BLOWING UP THE POWER OF SINGULAR CARDINAL OF UNCOUNTABLE COFINALITY WITH COLLAPSES

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Abstract. The *Singular Cardinal Hypothesis* (SCH) is one of the most classical combinatorial principles in set theory. It says that if κ is singular strong limit, then $2^\kappa = \kappa^+$. We prove that given a singular cardinal κ of *cofinality* η in the ground model, which is a limit of suitable large cardinals, there is a forcing extension such that κ becomes $\aleph_\eta$, and SCH fails at κ. Our large cardinal assumption is below the existence of a superstrong cardinal. In our model we also obtain a very good scale.

1. introduction

The *Singular Cardinal Hypothesis (SCH)* says, roughly speaking, that if θ is a singular cardinal then 2^θ has the smallest value consistent with the restrictions provable in ZFC: in particular SCH implies that if θ is a singular strong limit cardinal, then $2^\theta = \theta^+$. Failure of SCH is known to require some large cardinal hypotheses.

The first models for the failure of SCH were built by starting with a large cardinal κ, forcing that GCH fails at κ while preserving some large cardinal property of κ, and then making κ singular via Prikry-type forcing. Gitik and Magidor [1] gave a construction which starts with a large cardinal and simultaneously adds many cofinal ω-sequences in κ, while keeping it strong limit.

In subsequent work, Gitik and Magidor [2] gave a construction which starts with a cardinal κ that is the limit of ω many strong cardinals, and adds many cofinal subsets of κ while keeping κ strong limit. Recently Gitik [3] showed how to achieve a similar result when κ is a limit of large cardinals and has uncountable cofinality. In the resulting extension κ is quite large, for example it is a cardinal fixed point.

We prove the following theorem:

Theorem 1. *Assume GCH holds, and there is an increasing sequence of cardinals* $\langle \kappa_\alpha : \alpha < \eta \rangle$ *and an integer* $n \geq 2$ *such that, letting* $\lambda = (\sup_{\alpha<\eta} \kappa_\alpha)^{+n}$*:*

(1) $0 < \eta < \kappa_0$ *and* η *is limit.*
(2) There is a sequence of (κ_α, λ)*-extenders* $\langle E_\alpha : \alpha < \eta \rangle$ *such that:*
  (a) If $M_\alpha = \mathrm{Ult}(V, E_\alpha)$*, then* M_α *computes cardinals correctly up to and including* λ*, and* ${}^{\kappa_\alpha} M_\alpha \subseteq M_\alpha$*.*
  (b) If $j_{E_\alpha} : V \to M_\alpha$ *is the ultrapower map, then* $j_{E_\beta}(E_\alpha) \restriction \lambda = E_\alpha$ *for* $\alpha < \beta < \eta$*.*
  (c) There is a function $s_\alpha : \kappa_\alpha \to \kappa_\alpha$ *with* $j_{E_\alpha}(s_\alpha)(\kappa_\alpha) = \sup_{\alpha<\eta} \kappa_\alpha$*.*

Then there is a λ*-c.c. forcing poset such that in the generic extension, for each limit ordinal* $\beta < \eta$*,* $2^{\aleph_\beta} \geq \aleph_\beta^{+n}$*, and* $2^{\aleph_\eta} = \aleph_\eta^{+n}$*.*

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The condition "$j_{E_\beta}(E_\alpha) \restriction \lambda = E_\alpha$ for $\alpha < \beta < \eta$" is a form of *coherence* for the sequence of extenders: it implies that the sequence is *Mitchell increasing* in the sense that $E_\alpha \in \mathrm{Ult}(V, E_\beta)$ for $\alpha < \beta < \eta$. Gitik has conjectured that the existence of an uncountable Mitchell increasing sequence as above (with $n = 2$) is optimal to obtain failure of SCH at a singular cardinal of uncountable cofinality which is singular in the core model.

Our forcing construction is inspired by Gitik's construction [3] which blows up the powerset of a singular cardinal with any cofinality, and his recent work on collpasing generators [4].

We assume the reader is familiar with large cardinals and forcing. Our presentation of extender-based forcing follows the same lines as some work of Merimovich [5].

Our notation is mostly standard. For each sequence of ordinal length $X = \langle x_i : i < \alpha \rangle$ and $\beta < \alpha$, we write $X \restriction \beta = \langle x_i : i < \beta \rangle$, and $X \setminus \beta = \langle x_i : \beta \leq i < \alpha \rangle$.

The organization of this paper is as follows:

- In Section 2, we describe the assumptions we use to build the forcing. We then introduce the notions of domains and objects, explain the connection between objects and extenders, and do some extender analysis.
- In Section 3, we decsribe the forcing construction and explain how the extension works. After that, we describe the chain condition and closure of the forcing, and prove an *integration lemma*, which plays a vital role in proving the Prikry property.
- In Section 4, we prove the Prikry property and the strong Prikry property.
- In Section 5, we describe which cardinals are preserved.
- In Section 6, we analyze some scales. We use scales to show cardinal arithmetic, and also show that one of the scales derived from the forcing is very good.
- In Section 7, we show that the existence of a superstrong cardinal is sufficient to yield our initial assumptions for building the forcing.

2. PRELIMINARIES

We start with a ground model V in which GCH holds. Our assumptions are slightly more general than those of Theorem 1, because we will establish some of the properties of the main forcing (notably the Prikry property) by induction on η. In our assumptions λ is still of the form ρ^{+n}, but in our induction argument for the Prikry property, the proof requires extenders of longer lengths, hence we permit ρ to be larger than $(\sup_{\alpha<\eta} \kappa_\alpha)^{+n}$. Throughout the paper, we treat the case $n = 2$. One can modify our analysis to generalize for any larger n.

Fix an ordinal $\eta > 0$ (η can be finite), and a sequence of cardinals $\langle \kappa_\alpha : \alpha < \eta \rangle$ with $\eta < \kappa_0$. For each α with $0 < \alpha \leq \eta$, let $\overline{\kappa}_\alpha = \sup_{\beta<\alpha} \kappa_\beta$, and let $\overline{\kappa}_0$ be regular such that $\max\{\omega, \eta\} = \overline{\kappa}_0 < \kappa_0$. Let $\rho \geq \overline{\kappa}_\eta$, where ρ is either an inaccessible cardinal or a limit of inaccessible cardinals, and let $\lambda = \rho^{++}$.

Assume that :

1. For each α, κ_α carries a (κ_α, λ)-extender E_α.
2. Let $j_{E_\alpha} : V \to M_\alpha$ be derived from the extender E_α. Then M_α is closed under κ_α-sequences, and M_α computes cardinals correctly up to and including λ.

(3) There is a function $s_\alpha : \kappa_\alpha \to \kappa_\alpha$ with $j_{E_\alpha}(s_\alpha)(\kappa_\alpha) = \rho$, and $s_\alpha(\nu) > \max\{\nu, \overline{\kappa}_\alpha\}$ for all ν.
(4) The sequence of extenders $\langle E_\alpha : \alpha < \eta \rangle$ is pairwise *coherent* in the sense of [6], that is for $\alpha < \beta < \eta$, $j_{E_\beta}(E_\alpha) \restriction \lambda = E_\alpha$.

As we noted already, the coherence of the extender sequence implies the *Mitchell increasing property*, namely, for $\alpha < \beta < \eta$, $E_\alpha \in M_\beta$.

Observe that the sequence $\langle \kappa_\alpha : \alpha < \eta \rangle$ is **not** continuous. In general, for α with $\alpha + 1 \leq \eta$, $\overline{\kappa}_{\alpha+1} = \kappa_\alpha$. In addition, for each $\alpha < \eta$, $\overline{\kappa}_\alpha < \kappa_\alpha$, hence requirement 3 is sensible.

We believe that the coherence requirement can be dropped. The reason we include it in the assumptions is that it simplifies the definition of the forcing poset. Without this condition we have to fix functions $f_{\alpha,\beta}$ which represent E_α in $\mathrm{Ult}(V, E_\beta)$, which reduces readability. We note that using conditions 3 and 4, we have that $j_{E_\beta}(\zeta \mapsto E_\alpha \restriction s_\beta(\zeta)^{++})(\kappa_\beta) = E_\alpha$, so that E_α is represented in a very simple way.

The following representation of the extenders is due to Merimovich [5]. For each $\alpha < \eta$, we let an α-*domain* be a set $d \in [\lambda]^{\kappa_\alpha}$ such that $\kappa_\alpha + 1 \subseteq d$. In the original Merimovich context $\kappa_\alpha = \min(d)$, but it our case, it is more convenient to have $d \supseteq \kappa_\alpha + 1$. Define $\mathrm{mc}_\alpha(d)$ as $(j_{E_\alpha} \restriction d)^{-1}$. We represent $E_\alpha = \langle E_\alpha(d) : d \text{ is an } \alpha\text{-domain} \rangle$, where $X \in E_\alpha(d)$ iff $\mathrm{mc}_\alpha(d) \in j_{E_\alpha}(X)$. From this point, if there is no ambiguity about the value of α, we may drop α from the notation (refer to an α-domain as a domain, E_α as E, $\mathrm{mc}_\alpha(d)$ as $\mathrm{mc}(d)$ and so on). We define a set on which $E_\alpha(d)$ concentrates.

Definition 2. Let d be an α-domain. $\mathrm{OB}_\alpha(d)$ is the collection of d-α-*objects*, which are the functions μ such that

(1) $\mathrm{dom}(\mu) \subseteq d$, $\mathrm{ran}(\mu) \subseteq \kappa_\alpha$, and $\kappa_\alpha \in \mathrm{dom}(\mu)$,
(2) $|\mathrm{dom}(\mu)| = \mu(\kappa_\alpha)$ (which is below κ_α), and $\mu(\kappa_\alpha)$ is inaccessible.
(3) $\mathrm{dom}(\mu) \cap \kappa_\alpha = \mu(\kappa_\alpha)$.
(4) μ is order-preserving, and
(5) for $\beta \in \mathrm{dom}(\mu) \cap \kappa_\alpha$, $\mu(\beta) = \beta$.

As mentioned earlier, it is straightforward to check that $E_\alpha(d)$ concentrates on d-α-objects. It is easy to see that for each domain d, there is a unique $\alpha < \eta$ such that d is an α-domain. Hence we use the term "d-object" to refer to a d-α-object for the unique α such that d is an α-domain. The collection $\langle E_\alpha(d) : d \text{ is an } \alpha\text{-domain} \rangle$ comes with natural projections. For each pair of α-domains $d \subseteq d'$ define $\pi_{d',d} : \mathrm{OB}_\alpha(d') \to \mathrm{OB}_\alpha(d)$ by $\pi_{d',d}(\mu) = \mu \restriction d$: it is routine to check that $\pi_{d',d}$ projects $E_\alpha(d')$ to $E_\alpha(d)$. We do some analysis of extenders in Lemma 1 and Lemma 2.

