claim. Here it's even more obvious, since the Hilbert polynomial of a conic C_2 is $h_{C_2} = 2m + 1$, so if $C = C_4 \cup C_2$ with $\sharp(C_4 \cap C_2) = 2$ then

$$h_C(m) = (4m - 2) + (2m + 1) - 2 = 6m - 3$$

and clearly no quintic has such a Hilbert polynomial (smooth space sextics might, and these must have genus 4. These are canonical curves, and it is known that they are always given as a complete intersection of a cubic and a quadric hypersurfaces).

Having introduced the Hilbert scheme, we can mention some other fine moduli spaces whose construction uses that of the Hilbert scheme or follows closely to it.

1. **Hilbert scheme of subschemes.** Given a subscheme Z in $\mathbb{P}^r$ and an integral polynomial h there is a fine moduli space $\mathcal{H}^Z_{h,r}$ parameterizing subschemes of Z that are closed in $\mathbb{P}^r$ and have Hilbert polynomial h. This is most easily seen as a closed subscheme of $\mathcal{H}_{P,r}$, since a closed subscheme is given by closed conditions.

2. **Hilbert scheme of morphisms.** Given subschemes $X \subseteq \mathbb{P}^r$ and $Y \subseteq \mathbb{P}^t$ and an integer d there is a fine moduli space $\mathcal{H}_{X,Y,d}$ parameterizing morphisms $f : X \to Y$ that are locally given by polynomial maps of degree $\leq d$. This can be constructed as a Hilbert scheme of subschemes of $X \times Y$ which are the graphs of such morphisms.

3. **Hilbert scheme of projective bundles.** Given a $\mathbb{P}^r$-bundle $\mathcal{P} \to Z$ and an integral polynomial h there is a fine moduli space $\mathcal{H}_{\mathcal{P},h,r}$ parameterizing subschemes of $\mathcal{P}$ whose fibres over closed points of Z have Hilbert polynomial h. More generally,

4. **Relative Hilbert scheme.** Given a projective morphism $\pi : \mathcal{X} \to \mathbb{P}^r \times Z \to Z$ and an integral polynomial h there is a fine moduli space $\mathcal{H}_{\pi,h,r}$ parameterizing subschemes of $\mathcal{X}$ whose fibres over closed points of Z have Hilbert polynomial h. This can be thought of as a relative version of the usual Hilbert scheme.

Exercise 1.21 Show that for any $g \geq 3$ there is a constant $\varphi(g)$ such that any smooth curve C of genus g admits at most $\varphi(g)$ nonconstant morphisms to curves B of genus $h \geq 2$.

Solution: By Riemann-Hurwitz, given g there are only finitely many possibilities for the genus h, degree of $f : C_g \to C_h$ and for the degree of its divisor of ramification, so we may assume all these are fixed. We use a pluricanonical embedding of a curve C_g of genus g (and C_h of genus h) to some projective spaces, which are embeddings if we take a large enough multiple of the canonical bundle. Hence we may assume $C_g \subseteq \mathbb{P}^s$ and $C_h \subseteq \mathbb{P}^t$ are fixed. We use the fact that the Hilbert scheme of morphisms $f : C_g \to C_h$ of degree d or less exists and is quasi-projective. It suffices to show that it is zero-dimensional, hence finite. This we leave as an exercise.

1.1 Tangent Space to the Hilbert Scheme

Recall that if X is a scheme and $P \in X$ is a point then the *tangent space* to X at P is defined as

$$T_P X = \{ f : I \to X \mid f(0) = P \}$$

where $I = \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ is the spectrum of the ring of dual numbers and $0 \in I$ is the unique point, corresponding to the maximal ideal (ϵ). Turning to the case where $X = \mathcal{H}_{h,r}$ is the Hilbert scheme of closed subschemes of $\mathbb{P}^r$ having Hilbert polynomial h, and $P \in \mathcal{H}_{h,r}$ is a closed point, we get that $T_P\mathcal{H}_{h,r}$ is in 1-1 correspondence with flat proper families $\varphi : \mathcal{X} \to I$ whose fiber X_0 over $0 \in I$ is isomorphic to the closed subscheme of $\mathbb{P}^r$ described by the point $P = [X_0] \in \mathcal{H}_{h,r}$. Such families are called *first order deformations* of X_0.

