

EXERCISE 19 IN BASIC ALGEBRAIC TOPOLOGY

Problem 1. Let $X = A \cup B$, and let the chain complex $C(X)/(C(A) + C(B))$ be acyclic. Prove that the homomorphism $\gamma : H_q(A \cup B) \rightarrow H_{q-1}(A \cap B)$ in the Mayer-Vietoris homology sequence is the composition

$$H_q(A \cup B) \xleftarrow{\sim} H_q^{\{A,B\}}(A \cup B) \xrightarrow{\partial_*} H_{q-1}(A \cap B),$$

where ∂_* comes from the short exact sequence of chain complexes

$$0 \rightarrow C(A \cap B) \rightarrow C(A) \oplus C(B) \rightarrow C(A) + C(B) \rightarrow 0.$$

Problem 2. Let $Z \supset W$ be a topological pair, $Z = X \cup Y$, $W = A \cup B$, where $X \supset A$ and $Y \supset B$. Suppose that each of the coverings $Z = X \cup Y$ and $W = A \cup B$ satisfies the conditions of the Mayer-Vietoris theorem. Prove that there exists a long exact homology sequence

$$\begin{aligned} \dots \rightarrow H_q(X \cap Y, A \cap B) &\xrightarrow{\pm} H_q(X, A) \oplus H_q(Y, B) \xrightarrow{\mp} H_q(X \cup Y, A \cup B) \\ &\xrightarrow{\gamma} H_{q-1}(X \cap Y, A \cap B) \rightarrow \dots \end{aligned}$$

Good luck!