Lemma 1. *Suppose $0 \leq \alpha_{i_0} < \cdots < \alpha_{i_{k-1}} < \eta$, for each $j < k$, d_j is an α_{i_j}-domain, $A_j \in E_{\alpha_{i_j}}(d_j)$, and $F : \prod_{j<k} A_j \to 2$. Then there are $B_j \subseteq A_j$, $B_j \in E_{\alpha_{i_j}}(d_i)$ such that $F \restriction (\prod_{j<k} B_j)$ is constant.*

Proof. Induct on k. The case $k = 1$ is trivial. Suppose $k > 1$. For each $\mu \in A_{\alpha_{k-1}}$, let $F_\mu : \prod_{j<k-1} A_j \to 2$ be defined by $F_\mu(\vec{x}) = F(\vec{x}^\frown \langle \mu \rangle)$. Since $\kappa_{\alpha_{k-1}}$ is inaccessible, by completeness of $E_{\alpha_{k-1}}(d_{k-1})$, there exist B_{k-1} and F' such that $F_\mu = F'$ for all $\mu \in B_{k-1}$. Now apply the induction hypothesis to find $B_j \subseteq A_j$, $B_j \in E_{\alpha_{i_j}}(d_j)$ for $j < k-1$ with $F' \restriction (\prod_{j<k-1} B_j)$ constant. It is easy to see that $F \restriction (\prod_{j<k} B_j)$ is constant. ∎

Lemma 2. *For each $\alpha < \eta$ and each α-domain d, there is a set $A_\alpha(d)$ such that $A_\alpha(d) \in E_\alpha(d)$, and for each $\nu < \kappa_\alpha$, the size of $\{\mu \in A_\alpha(d) : \mu(\kappa_\alpha) = \nu\}$ is at most $s_\alpha(\nu)^{++}$.*

Proof. We drop the subscript α for this proof and fix a domain d. Let $d^* = d \setminus \kappa$. For each $\mu \in \mathrm{OB}(d)$, set $\mu^* = \mu \restriction (\mathrm{dom}(\mu) \setminus \kappa)$, and let $\mathrm{mc}^*(d) = \mathrm{mc}(d) \restriction (\mathrm{dom}(\mathrm{mc}(d)) \setminus \kappa)$. Enumerate d^* as $\langle \delta_i : i < \kappa \rangle$. Let B_d be the set of $\mu \in \mathrm{OB}(d)$ such that $\mathrm{dom}(\mu^*) = \{\delta_i : i < \mu(\kappa)\}$. We claim that $B_d \in E(d)$. The point is that $\mathrm{dom}(\mathrm{mc}^*(d)) = j[d^*] = \{j(\delta)_i : i < \kappa\}$, $\mathrm{mc}^*(d)(j(\kappa)) = \kappa$, and $j(\langle \delta_i : i < \kappa \rangle) \restriction \kappa = \langle j(\delta_i) : i < \kappa \rangle$.

Let $\vec{t} = \langle t_\delta : \delta < \kappa \rangle$ be an enumeration of $[\kappa]^{<\kappa}$ such that whenever $\delta < \kappa$ is a closure point of the map $\gamma \mapsto s(\gamma)^{++}$, $\langle t_\beta : \beta \in [\delta, s(\delta)^{++}) \rangle$ enumerates the set of $t \in [s(\delta)^{++}]^{\leq \delta}$ with $\min(t) = \delta$. Let $j(\vec{t}) = \vec{T}$. Since κ is a closure point for $j(\gamma \mapsto s(\gamma)^{++})$, and $j(s)(\kappa)^{++} = \lambda$, we have that $\langle T_\delta \sqcup \kappa : \kappa \leq \delta < \lambda \rangle$ enumerates all domains. Choose δ so that $T_\delta = d^*$. For each $\beta \in d^*$, let $i = \mathrm{ot}(d^* \cap \beta) < \kappa^+$, and let π_β be the function which takes $\nu < \kappa$ to the $f_i(\min(t_\nu))^{\mathrm{th}}$ element of t_ν where f_i is the i^{th} canonical function. Then $j(\pi_\beta)(\delta)$ is the $j(f_i)(\kappa)^{\mathrm{th}}$ element of T_δ, which is β.

Define

$$A_d = \{\mu \in B_d : \exists \xi < s(\mu(\kappa))^{++} \, \forall \beta \in \mathrm{dom}(\mu^*), \mu^*(\beta) = \pi_\beta(\xi)\}$$

We claim that $A_d \in E(d)$. Note that $\delta < \lambda = (j(s)(\mathrm{mc}(d)(j(\kappa))))^{++}$ and for each $\beta \in d^*$, $\mathrm{mc}^*(d)(j(\beta)) = \beta = j(\pi_\beta)(\delta)$.

The conclusion of the proof follows from the fact that given $\tau \in A_d$ with $\tau(\kappa) = \nu$, $\mathrm{dom}(\tau) = \nu \cup \{\delta_i : i < \nu\}$, $\tau(\beta) = \beta$ for all $\beta < \nu$, and there is $\xi < s(\nu)^{++}$ such that for $\beta \in \mathrm{dom}(\mu) \setminus \kappa$, $\mu(\beta) = \pi_\beta(\xi)$.

Hence ν and ξ completely determine τ. For every ν there are only $s(\nu)^{++}$ possible values for ξ. Hence A_d works as required. □

In the sequel we define $A_\alpha(d)$ to be the measure one set from the conclusion of Lemma 2.

3. The forcing

The forcing is constructed using ideas from Gitik's recent preprints [3] and [4]. To apply induction hypothesis, in this section, we assume $\max\{\omega, \eta\} \leq \overline{\kappa}_0 < \kappa_0$. Recall that $\rho \geq \overline{\kappa}_\eta$ and $\lambda = \rho^{++}$. We introduce some notation here. For each pair of cardinals κ and θ, let $\mathcal{A}(\kappa, \theta)$ be the set of partial functions from θ to κ with domains which have size κ and contain $\kappa + 1$, ordered by extension. Also define $E_\alpha(\kappa_\alpha)$ by $X \in E_\alpha(\kappa_\alpha)$ iff $\kappa_\alpha \in j_{E_\alpha}(X)$. This is just a normal measure on κ_α. If d is an α-domain and $A \in E_\alpha(d)$ then we define $A(\kappa_\alpha) = \{\mu(\kappa_\alpha) : \mu \in A\}$, and note that easily $A(\kappa_\alpha) \in E_\alpha(\kappa_\alpha)$. Assume all the hypotheses from the beginning of Section 2.

Definition 3. The forcing $\mathbb{P}_{\langle E_\alpha : \alpha < \eta \rangle}$ consists of sequences $p = \langle p_\alpha : \alpha < \eta \rangle$ such that for a finite subset $\mathrm{supp}(p)$ of η (the *support of* p):

$$p_\alpha = \begin{cases} \langle f_\alpha, \lambda_\alpha, h^0_\alpha, h^1_\alpha, h^2_\alpha \rangle & \text{if } \alpha \in \mathrm{supp}(p), \\ \langle f_\alpha, A_\alpha, H^0_\alpha, H^1_\alpha, H^2_\alpha \rangle & \text{otherwise,} \end{cases}$$

where:

(1) $\overline{\kappa}_\alpha < \lambda_\alpha < \kappa_\alpha$ for all $\alpha \in \mathrm{supp}(p)$.

(2) $f_\alpha \in \mathcal{A}(\kappa_\alpha, \lambda_{\alpha^*})$ where $\alpha^* = \min(\operatorname{supp}(p) \setminus (\alpha+1))$ if it exists, otherwise $\lambda_{\alpha^*} = \lambda$. We denote $\operatorname{dom}(f_\alpha)$ by d_α, and note that d_α is an α-domain.
(3) $\{d_\alpha : \alpha < \eta\}$ is $\subseteq$-increasing.
(4) If $\alpha \in \operatorname{supp}(p)$ then:
   - $f_\alpha(\kappa_\alpha)$ is inaccessible and $f_\alpha(\kappa_\alpha) > \overline{\kappa}_\alpha$.
   - $\lambda_\alpha = \rho_\alpha^{++}$ where $\rho_\alpha = s_\alpha(f_\alpha(\kappa_\alpha))$, so that in particular $\overline{\kappa}_\alpha < \overline{\kappa}_\alpha^+ < f_\alpha(\kappa_\alpha) < \rho_\alpha < \rho_\alpha^+ < \lambda_\alpha < \lambda_\alpha^+ < \kappa_\alpha$.
(5) If $\alpha \notin \operatorname{supp}(p)$, then $A_\alpha \in E_\alpha(d_\alpha)$, and $A_\alpha \subseteq A_\alpha(d_\alpha)$.
(6) If $\alpha \in \operatorname{supp}(p)$, then $h^0_\alpha \in \operatorname{Col}(\overline{\kappa}_\alpha^+, < f_\alpha(\kappa_\alpha)), h^1_\alpha \in \operatorname{Col}(f_\alpha(\kappa_\alpha), \rho_\alpha^+)$, and $h^2_\alpha \in \operatorname{Col}(\lambda_\alpha^+, < \kappa_\alpha)$.
(7) If $\alpha \notin \operatorname{supp}(p)$, then
   (a) $\operatorname{dom}(H^0_\alpha) = \operatorname{dom}(H^1_\alpha) = A_\alpha$.
   (b) For $\tau \in \operatorname{dom}(H^0_\alpha)$ let $\nu = \tau(\kappa_\alpha)$, then $H^0_\alpha(\tau) \in \operatorname{Col}(\overline{\kappa}_\alpha^+, < \nu)$, and $H^1_\alpha(\tau) \in \operatorname{Col}(\nu, s_\alpha(\nu)^+)$.
   (c) $\operatorname{dom}(H^2_\alpha) = A_\alpha(\kappa_\alpha)$.
   (d) For each ordinal $\nu \in \operatorname{dom}(H^2_\alpha)$, $H^2_\alpha(\nu) \in \operatorname{Col}(s_\alpha(\nu)^{+3}, < \kappa_\alpha)$.

Readers of Gitik's preprint [4] will notice a few differences in the collapsing parts, in particular H^0_α and H^1_α are defined with domains in $E_\alpha(d_\alpha)$ rather than a projected measure. This is not necessary here because the pattern of collapses is a bit different, the projection in [4] is needed for the chain condition but this is not necessary here and hence permits our conditions to be simpler.

There are two main reasons why our situation is simpler than that in [4]. The first one is that as in [3] the "impure" part of a condition is rather small. The other is that we are defining the forcing using Merimovich's machinery of objects, which simplifies the ultrapower analysis. Both of these factors play a role in the chain condition analysis in Theorem 3 below.

To clarify that all the requirements are reasonable, fix an $\alpha < \eta$. If α^* exists as in 2, then $\kappa_\alpha \leq \overline{\kappa}_{\alpha^*} < \lambda_{\alpha^*} < \kappa_{\alpha^*}$. The forcing $\mathcal{A}(\kappa_\alpha, \lambda_{\alpha^*})$ is equivalent to the forcing adding λ_{α^*}-subsets of κ_α^+.

The fact that for $\alpha \in \operatorname{supp}(p)$, $f_\alpha(\kappa_\alpha)$ is inaccessible will eventually guarantee that the forcing preserves some cardinals at each coordinate, see Lemma 8.

When we discuss multiple conditions, we usually put a superscript on every component with the name of the condition. Also a condition is officially a sequence of sequences, but we sometimes drop the angle brackets if it is clear from the context. For example, when $\eta = 1$, we may write $p = \langle f^p_0, A^p_0, (H^0_0)^p, (H^1_0)^p, (H^2_0)^p \rangle$ instead of having two pairs of brackets. As one might expect, objects in A_α are used to extend a condition in a meaningful way. We will restrict our attention to the valid objects (which we call *addable*) in Definition 5.