More generally, given a scheme Y and a closed subscheme $X_0 \subseteq Y$ we look at *first-order embedded deformations* of X_0 in Y, which are flat families $\varphi : \mathcal{X} \to I$ ($\mathcal{X} \subseteq Y \times I$ is a closed embedding over I) such that the fibre over $0 \in I$ is X_0.

In case $Y \cong \operatorname{Spec} R$ is affine we have the following result:

Proposition 1.1.1 *Let R be a finitely generated $\mathbb{C}$-algebra and let $I \subseteq R$ be an ideal, then the first-order embedded deformations of $X_0 = \operatorname{Spec} R/I$ in $Y = \operatorname{Spec} R$ are in 1-1 correspondence with*

$$\operatorname{Hom}_{R/I}(I/I^2, R/I) = \operatorname{Hom}_R(I, R/I).$$

This proposition follows from the following algebraic lemma (*flatness-by-relations*):

Lemma 1.1.1 *Let $\varphi : A \to B$ be a flat homomorphism of Noetherian rings, where either A is a local Artinian ring or A and B are both local rings and φ is a local homomorphism. Let $J \subseteq B$ be an ideal and denote $C = B/J$. Let $k = A/\mathfrak{m}_A$ be the residue field of A at its maximal ideal $\mathfrak{m}_A$ and let $B_k = B \otimes_A k$. Then C is flat over A if and only if there are generators $F_1, \dots, F_h$ of J whose images $f_1, \dots, f_h$ in B_k have the following property: every relation among the f_i extends to a relation among the F_i.*

Proof of the proposition. We have to classify the ideals $J \subseteq R[\epsilon]$ such that $R[\epsilon]/J$ is flat over $\mathbb{C}[\epsilon]$ and $J/(R\epsilon \cap J) = I$. Given such J, consider the short exact sequence

$$0 \longrightarrow J \longrightarrow R[\epsilon] \longrightarrow R[\epsilon]/J \longrightarrow 0.$$

Since both terms on the right are assumed to be flat over $\mathbb{C}[\epsilon]$ so is J. Now tensor this sequence with the $\mathbb{C}[\epsilon]$-module $\mathbb{C} = \mathbb{C}[\epsilon]/(\epsilon)$, to get

$$\cdots \to \operatorname{Tor}_1^{\mathbb{C}[\epsilon]}(R[\epsilon]/J, \mathbb{C}) \to J \otimes_{\mathbb{C}[\epsilon]} \mathbb{C} \to R \to R/I \to 0.$$

The first term vanishes, since $R[\epsilon]/J$ is flat. This shows that $J \otimes \mathbb{C} = J/\epsilon J \cong I$, that is, $\epsilon J = R\epsilon \cap J$.

Now let $i \in I$. There is a $j \in J$ such that $j = i - \epsilon h$ for some $h \in R$. h is uniquely determined modulo I, since if $i = 0$ then $\epsilon h \in J \cap R\epsilon = \epsilon J = \epsilon I \subseteq I$. The map $i \to h$ is also R-linear, so we get a homomorphism $\alpha : I \to R/I$. Conversely, given a homomorphism $\alpha : I \to R/I$ choose generators $f_1, \dots, f_n \in I$ and let $\alpha(f_i) = g_i + I$, for $g_i \in R$. Set $F_i = f_i - \epsilon g_i$ and put $J = (F_1, \dots, F_n)$. Clearly $J/(R\epsilon \cap J) = I$. We prove that $R[\epsilon]/J$ is flat over $\mathbb{C}[\epsilon]$, using the algebraic lemma. Let $\sum_i a_i f_i = 0$ be any relation among the f_i. It follows that $\sum_i a_i \alpha(f_i) = 0$, so $\sum_i a_i g_i \in I$. We can write, then $\sum a_i g_i = \sum b_i f_i$ for some $b_i \in R$. Thus

$$\sum_i (a_i + \epsilon b_i) F_i = \sum_i a_i f_i + \epsilon \left(\sum_i b_i f_i - \sum_i a_i g_i \right) = 0$$

is a relation among the F_i which extends the relation $\sum_i a_i f_i = 0$.

