Exercise 24 in Basic Algebraic Topology

Problem 1. Describe the action of the homomorphisms ∂_* and δ^* in the exact coefficient sequences

$$... \to H_q(X) \to H_q(X) \to H_q(X, \mathbb{Z}/2) \xrightarrow{\partial_*} H_{d-1}(X) \to ...$$

$$... \to H^q(X, \mathbb{Z}) \to H^q(X, \mathbb{Z}) \to H^q(X, \mathbb{Z}/2) \xrightarrow{\delta^*} H^{q+1}(X, \mathbb{Z}) \to ...$$

corresponding to the exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$.

Problem 2. Let $0 \to F_1 \to F_2 \to B \to 0$ be an exact sequence of abelian groups (a resolution of B), F_1, F_2 free groups, A any abelian group.

(1) Prove that the sequence

$$A \otimes F_1 \to A \otimes F_2 \to A \otimes B \to 0$$

is exact, in particular, $A \otimes B \simeq A \otimes F_2/\mathrm{Im}(A \otimes F_1)$.

(2) Prove that $\mathrm{Tor}(A, B) := \mathrm{Ker}(A \otimes F_1 \to A \otimes F_2)$ does not depend on the choice of a free resolution of B, i.e. for any other free resolution $0 \to F_1' \to F_2' \to B \to 0$ there exists a canonical isomorphism

$$\mathrm{Ker}(A \otimes F_1 \to A \otimes F_2) \simeq \mathrm{Ker}(A \otimes F_1' \to A \otimes F_2') \ .$$

(3) Prove that

1. $\mathrm{Tor}(A, B) \simeq \mathrm{Tor}(B, A)$,
2. $\mathrm{Tor}(A \oplus A', B) \simeq \mathrm{Tor}(A, B) \oplus \mathrm{Tor}(A', B)$,
3. $\mathrm{Tor}(A, B) = 0$ as A is free or $\mathbb{C}, \mathbb{R}, \mathbb{Q}$,
4. $\mathrm{Tor}(\mathbb{Z}/m, \mathbb{Z}/n) \simeq \mathbb{Z}/(m, n)$.

Problem 3. Let $0 \to F_1 \to F_2 \to A \to 0$ be a free abelian resolution of an abelian group A.

(1) Prove that, for any abelian group B, the sequence

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(F_2, B) \to \mathrm{Hom}(F_1, B)$$

is exact, in particular $\mathrm{Hom}(A, B) = \mathrm{Ker}(\mathrm{Hom}(F_2, B) \to \mathrm{Hom}(F_1, B))$.

(2) Prove that $\mathrm{Ext}(A, B) := \mathrm{Hom}(F_1, B)/\mathrm{Im}(\mathrm{Hom}(F_2, B))$ does not depend on the choice of a free resolution of A, i.e., for any other free resolution $0 \to F_1' \to F_2' \to A \to 0$, there exists a canonical isomorphism

$$\mathrm{Hom}(F_1, B)/\mathrm{Im}(\mathrm{Hom}(F_2, B)) \simeq \mathrm{Hom}(F_1', B)/\mathrm{Im}(\mathrm{Hom}(F_2', B)) \ .$$

(3) Prove that

1. $\mathrm{Ext}(A \oplus A', B) \simeq \mathrm{Ext}(A, B) \oplus \mathrm{Ext}(A', B)$, and $\mathrm{Ext}(A, B \oplus B') \simeq \mathrm{Ext}(A, B) \oplus \mathrm{Ext}(A, B')$,
2. $\mathrm{Ext}(\mathbb{Z}, B) = 0$, $\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}/n) \simeq \mathbb{Z}/(m, n)$, $\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}) \simeq \mathbb{Z}/m$,
3. $\mathrm{Ext}(A, B) = 0$ as $B = \mathbb{C}, \mathbb{R}, \mathbb{Q}$.

Problem 4. Prove that, for any field K and any topological space X,

$$H^q(X; K) \simeq \mathrm{Hom}(H_q(X; K), K) \ .$$

Good luck!