We give some conventions here. The *pure part* of p is the p_α's for $\alpha \notin \operatorname{supp}(p)$. The rest is called the *impure part* of p. We refer the first coordinate f_α of each p_α as the *Cohen part*, and the last three coordinates of each p_α as the *collapse parts*. A condition p is *pure* if $\operatorname{supp}(p) = \emptyset$. For each α-object μ, let $\rho_\alpha(\mu) = s_\alpha(\mu(\kappa_\alpha))$, $\lambda_\alpha(\mu) = \rho_\alpha(\mu)^{++}$. For each $\nu < \kappa_\alpha$, let $\rho_\alpha(\nu) = s_\alpha(\nu)$, and $\lambda_\alpha(\nu) = \rho_\alpha(\nu)^{++}$.

As usual, we drop the subscript α if it is clear from the context. Similarly we sometimes drop the subscript $\langle E_\alpha : \alpha < \eta \rangle$ from $\mathbb{P}_{\langle E_\alpha : \alpha < \eta \rangle}$. We define $\mathbb{P} \restriction \alpha$ to be $\{p \restriction \alpha : p \in \mathbb{P}\}$, and $\mathbb{P} \setminus \alpha$ to be $\{p \setminus \alpha : p \in \mathbb{P}\}$. The notions in Definition 3 refer to the corresponding components. For instance, f^p_0 is the Cohen part of p_0, the first entry of p, and d^p_0 is just $\operatorname{dom}(f^p_0)$.

We now define the concept of *direct extension*.

Definition 4. Let $p, q \in \mathbb{P}$. p is a **direct extension** of q, denote $p \leq^* q$, if and only if:

(1) $\operatorname{supp}(p) = \operatorname{supp}(q)$.
(2) For each α, $f^p_\alpha \leq f^q_\alpha$.
(3) If $\alpha \in \operatorname{supp}(p)$, then $\lambda^p_\alpha = \lambda^q_\alpha$.
(4) If $\alpha \in \operatorname{supp}(p)$, then $(h^l_\alpha)^p \leq (h^l_\alpha)^q$ for $l < 3$.
(5) If $\alpha \notin \operatorname{supp}(p)$, then
  (a) $\pi_{d^p_\alpha, d^q_\alpha}[A^p_\alpha] \subseteq A^q_\alpha$. As a consequence $\pi_{d^p_\alpha, d^q_\alpha}[\operatorname{dom}(H^l_\alpha)^p] \subseteq \operatorname{dom}(H^l_\alpha)^q$ for $l < 2$ and $\operatorname{dom}(H^2_\alpha)^p \subseteq \operatorname{dom}(H^2_\alpha)^q$.
  (b) For $l < 2$, $(H^l_\alpha)^p(\tau) \leq (H^l_\alpha)^q(\pi_{d^p_\alpha, d^q_\alpha}(\tau))$ for all relevant τ.
  (c) For all relevant ν, $(H^2_\alpha)^p(\nu) \leq (H^2_\alpha)^q(\nu)$.

We often express property 5a as "A^p_α projects down to A^q_α". To see what the extensions of a given condition look like in general, we first restrict the kind of objects which are allowed to be used to extend a condition.

The following definition and other definitions later on involves composition of functions. We note here that compositions are done partially, meaning for functions f and g, $f \circ g$ has the domain $\{x \in \operatorname{dom}(g) : g(x) \in \operatorname{dom}(f)\}$ and $f \circ g(x) = f(g(x))$.

Definition 5. Let $p \in \mathbb{P}$, $\alpha \notin \operatorname{supp}(p)$, and $\mu \in A^p_\alpha$. μ is **addable** to p if:

(1) $\overline{\kappa}_\alpha < \mu(\kappa_\alpha)$, and $\mu(\kappa_\alpha)$ is inaccessible.
(2) $\bigcup\limits_{\beta < \alpha} d_\beta \subseteq \operatorname{dom}(\mu)$ and $\mu \restriction (\overline{\kappa}_\alpha + 1)$ is an identity function.
(3) For each $\beta \in (\max(\operatorname{supp}(p) \cap \alpha), \alpha)$,
  (a) $\mu[d_\beta] \subseteq \lambda_\alpha(\mu)$.
  (b) $\{\psi \circ \mu^{-1} : \psi \in A_\beta\} \in E_\beta(\mu[d_\beta])$.
  (c) $\mu(\kappa_\beta) = \kappa_\beta$.

We denote the set in condition 3b as $A_\beta \circ \mu^{-1}$. It is important that almost every $\mu \in A_\alpha$ is addable, and the proof illustrates the role of coherence, so we sketch it. We write mc_α for $\operatorname{mc}_\alpha(d_\alpha)$.

- $j_{E_\alpha}(\overline{\kappa}_\alpha) = \overline{\kappa}_\alpha < \kappa_\alpha = \operatorname{mc}_\alpha(j_{E_\alpha}(\kappa_\alpha))$, and κ_α is inaccessible in M_α.
- If $D = \bigcup_{\beta<\alpha} d_\beta$ then $|D| \leq \overline{\kappa}_\alpha$ and $D \subseteq d_\alpha$, so $j_{E_\alpha}(D) = j_{E_\alpha}[D] \subseteq \operatorname{dom}(\operatorname{mc}_\alpha)$.
- By the previous item $j_{E_\alpha}(d_\beta) \subseteq \operatorname{dom}(\operatorname{mc}_\alpha)$, and easily $\operatorname{rge}(\operatorname{mc}_\alpha) \subseteq \lambda$. By the choice of s_α, $j_{E_\alpha}(s_\alpha)(\operatorname{mc}_\alpha(j_{E_\alpha}(\kappa_\alpha))) = \rho$. Now $\lambda = \rho^{++}$ and so easily $j_{E_\alpha}(\lambda_\alpha)(\operatorname{mc}_\alpha(j_{E_\alpha}(\kappa_\alpha))) = \lambda$.
- Note that for each $\psi \in A_\beta$, $\operatorname{dom}(\psi) \subseteq d_\beta \subseteq \operatorname{dom}(\mu)$, so that $\operatorname{dom}(\psi \circ \mu^{-1}) = \mu[\operatorname{dom}(\psi)]$. By a routine calculation, $\operatorname{mc}_\alpha[j_{E_\alpha}(\operatorname{dom}(\psi))] = \operatorname{dom}(\psi)$, and $j_{E_\alpha}(\psi) \circ \operatorname{mc}_\alpha^{-1}$ maps $\gamma \in \operatorname{dom}(\psi)$ to $\psi(\gamma)$, that is $j_{E_\alpha}(\psi) \circ \operatorname{mc}_\alpha^{-1} = \psi$. Since $|A_\beta| = 2^{\kappa_\beta} < \kappa_\alpha$, $j_{E_\alpha}(A_\beta) = j_{E_\alpha}[A_\beta]$ and so $\{\Psi \circ \operatorname{mc}_\alpha^{-1} : \Psi \in j_{E_\alpha}(A_\beta)\} = A_\beta$. Now $A_\beta \in E_\beta(d_\beta) = j_{E_\alpha}(E_\beta)(d_\beta)$ by Remark 4.

It will become clearer why we restrict ourselves to the addable objects, once we define the notion of one-step extension in Definition 6. To define a non-direct extension, we introduce more notation. If $f \in \mathcal{A}(\kappa_\alpha, \lambda)$ with domain d, and μ is an α-d-object, we define $f \oplus \mu$ to be the function $g \in \mathcal{A}(\kappa_\alpha, \lambda)$ such that $\operatorname{dom}(g) =$

$\mathrm{dom}(f)$, and

$$g(\gamma) = \begin{cases} \mu(\gamma) & \text{if } \gamma \in \mathrm{dom}(\mu), \\ f(\gamma) & \text{otherwise.} \end{cases}$$

Note that we obtain the function g by simply overwrite previous values by μ. For each condition p, $\beta < \alpha$, and $\mu \in E_\alpha(d^p_\alpha)$, we define $(f^p_\beta)_\mu$ as $f^p_\beta \circ \mu^{-1}$. For $l < 2$ we define $(H^l_\beta)^p_\mu$ with domain $A^p_\beta \circ \mu^{-1}$, and $(H^l_\beta)^p_\mu(\xi') = (H^l_\beta)^p(\xi' \circ \mu)$.

Definition 6. (one-step extension) Fix a condition $p \in \mathbb{P}$, $\alpha \notin \mathrm{supp}(p)$, and an addable object $\mu \in A^p_\alpha$. The **one-step extension** of p by μ, denoted by $p + \mu$ is the condition q, where

1. $\mathrm{supp}(q) = \mathrm{supp}(p) \cup \{\alpha\}$.
2. $q \restriction [0, \max(\mathrm{supp}(p) \cap \alpha)) = p \restriction [0, \max(\mathrm{supp}(p) \cap \alpha))$, and $q \setminus \alpha = p \setminus \alpha$.
3. At the α-th coordinate, we have
   (a) $f^q_\alpha = f^p_\alpha \oplus \mu$.
   (b) $\lambda^q_\alpha = \lambda_\alpha(\mu)$.
   (c) for $l < 2$, $(h^l_\alpha)^q = (H^l_\alpha)^p(\mu)$.
   (d) $(h^2_\alpha)^q = (H^2_\alpha)^p(\mu(\kappa_\alpha))$.
4. fix $\beta \in [\max(\mathrm{supp}(p) \cap \alpha), \alpha)$. Then
   (a) $f^q_\beta = (f^p_\beta)_\mu$.
   (b) $A^q_\beta = A^p_\beta \circ \mu^{-1}$.
   (c) For $l < 2$, $(H^l_\beta)^q = (H^l_\beta)^p_\mu$.
   (d) $(H^2_\beta)^q = (H^2_\beta)^p$.

Let's warm up and get familiar with our notations by showing that if p is pure and $\mu \in A^p_\alpha$ is addable, then $q := p + \mu$ is indeed a condition, as in Definition 3

- (1) $\mathrm{supp}(q) = \{\alpha\}$. $\lambda^q_\alpha = \lambda_\alpha(\mu) = s_\alpha(\mu(\kappa_\alpha))^{++} < \kappa_\alpha$. By our assumption of s_α, $s_\alpha(\mu(\kappa_\alpha))^{++} > s_\alpha(\mu(\kappa_\alpha)) > \overline{\kappa}_\alpha$.
- (2),(3) It is easy to see that for $\beta \geq \alpha$, $\lambda_{\beta^*} = \lambda$. The part above α is not affected, and $\mathrm{dom}(f^p_\alpha \oplus \mu) = \mathrm{dom}(f^p_\alpha)$. Hence $\mathrm{dom}(f^q_\beta) = \mathrm{dom}(f^p_\beta)$. For $\beta < \alpha$, $\beta^* = \alpha$, and $\mathrm{dom}(f^q_\beta) = \mathrm{dom}(f^p_\beta \circ \mu^{-1}) = \mu[d^p_\beta] \subseteq \lambda_\alpha(\mu) = \lambda^q_\alpha$. Since μ fixes ordinals below $\kappa_\beta + 1$, d^q_β is a β-domain. Also $\lambda_\alpha(\mu) < \kappa_\alpha + 1 \subseteq \mathrm{dom}(f^q_\alpha)$. Hence the domains in q are $\subseteq$-increasing.
- (4) $f^q_\alpha(\kappa_\alpha) = \mu(\kappa_\alpha) > \overline{\kappa}_\alpha$, and $\mu(\kappa_\alpha)$ is inaccessible. The rest follows by our definitions.

The rest of the proof is trivial for $\beta > \alpha$, and we assume $\beta < \alpha$.