The maps constructed are easily seen to be inverses of each other, so the proof is complete. ■

The result of this discussion is that the tangent space to the Hilbert scheme $\mathcal{H}_{h,r}$ at a point $P = [X]$ corresponds to a global section of the *normal sheaf* of X on $\mathbb{P}^r$,

$$\mathcal{N}_{X/\mathbb{P}^r} = \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X),$$

where $\mathcal{I}$ is the ideal sheaf of X in $\mathbb{P}^r$. We write

$$T_{[X]}\mathcal{H}_{h,r} = H^0(X, \mathcal{N}_{X/\mathbb{P}^r}).$$

Exercise 1.26 Determine the normal bundle to the rational normal curve $C \subset \mathbb{P}^r$ and show, by computing its h^0, that the Hilbert scheme parameterizing such curves is smooth at any point corresponding to a rational normal curve.

Solution: C is a rational curve of degree $d = r$ in $\mathbb{P}^r$, hence has Hilbert polynomial $h_C(m) = rm + 1$. Consider the exact sequence

$$0 \longrightarrow T_C \longrightarrow T_{\mathbb{P}^r} \otimes \mathcal{O}_C \longrightarrow \mathcal{N}_{C/\mathbb{P}^r} \longrightarrow 0,$$

that gives

$$\deg \mathcal{N}_{C/\mathbb{P}^r} = d(r+1) + 2g - 2 = r(r+1) - 2 > 2g - 2 = -2$$

so, by Riemann-Roch, we get

$$h^0(\mathcal{N}_{C/\mathbb{P}^r}) = \deg \mathcal{N}_{C/\mathbb{P}^r} - (r-1)(g-1) = r(r+1) - 2 + r - 1 = r^2 + 2r - 3.$$

On the other hand, C is the image of $\mathbb{P}^1$ by an r-uple embedding, up to an automorphism of $\mathbb{P}^r$. Since the automorphism group of $\mathbb{P}^r$ is $PGL(r+1)$ it has order $(r+1)^2 - 1 = r^2 + 2r$. Subtracting the order of the automorphism group of $\mathbb{P}^1$, which is 3, we get that the family of rational normal curves has dimension at least $r^2 + 2r - 3$. Since this is also the result for h^0 we must have, then $\dim_{[C]} \mathcal{H}_{rm+1,r} = \dim T_{[C]}\mathcal{H}_{rm+1,r}$, i.e., the Hilbert scheme is smooth at $[C]$.

Note that the Hilbert scheme parameterizes also very nasty schemes. For example, in the case of twisted cubics (which are rational normal curves in $\mathbb{P}^3$), the Hilbert polynomial is $h_C(m) = 3m+1$, which is also the Hilbert polynomial of a plane cubic plus an extra point. The component D of $\mathcal{H}_{3m+1,3}$ parameterizing twisted cubics has dimension 12, as we have seen, and is smooth at any point $[C]$ correpsonding to such a curve. However, there is another component E parameterizing plane cubics plus a point in $\mathbb{P}^3$. This has dimension $3 + 9 + 3 = 15$ (3 for the choice of plane in $\mathbb{P}^3$, since indeed no plane cubic can lie on two such planes, their intersection is a line and of degree 1; 9 for the family of cubics in that plane; and 3 more for the point).

