Exercise 1 in Basic Algebraic Topology

Problem 1. Construct a surjective continuous map $\psi : D^{2n} \to \mathbb{C}P^n$ which is one-to-one on $\mathrm{Int}(D^{2n})$.

Problem 2. Prove that $\mathbb{R}P^n$ resp. $\mathbb{C}P^n$ is representable as the disjoint union $V_0 \cup V_1 \cup ... \cup V_n$, where V_i is homeomorphic to $\mathbb{R}^i$, $i = 0, 1, ..., n$, resp. to $\mathbb{C}^i \simeq \mathbb{R}^{2i}$, $i = 0, 1, ..., n$.

Problem 3. Prove that the Plücker maps $\pi : Gr(n,k) \to \mathbb{R}P^N$ and $\pi_{\mathbb{C}} : \mathbb{C}Gr(n,k) \to \mathbb{C}P^N$ embed the Grassmann into the projective space as a smooth submanifold, where $N = \begin{pmatrix} n \\ k \end{pmatrix} - 1$,

$$\pi(L) = \{\pi_{i_1 \dots i_k} = |M_{i_1 \dots i_k}|,\ 1 \le i_1 < ... < i_k \le n\}$$

with $M_{i_1 \dots i_k}$ the $k \times k$ minor built of the i_1-th, ..., i_k-th columns of the matrix of the coordinates of a basis for L. Show that $\mathrm{Im}(\pi)$ is defined by the equations

$$\pi_{i_1 \dots i_k} \pi_{j_1 \dots j_k} = \sum_{r=1}^{k} \pi_{i_1 \dots i_{r-1} j_1 i_{r+1} \dots i_k} \pi_{i_r j_2 \dots j_k} ,$$

$$1 \le i_1 < ... < i_k \le n, \quad 1 \le j_1 < ... < j_k \le n .$$

Hint: (i) Show that the sequences $\pi(L) = \{\pi_{i_1 \dots i_k}\}_{1 \le i_1 < \dots < i_k \le n}$ built from different bases of L are proportional. (ii) Show that the image of π is a submanifold using local coordinates. (iii) The same for equations of $\mathrm{Im}(\pi)$.

Problem 4. Prove that the spaces $\mathbb{R}^\infty, \mathbb{C}^\infty, D^\infty, S^\infty$ are not metrisable.
Hint: Show that there is no countable base of neighborhoods at a point.

Problem 5. Prove that the closed ball $D = \{\sum x_i^2 \le 1\}$ is compact in $\prod_{n=1}^\infty \mathbb{R}$, and is not compact in l_2 and in $\mathbb{R}^\infty$.

Problem 6. Define inductive limits

$$Gr(\infty, k) = \bigcup_{n=k+1}^{\infty} Gr(n,k), \quad Gr(\infty,\infty) = \bigcup_{k=1}^{\infty} Gr(\infty, k) ,$$

using suitable ascending sequences

$$Gr(k+1,k) \subset ... \subset Gr(n,k) \subset Gr(n+1,k) \subset ...,$$

$$Gr(\infty,1) \subset ... \subset Gr(\infty,k) \subset Gr(\infty,k+1) \subset ...$$

Good luck!

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