- (5) Follows from addability of μ.
- (6) Follows from (7) in Definition 3.
- (7) For $l = 0, 1$, $\mathrm{dom}(H^l_\beta)^q = \mathrm{dom}(H^l_\beta)^p_\mu = A^p_\beta \circ \mu^{-1} = A^q_\beta$. Since $d^p_\beta \subseteq \mathrm{dom}(\mu)$, for $\psi \in A^p_\beta$, and $l = 0, 1$, $\mathrm{dom}(\psi) \subseteq d^p_\beta \subseteq \mathrm{dom}(\mu)$, so $(H^l_\beta)^q(\psi \circ \mu^{-1}) = (H^l_\beta)^p(\psi \circ \mu^{-1} \circ \mu) = (H^l_\beta)^p(\psi)$. $\mathrm{dom}(H^2_\beta)^p = A^p_\beta(\kappa_\beta) = A^p_\beta \circ \mu^{-1}(\kappa_\beta)$ since μ fixes κ_β. Also all the collapses fall into the right types, since μ fixes κ_β as well.

For each $\beta < \alpha$, we sometimes write q_β as $(p_\beta)_\mu$. When we perform a one-step extension of p by μ from the α-th coordinate, we call the part of the resulting condition before α "*$p \restriction \alpha$ squished by μ*". Occasionally, it is possible that we squish $p \restriction \alpha$ by some $\tau \in E_\alpha(d')$ for some $d' \supseteq d$, where $d = \mathrm{dom}(f^p_\alpha)$. In this case

$(p_\beta)_\tau$ is just $(p_\beta)_{\tau\restriction d}$. For each sequence of objects $\vec{\mu} = \langle \mu_0, \mu_1, \ldots, \mu_{k-1}\rangle$, we define $p + \vec{\mu}$ recursively as $(p + \mu_0) + \langle \mu_1, \ldots, \mu_{k-1}\rangle$ when μ_0 is addable to p and for every $0 < i < k$, μ_i is addable to $p + \langle \mu_0, \ldots, \mu_{i-1}\rangle$.

We define $p \leq q$ if $p \leq^* q + \vec{\mu}$ for some sequence of objects $\vec{\mu}$. We refer the readers to Lemma 2.4 in [6] for some straightforward technical lemmas on transitivity and commutativity of the choices of the orders of the objects we add to a condition. Briefly, we can commute the order of the extensions.

For each condition $p \in \mathbb{P}_{\langle E_\alpha:\alpha<\eta\rangle}$ with $\alpha \in \mathrm{supp}(p)$, we can see $\mathbb{P}/p \cong (\mathbb{P}/p) \restriction \alpha \times (\mathbb{P}/p) \setminus \alpha$, where the first factor can be regarded as $(\mathbb{P}/p)_{\langle E_\beta \restriction \lambda_\alpha:\beta<\alpha\rangle}$. The second factor can be regarded as $(\mathbb{P}/p \setminus \alpha)_{\langle E_\beta:\alpha\leq\beta<\eta\rangle}$. These two factors are independent of each other, despite the connection between d_α's on different levels in clause 3 of Definition 3. The point is that $d_\alpha^p \supseteq \kappa_\alpha + 1$ and $\lambda_\alpha < \kappa_\alpha$. Hence any kinds of extensions in the first factor do not harm property 3 in Definition 3.

We begin our analysis of the poset by computing its chain condition.

Lemma 3. *If η is a limit ordinal, then $\mathbb{P}_{\langle E_\alpha:\alpha<\eta\rangle}$ has the λ-chain condition.*

Proof. Suppose the conditions $\langle p^i : i < \lambda\rangle$ are given. The first step is to show that without loss of generality, we can assume p^i is pure for all i. Since $|[\eta]^{<\omega}| < \lambda$, we assume every condition has the same support s. Suppose $s \neq \emptyset$, and let $\alpha = \max(s)$.

Each condition breaks into two pieces, with the first piece in $\mathbb{P}_{\langle E_\beta\restriction\lambda_\alpha:\beta<\alpha\rangle}$ for some $\lambda_\alpha < \kappa_\alpha$. Since the number of the conditions in the first factor is at most $\lambda_\alpha < \lambda$, we assume that every condition's first factor is the same, and also every condition has the same λ_α. Now for each $i < \lambda$, $\mathrm{dom}(f_\alpha^{p^i}) \in [\lambda]^{\kappa_\alpha}$. We thin out the collection $\{p^i : i < \lambda\}$ so that $\{\mathrm{dom}(f_\alpha^{p^i}) : i < \lambda\}$ form a Δ-system. Then thin out the collection again so that the α-th Cohen parts are all compatible. Since all the α-th collapse parts are small, by the Pigeonhole Principle we can also thin out the collection so that every condition has the same collapses at the α-th coordinate. Hence the initial segments up to and including α are compatible.

Since we can shrink the collection $\{p^i : i < \lambda\}$ so that the impure parts are compatible, we now may assume all the conditions p^i are pure. Since $|\bigcup_{i<\eta} \mathrm{dom}(f_\alpha^{p^i})| \leq \overline{\kappa}_\eta \leq \rho$, by a similar Δ-system argument, we can thin out so that the Cohen conditions $f_\alpha^{p^i}$ are pairwise compatible in $\mathcal{A}(\kappa_\alpha, \lambda)$ for each $\alpha < \eta$. Measure one sets are always compatible. It remains to find a sub-collection of $\{p^i : i < \lambda\}$ so that all the collapse parts $(H_\alpha^l)^{p^i}$ are compatible. Note that the collapses are no longer small.

For $l = 0$ and $l = 1$, $(H_\alpha^l)^{p^i}$ represents $j_{E_\alpha}((H_\alpha^l)^{p^i})(\mathrm{mc}_\alpha(d_\alpha))$, which is a condition in $\mathrm{Col}(\overline{\kappa}_\alpha^+, <\kappa_\alpha)$ when $l = 0$, and a condition in $\mathrm{Col}(\kappa_\alpha, \rho^+)$ for $l = 1$. Recall that $\lambda = \rho^{++}$. These two collapse forcings both have size ρ^+ in V. The situation is slightly different for $(H_\alpha^2)^{p^i}$, a function whose domain is in $E_\alpha(\kappa_\alpha)$: if we let $i_\alpha : V \to N_\alpha = Ult(V, E_\alpha(\kappa_\alpha))$, then we can view this function as representing $i_\alpha((H_\alpha^2)^{p^i})(\kappa_\alpha)$, a condition in $\mathrm{Col}(i_\alpha(s_\alpha)(\kappa_\alpha)^{+++}, < i_\alpha(\kappa_\alpha))^{N_\alpha}$. This poset has size 2^{κ_α} in V.

Recalling that $\eta < \kappa_0$, it follows that there are at most ρ^+ possibilities for the sequence of conditions $\langle j_{E_\alpha}((H_\alpha^0)^{p^i})(\mathrm{mc}_\alpha(d_\alpha)), j_{E_\alpha}((H_\alpha^1)^{p^i})(\mathrm{mc}_\alpha(d_\alpha)), i_\alpha((H_\alpha^2)^{p^i})(\kappa_\alpha) : \alpha < \eta\rangle$. Hence we can thin out the collection of the conditions so that for all $\alpha < \eta$, $i < \lambda$, and $l < 3$, the functions $(H_\alpha^l)^{p^i}$ represent the same function in M_α or N_α. It is now easy to show that the resulting collection of conditions is pairwise compatible. $\square$

The following lemma is straightforward:

Lemma 4. *$((\mathbb{P}/p) \setminus \alpha, \leq^*)$ is $\overline{\kappa}_\alpha^+$-closed.*

The proof of the Prikry property here requires some "integration" of the conditions. The following lemmas (Lemma 5 and Lemma 6) shows that we can assemble the conditions properly. We state the lemmas for a pure condition. In general, a condition may not be pure, and each impure coordinate divides the condition into blocks. The lemmas can be stated in each separate block individually.

Lemma 5. *Let $p = \langle p_\alpha : \alpha < \eta \rangle$ be a pure condition and $\alpha \notin \operatorname{supp}(p)$. Let $d = \operatorname{dom}(f_\alpha^p)$. Set $q = j_{E_\alpha}(p) + \operatorname{mc}_\alpha(d)$. Then $q \restriction \alpha = p \restriction \alpha$.*

Proof. Without loss of generality, we may assume p is pure, otherwise the part $p \restriction \max(\operatorname{supp}(p) \cap \alpha)$ is not affected by α-objects.

Fix $\beta < \alpha$. Since $|d_\beta^p| < \kappa_\alpha$, $j_{E_\alpha}(d_\beta^p) = j_{E_\alpha}[d_\beta^p]$. Hence $d_\beta^q = \operatorname{mc}_\beta(d)[j_{E_\alpha}(d_\beta^p)] = d_\beta^p$. Furthermore, for $\gamma \in d_\beta^q$, $f_\beta^q(\gamma) = j_{E_\alpha}(f_\beta^p) \circ (\operatorname{mc}_\beta(d))^{-1}(\gamma) = j_{E_\alpha}(f_\beta^p)(j_{E_\alpha}(\gamma)) = j_{E_\alpha}(f_\beta^p(\gamma)) = f_\beta^p(\gamma)$. This shows that $f_\beta^q = f_\beta^p$. Similar calculation shows that other components in q_β are the same as the corresponding components in p_β. □

Further calculations show

Lemma 6. *Assume $\langle p_\alpha : \alpha < \eta \rangle$ is pure. Fix $\alpha < \eta$. $d = \operatorname{dom}(f_\alpha^p)$. Let $f \leq f_\alpha^p$ with $\operatorname{dom}(f) = d'$. Let $A \in E_\alpha(d')$, and A projects down to A_α^p. For $\tau \in A$, denote $(p \restriction \alpha)_\tau = (p \restriction \alpha)_{\tau \restriction d} = (p + (\tau \restriction d)) \restriction \alpha$. Suppose for each $\tau \in A$, there is a condition $t(\tau) \leq^* (p \restriction \alpha)_\tau$, $h^0(\tau) \leq (H_\alpha^0)^p(\tau \restriction d)$, and $h^1(\tau) \leq (H_\alpha^1)^p(\tau \restriction d)$. Then there is a condition $q \leq^* p$ such that if $\psi \in A_\alpha^q$, $\tau = \psi \restriction d'$, and $\nu = \tau(\kappa_\alpha) = \psi(\kappa_\alpha)$, we have:*

1. *If $\beta < \alpha$ and g is the Cohen part of $(q + \psi)_\beta$, then $g = f_\beta^{t(\tau)}$.*
2. *$(q + \psi) \restriction \alpha \leq^* t(\tau)$.*
3. *For $l = 0, 1$, $(H_\alpha^l)^q(\psi) = h^l(\tau)$.*
4. *$(H_\alpha^2)^q(\nu) = (H_\alpha^2)^p(\nu)$.*
5. *$f_\alpha^q \leq f$.*

Proof. Set $r = j_{E_\alpha}(t)(\operatorname{mc}_\alpha(d'))$, where t is considered as a function $\tau \mapsto t(\tau)$. Note that r is a condition in the forcing up to α by a routine calculation and coherence of the extenders. We recall that $r = \langle r_\beta : \beta < \alpha \rangle$, where for each $\beta < \alpha$,

$$r_\beta = \langle f_\beta^r, A_\beta^r, (H_\beta^0)^r, (H_\beta^1)^r, (H_\beta^2)^r \rangle.$$

Fix $\beta < \alpha$. Here are some properties of r_β:

(A1) (a) Let $x_\beta = d_\beta^r \cap \kappa_\alpha$. Then x_β is a bounded subset of κ_α.
(b) Let $\gamma_\beta = \operatorname{ot}(d_\beta^r \setminus \kappa_\alpha)$. Then $\gamma_\beta < \kappa_\beta^+ < \kappa_\alpha$.
(c) $A_\beta^r \in E_\beta(d_\beta^r)$.