A general point in $D \cap E$ parameterizes a degeneration of a twisted cubic into a a plane rational cubic plus an embedded point at the singular point. Indeed, consider the family of curves X_a, $a \in \mathbb{C}$ given in affine coordinates by the parametrization:

$$X_a : \begin{cases} x = t^2 - 1 \\ y = t(t^2 - 1) \\ z = at. \end{cases}$$

For $a \neq 0$ we get $t = z/a$, $t^2 = x+1$, $t^3 = y + z/a$, which we recognize as a twisted cubic. Calculating the ideal of X_a amounts to eliminating t, and we get

$$I(X_a) = \left(a^2(x+1) - z^2, ax(x+1) - yz, xz - ay, y^2 - x^2(x+1)\right).$$

We obtain the flat limit of this family at $a = 0$ by putting $a = 0$, so the resulting cubic X_0 has defining ideal

$$I(X_0) = \left(z^2, yz, xz, y^2 - x^2(x+1)\right).$$

This is the ideal sheaf of the nodal plane cubic $y^2 = x^2(x+1)$ plus an embedded point at the origin.

At such a point $[X_0]$ it will clearly hold that $\dim T_{[X_0]}\mathcal{H}_{3m+1,3} > 15$. It is not at all clear whether the components D, E are smooth at such points, and this was only recently proved.

Exercise 1.28 Prove that at a general point $[X_0]$ along $D \cap E$ we have $\dim T_{[X_0]}\mathcal{H}_{3m+1,3} = 16$. As a hint you may assume that the integer m_0 used in the construction of this Hilbert scheme is 4.

Solution: It is easiest to use the characterization of the tangent space as the linear space

$$\operatorname{Hom}_{S/I}(I/I^2, S/I) \cong \operatorname{Hom}_S(I, S/I),$$

where $S = \mathbb{C}[x,y,z,w]$, $I = I(X_0)$ given above and Hom_S is the set of degree-preserving S-homomorphisms. To compute its dimension write $I = (xz, yz, z^2, f)$ where $f = f(x,y,w) = y^2w - x^2(x+w)$ is the homogeneous equation of the nodal plane cubic. Let $J = (z, f)$ be the ideal of this reduced cubic, and define $K = J/I = z\mathbb{C}[w]$. We have an exact sequence $0 \to I \to J \to K \to 0$, and after applying $\operatorname{Hom}_S(-, S/I)$ we get the following long exact sequence:

$$0 \to \operatorname{Hom}_S(K, S/I) \to \operatorname{Hom}_S(J, S/I) \to \operatorname{Hom}_S(I, S/I) \to \operatorname{Ext}^1_S(K, S/I) \to \operatorname{Ext}^1_S(J, S/I) \to \cdots$$

Consider the following presentation of K:

$$0 \to S(-4) \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} S(-3)^3 \xrightarrow{\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}} S(-2)^3 \xrightarrow{(x,y,z)} S(-1) \longrightarrow K \to 0.$$

We deduce that $\operatorname{Hom}_S(K, S/I) \cong K_1$ is 1-dimensional and $\operatorname{Ext}^1_S(K, S/I) \subseteq (S/I)^3_2$ is 4-dimensional, generated by

$$\left\{ \begin{pmatrix} zw \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ zw \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ zw \end{pmatrix}, \begin{pmatrix} x(x+w) \\ yw \\ 0 \end{pmatrix} \right\}.$$

We write also a presentation of J:

$$0 \to S(-4) \xrightarrow{\begin{pmatrix} f \\ -z \end{pmatrix}} S(-1) \oplus S(-3) \xrightarrow{(z,f)} J \to 0$$

which shows that $\operatorname{Hom}_S(J, S/I) = (S/I)_1 \oplus M_3$, where $M = (x,y,z)/I$. We have $\dim(S/I)_1 = 4$ and $\dim M_3 = 9$, so $\dim \operatorname{Hom}_S(J, S/I) = 13$. Finally $\operatorname{Ext}^1_S(J, S/I) \cong (S/J)_4$, so the map $\operatorname{Ext}^1_S(K, S/I) \to \operatorname{Ext}^1_S(J, S/I)$ is the zero map. We therefore have that

$$\dim \operatorname{Hom}_S(I, S/I) = -1 + 13 + 4 = 16,$$

as was to be shown.