(A1)a and (A1)b follow from the fact that $\operatorname{dom}(f_\beta^{t(\tau)})$ has size κ_β for all τ. Fix an increasing enumeration of $d_\beta^r \setminus \kappa_\alpha$ as $\{\xi_{\beta,i}^r : i < \gamma_\beta\}$. For each $\tau \in A_\alpha^p$, let

$$t(\tau) = \langle \langle f_\beta^\tau, A_\beta^\tau, (H_\beta^0)^\tau, (H_\beta^1)^\tau, (H_\beta^2)^\tau \rangle : \beta < \alpha \rangle$$

and let $d_\beta^\tau = \operatorname{dom}(f_\beta^\tau)$.

We record a few equations for each $\beta < \alpha$:

(i) If $l=0,1$, then by coherence (see Lemma 2.4 in [6]), we have

$$\begin{aligned} j_{E_\beta}((H^l_\beta)^r)(\mathrm{mc}_\beta(d^r_\beta)) &= j_{E_\beta}(j_{E_\alpha}(\tau\mapsto (H^l_\beta)^\tau))(j_{E_\alpha}(\tau\mapsto \mathrm{mc}_\beta(d^\tau_\beta))) \\ &= j_{E_\alpha}(\tau\mapsto j_{E_\beta}(H^l_\beta)^\tau(\mathrm{mc}_\beta(d^\tau_\beta))). \end{aligned}$$

(ii)

$$\begin{aligned} j_{E_\beta}((H^2_\beta)^r)(\kappa_\beta) &= j_{E_\beta}(j_{E_\alpha}(\tau\mapsto (H^2_\beta)^\tau))(j_{E_\alpha}(\tau\mapsto \kappa_\beta)) \\ &= j_{E_\alpha}(\tau\mapsto j_{E_\beta}(H^2_\beta)^\tau(\kappa_\beta)). \end{aligned}$$

Note that by Lemma 5, $A^r_\beta = j_{E_\alpha}(A_\beta r_{\mathrm{mc}_\alpha(d^r_\beta)}$. We shrink A to A^* so that for every $\tau\in A^*$, and every $\beta<\alpha$, the following statements hold:

(A2) (a) $d^\tau_\beta\cap\kappa_\alpha = x_\beta$.
(b) $\mathrm{ot}(d^\tau_\beta\setminus\kappa_\alpha)=\gamma_\beta$.
(c) For $\xi\in d^\tau_\beta\cap\kappa_\alpha$, $f^\tau_\beta(\xi)=f^r_\beta(\xi)$.
(d) Let $\{\xi^\tau_{\beta,i} : i<\gamma_\beta\}$ be the increasing enumeration of $d^\tau_\beta\setminus\kappa_\alpha$, then for all $i<\gamma_\beta$, $f^\tau_\beta(\xi^\tau_{\beta,i})=f^r_\beta(\xi^r_{\beta,i})$.
(e) $A^\tau_\beta = A^r_\beta\circ\tau^{-1}$.
(f) For $l=0,1$, $j_{E_\beta}((H^l_\beta)^\tau)(\mathrm{mc}_\beta(d^\tau_\beta)) = j_{E_\beta}((H^l_\beta)^r)(\mathrm{mc}_\beta(d^r_\beta))$.
(g) $j_{E_\beta}((H^2_\beta)^\tau)(\kappa_\beta) = j_{E_\beta}((H^2_\beta)^r)(\kappa_\beta)$.

It is tempting to think we will set q_β to be r_β for all $\beta<\alpha$, but we need to make a tiny cosmetic modification. The property of τ in (A2)f and (A2)g makes sure that for each $\tau\in A^*$, $\beta<\alpha$, and $l=0,1$,

$$C_{\beta,l} := \{\psi\in E_\beta(d^\tau_\beta) : (H^l_\beta)^\tau(\psi) = (H^l_\beta)^r(\psi\circ\tau)\}\in E_\beta(d^\tau_\beta),$$

and

$$C_{\beta,2} := \{\psi\in E_\beta(d^\tau_\beta) : (H^2_\beta)^\tau(\psi(\kappa_\beta)) = (H^2_\beta)^r(\psi(\kappa_\beta))\}\in E_\beta(d^\tau_\beta).$$

Shrink all measure one sets appearing in $t(\tau)$ so that every β-object belongs to $C_{\beta,0}\cap C_{\beta,1}\cap C_{\beta,2}$. Restrict the collapsing functions in the natural way, and call the result $t^*(\tau)$. Instead of integrating the function t to obtain r, we will integrate the function t^* in the same manner to obtain $\langle q_\beta : \beta<\alpha\rangle$.

Since we just shrank the measure one sets and restricted the collapses, all the properties in the (A2)-list hold, except that the property (A2)e is weakened to: $(A^\tau_\beta)^*\subseteq A^r_\beta\circ\tau^{-1}$, where $(A^\tau_\beta)^*$ is the measure one set of the βth-coordinate of $t^*(\mu)$

Now for $\beta<\alpha$, let $q_\beta = j_{E_\alpha}(t^*)(\mathrm{mc}_\alpha(d'))$. Note that q_β is almost identical to r_β, except the measure one sets: $A^q_\beta\subseteq A^r_\beta$. We now define q_α. Set $q_\alpha=\langle f^q_\alpha, A^q_\alpha, (H^0_\alpha)^q, (H^1_\alpha)^q, (H^2_\alpha)^q\rangle$ as follows:

(A3) (a) f^q_α has domain $d'\cup\bigcup\limits_{\beta<\alpha}(x_\beta\cup\{\xi^r_{\beta,i} : i<\gamma_\beta\})$. This is just $d'\cup\bigcup_{\beta<\alpha} d^r_\beta$.
(b) For each $\xi\in\mathrm{dom}(f^q_\alpha)$, if $\xi\in d'$, $f^q_\alpha(\xi)=f(\xi)$, otherwise $f^q_\alpha(\xi)=0$.
(c) $A^q_\alpha\in E_\alpha(\mathrm{dom}(f^q_\alpha))$, where $\psi\in A^q_\alpha$ iff
  (i) $\tau=\psi\restriction d'\in A^*$,
  (ii) $\bigcup\limits_{\beta<\alpha}(x_\beta\cup\{\xi^r_{\beta,i} : i<\gamma_\beta\})\subseteq\mathrm{dom}(\psi)$,
  (iii) for each $\beta<\alpha$ and each $\xi\in x_\beta$, $\psi(\xi)=\xi$,
  (iv) for each β and i, $\psi(\xi^r_{\beta,i})=\xi^\tau_{\beta,i}$, where τ is defined as in (A3)(c)i.

(d) For $l = 0, 1$, set $(H^l_\alpha)^q(\psi) = h^l(\tau)$.
(e) $(H^2_\alpha)^q = (H^2_\alpha)^p \upharpoonright A^q_\alpha(\kappa_\alpha)$.

For $\beta > \alpha$, extend p_β to $q_\beta = \langle f^q_\beta, A^q_\beta, (H^0)^q_\beta, (H^1)^q_\beta, (H^2)^q_\beta\rangle$ in the obvious way, meaning:

(A4) (a) Extend the f^p_β to f^q_β with domain $d^q_\beta := \mathrm{dom}(f^q_\alpha) \cup d'$, $f^q_\beta(\gamma) = f^p_\beta(\gamma)$ if $\gamma \in d^p_\beta$, otherwise $f^q_\beta(\gamma) = 0$.
(b) The measure one set in q_β projects down the measure one set in p_β, i.e. $A^{q_\beta} \upharpoonright d^p_\beta \subseteq A^p_\beta$. Also $A^q_\beta \subseteq A_\beta(d^q_\beta)$.
(c) Restrict the collapses based on the measure one set we just defined, i.e. for $l = 0, 1$, $(H^l)^q_\beta(\tau) = (H^0)^p_\beta(\tau \upharpoonright d^p_\beta)$, and $(H^2)^q_\beta = (H^2)^p_\beta \upharpoonright A^q_\beta(\kappa_\beta)$.

Let $q = \langle q_\beta : \beta < \eta\rangle$. We claim q satisfies the conclusion of Lemma 6. It is easy to see that $q \setminus \alpha$ is $\leq^*$-below $p \setminus \alpha$. For $\tau \in A^*$, $t^*(\tau) \leq^* t(\tau) \leq^* (p + \tau \upharpoonright d) \upharpoonright \alpha$, and so $q \upharpoonright \alpha \leq^* (j_{E_\alpha}(p) + \mathrm{mc}_\alpha(d)) \upharpoonright \alpha$. By Lemma 5, the last term belongs to $j_{E_\alpha}(\mathbb{P}_{\langle E_\beta:\beta<\alpha\rangle}) \upharpoonright \lambda = \mathbb{P}_{\langle E_\alpha:\beta<\alpha\rangle}$, and is equal to $p \upharpoonright \alpha$. Hence $q \leq^* p$.

Now we check that q satisfies all the properties listed in Lemma 6. Fix $\psi \in A^q_\alpha$, $\tau = \psi \upharpoonright d'$, $\mu = \psi \upharpoonright d$, and $\nu = \mu(\kappa_\alpha) = \tau(\kappa_\alpha) = \psi(\kappa_\alpha)$.

Requirement 1: Fix $\beta < \alpha$. The Cohen part of $(q + \tau)_\beta$ is $f^q_\beta \circ \psi^{-1} = f^r_\beta \circ \psi^{-1}$. By (A3)(c)ii and (A3)(c)iii, and (A3)(c)iv, $\mathrm{dom}(f^r_\beta \circ \psi^{-1}) = \psi[d^r_\beta] = \psi[x_\beta \cup \{\xi^r_{\beta,i} : i < \gamma_\beta\}] = x_\beta \cup \{\xi^\tau_{\beta,i} : i < \gamma_\beta\}$. The last term is equal to $\mathrm{dom}(f^\tau_\beta)$ by (A2)a and (A2)b. From (A3)(c)iv and (A2)c, if $\xi \in x_\beta$, $f^r_\beta \circ \psi^{-1}(\xi) = f^r_\beta(\xi) = f^\tau_\beta(\xi)$. From (A2)d, for $i < \gamma_\beta$, $f^r_\beta \circ\ \psi^{-1}(\xi^\tau_{\beta,i}) = f^r_\beta(\xi^r_{\beta,i}) = f^\tau_\beta(\xi^\mu_{\beta,i})$. The proof for requirement 1 is done.

Requirement 2: Fix $\beta < \alpha$. From (A2)e, we have $A^q_\beta \circ \psi^{-1} \subseteq A^r_\beta \circ \psi^{-1} = A^r_\beta \circ \tau^{-1} = A^\tau_\beta$. By (A3)(c)i, $\tau \in A^*$. For $\sigma \in A^q_\beta \circ \psi^{-1}$, $\sigma \in (A^\tau_\beta)^*$, so $l = 0, 1$, $(H^l_\beta)^{q+\psi}(\sigma) = (H^l_\beta)^q(\sigma \circ \psi) = (H^l_\beta)^r(\sigma \circ \tau) = (H^l_\beta)^\tau(\sigma)$. Similarly $(H^l_\beta)^{q+\psi}(\sigma(\kappa_\beta)) = (H^2_\beta)^\tau(\sigma(\kappa_\beta))$.

Requirement 3: From (A3)d, for $l = 0, 1$, $(H^l_\alpha)^q(\psi) = h^l(\tau)$.

Requirement 4: Straightforward from (A3)e.

Requirement 5: Follows from (A3)b.

This completes the proof.

□

4. PRIKRY PROPERTY

Theorem 7. $(\mathbb{P}, \leq, \leq^*)$ *has the Prikry property, that is to say for any boolean value b and any condition $p \in \mathbb{P}$, there is a condition $p' \leq^* p$ such that p' decides b.*

Our proof of the Prikry property follows the same lines as the proof in [3], but the collapse parts introduce additional challenges which we briefly explain. For each component p_α, the collapse parts involve three collapsing forcings, where the chain condition of the first two collapse forcings is very close to the closure of the last collapse forcing. For $\alpha > 0$, the collapse parts at α are only $\overline{\kappa}^+_\alpha$-closed, while the Cohen part is κ^+_α-closed. This all makes it natural to group together components of a forcing condition which live on different levels, which explains why our inductive hypothesis concerns a product $\mathbb{P} \times \mathbb{A}$: the intuition is that the factor $\mathbb{A}$ anticipates some collapsing at the top level when we add another level to $\mathbb{P}$.

Proof. We consider 3 cases:

(1) $\eta = 1$.
(2) $\eta > 1$ is a successor ordinal.
(3) η is limit.

Throughout these 3 cases, we assume for simplicity that the condition p is pure. We prove a stronger statement by induction on η: Suppose $\mathbb{P}$ has length η, and $(\mathbb{A}, \leq)$ is a $\overline{\kappa}_\eta^+$-closed forcing poset. Define $(p,a) \leq^* (p',a')$ in $\mathbb{P} \times \mathbb{A}$ iff $p \leq^* p'$ and $a \leq a'$. Then $(\mathbb{P} \times \mathbb{A}, \leq, \leq^*)$ also has the Prikry property.

CASE 1: $\eta = 1$

We drop the subscript 0 for simplicity, that is $\kappa = \kappa_0$, $E = E_0$, $\mathrm{mc} = \mathrm{mc}_0$, $s = s_0$ and so on. Note here that $\mathbb{A}$ is κ^+-closed. Let b be a boolean value. Let $p = \langle f, A, H^0, H^1, H^2 \rangle \in \mathbb{P}$, and $a \in \mathbb{A}$. Let θ be a sufficiently large regular cardinal, $N \prec H_\theta$ such that ${}^{<\kappa}N \subseteq N$, $|N| = \kappa$ and $p, a, \mathbb{P}, \mathbb{A}, b \in N$. Enumerate the dense open subsets of $(\mathcal{A}(\kappa,\lambda) \times \mathbb{A}) \cap N$ as $\{D_i : i < \kappa\}$. Build a decreasing sequence $\{(f_i', a_i) : i < \kappa\}$ in $(\mathcal{A}(\kappa,\lambda) \times \mathbb{A}) \cap N$, where $f_0' = f$, $a_0 = a$ and $(f_{i+1}', a_{i+1}) \in D_i$ for all i. Let $f' = \bigcup_{i<\kappa} f_i'$, and a' be a lower bound of $\{a_i : i < \kappa\}$. Set $d' = \mathrm{dom}(f')$. By a simple density argument, we have $d' = N \cap \lambda$. We see that $[d']^{<\kappa} \subseteq N$. Let $A' \in E(d')$ be such that $A' \subseteq A(d')$ and A' projects down to a subset of A. Recall the property of $A(d') : \{\mu \in A(d') : \mu(\kappa) = \nu\}$ has size at most $s(\nu)^{++}$. For each $\mu \in A'$, $\mathrm{dom}(\mu) \subseteq d' \subseteq N$ and $|\mathrm{dom}(\mu)| \leq \mu(\kappa) < \kappa$, hence $\mu \in N$.

Let $\nu \in A'(\kappa)$. By Lemma 2, let $\{\mu_j, h_j^0, h_j^1\}_{j<\lambda(\nu)}$ be an enumeration in N of the triples μ, h^0, h^1 where $\mu \in A'$, $\mu(\kappa) = \nu$, $h^0 \in \mathrm{Col}(\omega_1, <\nu)$, and $h^1 \in \mathrm{Col}(\nu, \rho(\nu)^+)$, respectively.

Define D_ν as the collection of $(g,x) \in \mathcal{A}(\kappa,\lambda) \times \mathbb{A}$ such that there is an $h \in \mathrm{Col}(\lambda(\nu)^+, <\kappa)$ with $h \leq H^2(\nu)$ meeting the following requirements:

(1) For all $j < \lambda(\nu)$, $\mathrm{dom}(\mu_j) \subset \mathrm{dom}(g)$.
(2) For all $j < \lambda(\nu)$,
   - EITHER $(\langle g \oplus \mu_j, \lambda(\nu), h_j^0, h_j^1, h \rangle, x)$ decides b,
   - OR there are no $g' \leq g$, $h' \leq h$, and $x' \leq x$ such that $(\langle g' \oplus \mu_j, \lambda(\nu), h_j^0, h_j^1, h' \rangle, x')$ decides b.

Claim 7.1. *D_ν is a dense open subset of $\mathcal{A}(\kappa,\lambda) \times \mathbb{A}$ and $D_\nu \in N$.*

Proof: Since D_ν is defined using parameters in N, $D_\nu \in N$. It is easy to check that D_ν is open. Now we check the density for D_ν. Let $g \in \mathcal{A}(\kappa,\lambda)$ and $x \in \mathbb{A}$. Because $\mathcal{A}(\kappa,\lambda)$ is κ^+-closed, we may assume g meets the first requirement. Build sequences $\vec{g} = \{g_j\}_{j \leq \lambda(\nu)}$, $\vec{h} = \{h_j\}_{j \leq \lambda(\nu)}$, and $\vec{x} = \{x_j\}_{j \leq \lambda(\nu)}$ such that

- $\vec{g}$ is a decreasing sequence in $\mathcal{A}(\kappa,\lambda)$.
- $\vec{h}$ is a decreasing sequence in $\mathrm{Col}(\lambda(\nu)^+, <\kappa)$.
- $\vec{x}$ is a decreasing sequence in $\mathbb{A}$.
- $g_0 = g$, $h_0 = H^2(\nu)$ and $x_0 = x$.
- At each limit $j \leq \lambda(\nu)$, we take $g_j = \bigcup_{j'<j} g_{j'}$, and $h_j = \bigcup_{j'<j} h_{j'}$.
- At each limit $j \leq \lambda(\nu)$, take x_j as a lower bound of $\{x_{j'} : j' < j\}$.

Note that the construction proceeds to the end since $\mathcal{A}(\kappa,\lambda)$ and $\mathbb{A}$ are κ^+-closed, $\mathrm{Col}(\lambda(\nu)^+, <\kappa)$ is $\lambda(\nu)^+$-closed, and $\lambda(\nu) < \kappa$. Now suppose f_j, h_j, and x_j are constructed and $j < \lambda(\nu)$. Ask if there is a triple g',h',x' below g_j,h_j, and x_j, respectively, such that $(\langle g' \oplus \mu_j, \lambda(\nu), h_j^0, h_j^1, h' \rangle, x')$ decides b. If the answer is no, take $g_{j+1} = g_j$,$h_{j+1} = h_j$, and $x_{j+1} = x_j$. Otherwise, there are such g',h', and x'. Take $g_{j+1} = g'$, $h_{j+1} = h'$, and $x_{j+1} = x'$. From the construction, we see $(g_{\lambda(\nu)}, x_{\lambda(\nu)}) \leq (g, x)$ is in D_ν, as witnessed by $h_{\lambda(\nu)}$. ∎

By the construction of f' and a' we have $(f', a') \in D_\nu$ with a witness $h \in \mathrm{Col}(\rho(\nu)^{+3}, < \kappa)$. Define $(H^2)'(\nu) = h$.

We record the properties of f', a', and $(H^2)'$ here:
($\star$) For each $\mu \in A'$ with $\nu = \mu(\kappa)$, we have that for all $h^0 \in \mathrm{Col}(\omega_1, < \nu)$ and $h^1 \in \mathrm{Col}(\nu, \rho(\nu)^+)$,

- EITHER $(\langle f' \oplus \mu, \lambda(\nu), h^0, h^1, (H^2)'(\nu)\rangle, a')$ decides b,
- OR there are no $g \leq f'$, $h \leq (H^2)'(\nu)$, and $x \leq a'$ such that $(\langle g \oplus \mu, \lambda(\nu), h^0, h^1, h\rangle, x)$ decides b.

Now for each $\mu \in A'$, find $g(\mu) \leq f'$, $h^0(\mu) \leq H^0(\mu \restriction \mathrm{dom}(f))$, $h^1(\mu) \leq H^1(\mu \restriction \mathrm{dom}(f))$, $h^2(\mu) \leq (H^2)'(\mu(\kappa))$, and $x(\mu) \leq a'$ such that

$$(\langle g(\mu) \oplus \mu, \lambda(\mu), h^0(\mu), h^1(\mu), h^2(\mu)\rangle, x(\mu)) \text{ decides } b.$$

By property ($\star$), we have

$$(\langle f' \oplus \mu, \lambda(\nu), h^0(\mu), h^1(\mu), (H^2)'(\mu(\kappa))\rangle, a') \text{ decides } b.$$

Set

$$B_0 = \{\mu \in A' : (\langle f' \oplus \mu, \lambda(\mu), h^0(\mu), h^1(\mu), (H^2)'(\mu(\kappa))\rangle, a') \Vdash b\}.$$
$$B_1 = \{\mu \in A' : (\langle f' \oplus \mu, \lambda(\mu), h^0(\mu), h^1(\mu), (H^2)'(\mu(\kappa))\rangle, a') \Vdash \neg b\}.$$

We see that $A' = B_0 \sqcup B_1$. Choose $i_0 \in \{0, 1\}$ such that $B_{i_0} \in E(d')$. Define $p'' = \langle f'', A'', (H^0)'', (H^1)'', (H^2)''\rangle$ (note that f', A', and $(H^2)'$ are already defined) as follows:

- $f'' = f'$.
- $A'' = B_{i_0}$.
- For $l = 0, 1$, $\mathrm{dom}((H^l)'') = A''$ and $H^l(\mu) = h^l(\mu)$.
- $(H^2)'' = (H^2)' \restriction A''(\kappa)$.

Claim 7.2. *$(p'', a') \leq^* (p, a)$ and (p'', a') decides b.*

Proof: It is easy to see that $(p'', a') \leq^* (p, a)$. Let $q \leq p'', x \leq a'$ such that (q, x) decides b. Without loss of generality, $(q, x) \Vdash b$, and q is not pure. Hence $q \leq^* p'' + \mu$ for some $\mu \in A''$. Observe that $f^q \leq g \oplus \mu$ for some $g \leq f'$, and $(h^2)^q \leq (H^2)''(\mu(\kappa)) = (H^2)'(\mu(\kappa))$. From the properties ($\star$), we have

$$(\langle f' \oplus \mu, \lambda(\mu), (h^0)^q, (h^1)^q, (H^2)'(\mu(\kappa))\rangle, a') \Vdash b.$$

Since for $l = 0, 1$, $(h^l)^q \leq (H^l)''(\mu) = h^l(\mu)$, μ can't be in B_1. Hence $i_0 = 0$, and $A'' = B_0$.

By a similar argument, every extension of (p'', a') by $\mu' \in A''$ forces b. Since every extension of p'' has a further extension which is not pure, and that forces b, by a density argument, $(p'', a') \Vdash b$. ∎

CASE 2: $\eta > 1$ is a successor ordinal

The proofs for all successor ordinals $\eta > 1$ are essentially the same. For simplicity, assume $\eta = 2$. Hence $\mathbb{A}$ is κ_1^+-closed. Suppose for simplicity that $p = \langle p_0, p_1\rangle$ is pure. Write $p_0 = \langle f_0, A_0, H_0^0, H_0^1, H_0^2\rangle$, and $p_1 = \langle f_1, A_1, H_1^0, H_1^1, H_2^1\rangle$. Also let $a \in \mathbb{A}$.

Let θ be a sufficiently large regular cardinal. Let $N_1 \prec H_\theta$, $|N_1| = \kappa_1$, ${}^{<\kappa_1}N_1 \subseteq N_1$, and $p, a, \mathbb{P}, b \in N_1$. Enumerate the dense open subsets of $\mathcal{A}(\kappa_1, \lambda) \times \mathbb{A} \cap N_1$ as $\{D_i : i < \kappa\}$. Build a decreasing sequence $\{(f'_{1,i}, a_i) : i < \kappa_1\}$ in $(\mathcal{A}(\kappa_1, \lambda) \times \mathbb{A}) \cap N_1$ such that $f'_{1,0} = f_1$, $a_0 = a$, and $(f'_{1,i+1}, a_{i+1}) \in D_i$ for all i. Let $f'_1 = \bigcup_{i<\kappa_1} f'_{1,i}$ and $a' =$ a lower bound of $\{a_i : i < \kappa_1\}$. Set $d'_1 = \mathrm{dom}(f'_1)$, which is $N_1 \cap \lambda$. Let $A'_1 \in E_1(d'_1)$ be such that $A'_1 \subseteq A_1(d'_1)$ and A'_1 projects down to a subset of A_1. Similar to the one extender case, $A'_1 \subseteq N_1$.

Let $\nu \in A'_1(\kappa_1)$. By Lemma 2, let $\{t_j, h^0_j, h^1_j, \mu_j\}_{j<\lambda_1(\nu)}$ be an enumeration in N_1 of the quadruples t, μ, h^0, h^1 where $t \in \mathbb{P}_{E_0 \restriction \lambda_1(\nu)}$, $\mu \in A'$ with $\mu(\kappa_1) = \nu$, $h^0 \in \mathrm{Col}(\kappa_0^+, <\nu)$, and $h^1 \in \mathrm{Col}(\nu, \rho_1(\nu)^+)$, respectively.

Define D_ν as the collection of $(g, x) \in \mathcal{A}(\kappa_1, \lambda) \times \mathbb{A}$ such that there is an $h \in \mathrm{Col}(\lambda_1(\nu)^+, <\kappa_1)$ with $h \leq H^2_1(\nu)$ meeting the following requirements:

1. For all $j < \lambda_1(\nu)$, $\mathrm{dom}(\mu_j) \subset \mathrm{dom}(g)$.
2. For all $j < \lambda_1(\nu)$,
   - EITHER $(t_j{}^\frown\langle g \oplus \mu_j, \lambda_1(\nu), h^0_j, h^1_j, h\rangle, x)$ decides b,
   - OR there is no $g' \leq g$, $h' \leq h$ and $x' \leq x$ such that $(t_j{}^\frown\langle g' \oplus \mu_j, \lambda_1(\nu), h^0_j, h^1_j, h\rangle, x')$ decides b.

Similar to the one extender case, D_ν is a dense open subset of $\mathcal{A}(\kappa_1, \lambda) \times \mathbb{A}$ and $D_\nu \in N_1$. We have $(f'_1, a') \in D_\nu$ with a witness $h \in \mathrm{Col}(\rho_1(\nu)^{+3}, <\kappa_1)$. Define $(H^2_1)'(\nu) = h$.

We record some properties of $(H^2_1)'$:
($\star$) For each $\mu \in A'_1$ with $\nu = \mu(\kappa_1)$, we have that for all $h^0 \in \mathrm{Col}(\kappa_0^+, <\nu)$, $h^1 \in \mathrm{Col}(\nu, \rho_1(\nu)^+)$, and $t \in \mathbb{P}_{E_0 \restriction \lambda_1(\nu)}$:

- EITHER $(t^\frown\langle f'_1 \oplus \mu, \lambda_1(\nu), h^0, h^1, (H^2_1)'(\nu)\rangle, a')$ decides b,
- OR there is no $g \leq f'_1$, $h \leq (H^2_1)'(\nu)$, and $x \leq a'$ such that $(t^\frown\langle g \oplus \mu, \lambda_1(\nu), h^0, h^1, h\rangle, x)$ decides b.

Fix $\mu \in A'_1$. Let $\dot{G}$ be a canonical name for a generic object for $\mathbb{P}_{E_0 \restriction \lambda_1(\mu)} \times \mathrm{Col}(\kappa_0^+, <\mu(\kappa_1)) \times \mathrm{Col}(\mu(\kappa_1), \rho_1(\mu)^+)$. Since the product of the collapses $\mathrm{Col}(\kappa_0^+, <\mu(\kappa_1)) \times \mathrm{Col}(\mu(\kappa_1), \rho(\mu)^+)$ is κ_0^+-closed, we apply the induction hypothesis for the one extender case to the condition

$$((p_0)_\mu, (H^0_1)(\mu \restriction \mathrm{dom}(f_1)), (H^1_1)(\mu \restriction \mathrm{dom}(f_1)).$$

Let $t(\mu) \leq^* (p_0)_\mu$, $h^0(\mu) \leq (H^0_1)(\mu \restriction \mathrm{dom}(f_1))$, and $h^1(\mu) \leq (H^1_1)(\mu \restriction \mathrm{dom}(f_1))$ such that

$$\begin{gathered}(t(\mu), h^0(\mu), h^1(\mu)) \text{ decides if } \exists (t, (h^*)^0, (h^*)^1) \in \dot{G},\\ (t^\frown\langle f'_1, \lambda_1(\mu), (h^*)^0, (h^*)^1, (H^2_1)'(\mu(\kappa_1))\rangle, a') \parallel b.\end{gathered}$$

By strengthening $t(\mu)$ (under $\leq^*$), $h^0(\mu)$ and $h^1(\mu)$ further, we can assume that for each $\mu \in A'_1$, $(t(\mu), h^0(\mu), h^1(\mu))$ satisfies exactly one out of the following three mutually exclusive properties:

(P1) $(t(\mu), h^0(\mu), h^1(\mu)) \Vdash \exists (t, (h^0)^*), (h^1)^*) \in \dot{G}$,
$(t^\frown\langle f'_1 \oplus \mu, \lambda_1(\mu), (h^0)^*, (h^1)^*, (H^2_1)'(\mu(\kappa_1))\rangle, a') \Vdash b$.
(P2) $(t(\mu), h^0(\mu), h^1(\mu)) \Vdash \exists (t, (h^0)^*), (h^1)^*) \in \dot{G}$,
$(t^\frown\langle f'_1 \oplus \mu, \lambda_1(\mu), (h^0)^*, (h^1)^*, (H^2_1)'(\mu(\kappa_1))\rangle, a') \Vdash \neg b$.
(P3) $(t(\mu), h^0(\mu), h^1(\mu)) \Vdash \nexists (t, (h^0)^*), (h^1)^*) \in \dot{G}$,
$(t^\frown\langle f'_1 \oplus \mu, \lambda_1(\mu), (h^0)^*, (h^1)^*, (H^2_1)'(\mu(\kappa_1))\rangle, a') \parallel b$.

Shrink A_1' so that every $\mu \in A_1'$ falls into the same case above. Use Lemma 6 to find $q \leq^* p_0^\frown\langle f_1', A_1', H_1^0, H_1^1, (H_1^2)'\rangle$ such that for each $\tau \in A_1^q$ with $\mu = \tau \restriction d_1'$, we have $f_0^{q+\tau} = f_0^{t(\mu)}$, $(q+\tau) \restriction 1 \leq^* t(\mu)$, $(H_1^0)^q(\tau) = h^0(\mu)$, $(H_1^1)^q(\tau) = h^1(\mu)$, $(H_1^2)^q(\mu(\kappa_1)) = (H_1^2)'(\mu(\kappa_1))$, and $f_1^q \leq f_1'$.

Claim 7.3. *(q, a') decides b*

Proof: Let $(r, x) \leq (q, a')$ be such that $(r, x) \parallel b$. Without loss of generality $(r, x) \Vdash b$. Assume also $1 \in \mathrm{supp}(r)$. Then r is an extension of a 1-step extension of q by $\mu' \in A_1^q$ for some μ'.

Recall $d_1' = \mathrm{dom}(f_1')$. Set $\mu = \mu' \restriction d_1'$. We see that $f_1^r \restriction d_1' \leq f_1' \oplus \mu$, $(h_1^2)^r \leq (H_1^2)'(\mu(\kappa_1))$, and $x \leq a'$. By $(\star)$, we have $(r_0^\frown\langle f_1' \oplus \mu, \lambda_1(\mu), (h_1^0)^r, (h_1^1)^r, (H_1^2)'(\mu(\kappa_1))\rangle, a') \Vdash b$.

Since r is an extension of a one-step extension of q by μ', $r_0 \leq t(\mu)$, $(h_1^0)^r \leq h^0(\mu)$, and $(h_1^1)^r \leq h^1(\mu)$. By construction μ must fall into exactly one of the cases (P1), (P2), (P3) listed above: we claim that cases (P2) and (P3) are impossible. Suppose for contradiction that we are in case (P2), and force below $(r_0, (h_1^0)^r, (h_1^1)^r)$ to obtain a generic object G. Then $(r_0, (h_1^0)^r, (h_1^1)^r) \in G$ and $r_0^\frown\langle f_1' \oplus \mu, \lambda_1(\mu), (h_1^0)^r, (h_1^1)^r, (H_1^2)'(\mu(\kappa_1))\rangle, a') \Vdash b$, but $(r_0, (h_1^0)^r, (h_1^1)^r) \leq (t(\mu), h^0(\mu), h^1(\mu))$ and we are in case (P2) so that also G contains a triple $(t, (h^0)^*, (h^1)^*)$ with $(t^\frown\langle f_1' \oplus \mu, \lambda_1(\mu), (h^0)^*, (h^1)^*, (H_1^2)'(\mu(\kappa_1))\rangle, a') \Vdash \neg b$: this gives comparable conditions forcing b and $\neg b$, an immediate contradiction. Similarly we are not in case (P3), so $(t(\mu), h^0(\mu), h^1(\mu))$ must have the property (P1).

We claim already $r^\mu := (t(\mu)^\frown\langle f_1' \oplus \mu, \lambda_1(\mu), h^0(\mu), h^1(\mu), (H_1^2)'(\mu(\kappa_1))\rangle, a')$ forces b. Suppose not, find $(r', x') \leq r^\mu$ such that $(r', x') \Vdash \neg b$. Strengthen further, we assume $f_1^{r'} \restriction d_1' \leq f_1' \oplus \mu$, $(h_1^2)^{r'} \leq (H_1^2)'(\mu(\kappa_1))$, and $x' \leq a'$. By $(\star)$, we have

$$(\dagger) \qquad (r_0'{}^\frown\langle f_1' \oplus, \mu, \lambda_1(\mu), (h_1^0)^{r'}, (h_1^1)^{r'} (H_1^2)'(\mu(\kappa_1))\rangle, a') \Vdash \neg b.$$

Let G be a generic object for $\mathbb{P}_{E_0 \restriction \lambda_1(\mu)} \times \mathrm{Col}(\kappa_0^+, <\mu(\kappa_1)) \times \mathrm{Col}(\mu(\kappa_1), \rho_1(\mu)^+)$ containing $(r_0', (h_1^0)^{r'}, (h_1^1)^{r'})$. Since $(r_0', (h_1^0)^{r'}, (h_1^1)^{r'}) \leq (t(\mu), h^0(\mu), h^1(\mu))$, we have $(t(\mu), h^0(\mu), h^1(\mu)) \in G$. Then find $(t, (h^0)^*, (h^1)^*) \in G$ according to the property (P1), i.e.

$$(\dagger\dagger) \qquad (t^\frown\langle f_1' \oplus \mu, \lambda_1(\mu), (h^0)^*, (h^1)^*, (H^2)'(\mu(\kappa_1))\rangle, a') \Vdash b.$$

By the directedness of G, we may assume $t \leq r_0'$, $(h^0)^* \leq (h_1^0)^{r'}$, and $(h^1)^* \leq (h_1^1)^{r'}$. Thus, from $(\dagger)$, we have

$$(t^\frown\langle f_1' \oplus \mu, \lambda_1(\mu), (h^0)^*, (h^1)^*, (H_1^2)'(\mu(\kappa_1))\rangle, a') \Vdash \neg b,$$

which is a contradiction when comparing to $(\dagger\dagger)$.

With the way we shrank the measure one set A_1', we have that for every $\widetilde{\mu} \in A_1^q$,$(t(\widetilde{\mu} \restriction d_1'), h^0(\widetilde{\mu} \restriction d_1'), h^1(\widetilde{\mu} \restriction d_1'))$ has the property (P1). By the same proof, $(q + \widetilde{\mu}, a') \Vdash b$. Finally, we are going to show that $(q, a') \Vdash b$. If not, let $(q', x') \leq (q, a')$ such that $(q', x') \Vdash \neg b$, but we can extend further so that $1 \in \mathrm{supp}(q')$. Hence $(q', x') \leq (q + \widetilde{\mu}, a')$ for some $\widetilde{\mu} \in A_1^q$, but $(q + \widetilde{\mu}, a') \Vdash b$, a contradiction.

Hence, we have finished the proof for η a successor ordinal.

■

CASE 3: η is limit

The proofs for the limit cases are essentially the same, we may assume $\eta = \omega$. Suppose for simplicity that $p = \langle p_n : n < \omega \rangle$ is pure. Write

$$p_n = \langle f_n^p, A_n^p, (H_n^0)^p, (H_n^1)^p, (H_n^2)^p \rangle \text{ for each } n.$$

Recall $\mathbb{A}$ is $\overline{\kappa}_\omega^+$-closed. Let $a \in \mathbb{A}$. We will build inductively a $\leq^*$-sequence $\{(q^m, a^m) : m < \omega\}$ where $q^0 \leq^* p$, $a^0 \leq a$. We will show that (q, a'), the $\leq^*$-lower bound of the sequence $\{(q^m, a^m) : m < \omega\}$, will decide b. We divide our constructions into two parts: q^0, a^0, and q^m, a^m for positive m.

Construction of q^0: Define

$$\mathbb{Q}_0 = \{(f, \vec{r}) \in (\mathcal{A}(\kappa_0, \lambda)/f_0^p, \leq) \times ((\mathbb{P}/p) \setminus 1, \leq^*) : \text{dom}(f) \text{ is a subset of dom}(f_1^r)\ \}.$$

Here note that f_1^r is the first Cohen part of $\vec{r}$. Observe that $\mathbb{Q}_0$ is κ_0^+-closed. Let θ be a sufficiently large regular cardinal. Let $N_0 \prec H_\theta$ be such that $|N_0| = \kappa_0$, ${}^{<\kappa_0}N_0 \subseteq N_0$, $p, a, \mathbb{Q}_0, \mathbb{P}, \mathbb{A} \in N_0$. Enumerate dense open sets in $(\mathbb{Q}_0 \times \mathbb{A}) \cap N_0$ as $\{D_i : i < \kappa_0\}$. Build a $\mathbb{Q}_0 \times \mathbb{A}$-decreasing sequence $\{(f'_{0,i}, \vec{r}_{0,i}, a_i) : i < \kappa_0\}$, each in N_0, where $f'_{0,0} = f_0^p$, $\vec{r}_{0,0} = p \setminus 1$, $a_0 = a$, and for all i, $(f'_{0,i+1}, \vec{r}_{0,i+1}, a_{i+1}) \in D_i$. Let $(f'_0, \vec{r}_0, a^0)$ be a lower bound of the sequence $\{(f'_{0,i}, \vec{r}_i, a_i) : i < \kappa_0\}$. Set $d'_0 = \text{dom}(f'_0)$, which is just $N_0 \cap \lambda$. Let $A'_0 \in E_0(d'_0)$ be such that $A'_0 \subseteq A_0(d'_0)$ and A'_0 projects down to a subset of A_0^p. As usual, $A'_0 \subseteq N_0$.

Let $\nu \in A'_0(\kappa_0)$. By Lemma 2, let $\{\mu_j, h_j^0, h_j^1\}_{j < \lambda_0(\nu)}$ be an enumeration in N_0 of the triples μ, h^0, h^1 where $\mu \in A'_0$, $\mu(\kappa_0) = \nu$, $h^0 \in \text{Col}(\omega_1, < \nu)$, and $h^1 \in \text{Col}(\nu, \rho_0(\nu)^+)$, respectively.

Define D_ν as the collection of $(g, \vec{r}, x) \in \mathbb{Q}_0 \times \mathbb{A}$ such that there is an $h \in \text{Col}(\lambda_0(\nu)^+, < \kappa_0)$ with $h \leq (H_0^2)^p(\nu)$ meeting the following requirements:

(1) For all $j < \lambda_0(\nu)$, $\text{dom}(\mu_j) \subset \text{dom}(g)$.
(2) For all $j < \lambda_0(\nu)$,
  - EITHER $(\langle g \oplus \mu_j, \lambda_0(\nu), h_j^0, h_j^1, h \rangle^\frown \vec{r}, x)$ decides b,
  - OR there is no $g' \leq g$, $h' \leq h$, $\vec{r}' \leq^* \vec{r}$, and $x' \leq x$ such that $\text{dom}(g')$ is a subset of the Cohen part of $\vec{r}'$, and the condition $(\langle g' \oplus \mu_j, \lambda_0(\nu), h_j^0, h_j^1, h' \rangle^\frown \vec{r}', x')$ decides b.

Similar to the one-extender case, the set D_ν is a dense, open subset of $\mathbb{Q}_0 \times \mathbb{A}$, and $D_\nu \in N_0$. Hence we have $(f'_0, \vec{r}_0, a^0) \in D_\nu$ with a witness $h \in \text{Col}(\lambda_0(\nu)^+, < \kappa_0)$. Define $(H_0^2)'(\nu) = h$.

We record some properties of $(H_0^2)'$:

$(\star_0)$: For each $\mu \in A'_0$ with $\nu = \mu(\kappa_0)$, we have that for all $h^0 \in \text{Col}(\omega_1, < \nu)$ and $h^1 \in \text{Col}(\nu, \rho_0(\nu)^+)$,

- EITHER $(\langle f'_0 \oplus \mu, \lambda_0(\nu), h^0, h^1, (H_0^2)'(\nu) \rangle^\frown \vec{r}_0, a^0 \rangle$ decides b,
- OR there is no $g \leq f'_0, h \leq (H_0^2)'(\nu)$, $\vec{r} \leq^* \vec{r}_0$ such that $\text{dom}(g)$ is a subset of the Cohen part of $\vec{r}$, and $x \leq a^0$ such that $(\langle g \oplus \mu, \lambda_0(\nu), h^0, h^1, h \rangle^\frown \vec{r}, x)$ decides b.

Now for each $\mu \in A'_0$, find $h^0(\mu) \leq H^0(\mu \restriction d_0^p)$, $h^1(\mu) \leq H^1(\mu \restriction d_0^p)$, (if they exist), such that

$$(\langle f'_0 \oplus \mu, \lambda_0(\mu), h^0(\mu), h^1(\mu), (H_0^2)'(\mu(\kappa_0)) \rangle^\frown \vec{r}_0, a^0) \text{ decides b,}$$

otherwise, $h^0(\mu) = H_0^0(\mu \restriction d_0^p)$, $h^1(\mu) = H_0^1(\mu \restriction d_0^p)$.

Now set

$$B_0 = \{\mu \in A_0' : (\langle f_0' \oplus \mu, \lambda_0(\mu), h^0(\mu), h^1(\mu), (H_0^2)'(\mu(\kappa_0))\rangle^\frown \vec{r}_0, a^0) \Vdash b\}$$
$$B_1 = \{\mu \in A_0' : (\langle f_0' \oplus \mu, \lambda_0(\mu), h^0(\mu), h^1(\mu), (H_0^2)'(\mu(\kappa_0))\rangle^\frown \vec{r}_0, a^0) \Vdash \neg b\}$$
$$B_2 = \{\mu \in A_0' : (\langle f_0' \oplus \mu, \lambda_0(\mu), h^0(\mu), h^1(\mu), (H_0^2)'(\mu(\kappa_0))\rangle^\frown \vec{r}_0, a^0) \nparallel b\}$$

We see that $A_0' = B_0 \sqcup B_1 \sqcup B_2$. As in the case of one extender, choose $i_0 \in \{0,1,2\}$ such that $B_{i_0} \in E_0(d_0')$. Define $f_0'' = f_0'$. Let $A_0'' = B_{i_0}$. For $l = 0,1$, set $\mathrm{dom}(H_0^l)'' = A_0''$, and $(H_0^l)''(\mu) = h^l(\mu)$. Let $(H^2)'' = (H^2)' \restriction A_0''(\kappa_0)$.

Finally, extend $\vec{r}_0$ to $\vec{r}_0{}'$ in a natural way: make sure $\mathrm{dom}(f'')$ is a subset of the Cohen part in $\vec{r}_0'$, $f_m^{\vec{r}_0'}(\xi) = f_m^{\vec{r}_0}(\xi)$ if ξ is in $\mathrm{dom}(f_m^{\vec{r}_0})$, otherwise $f_m^{\vec{r}_0'}(\xi) = 0$. All measure one sets project down to the corresponding measure one sets in $\vec{r}_0$, meaning $A_m^{\vec{r}_0'} \restriction \mathrm{dom}(f_m^{\vec{r}_0}) \subseteq A_m^{\vec{r}_0}$. If μ belongs to $A_m^{\vec{r}_0'}$, define $(H_m^0)^{\vec{r}_0'}(\mu) = (H_m^0)^{\vec{r}_0}(\mu \restriction \mathrm{dom}(f_m^{\vec{r}_0}))$ $(H_m^1)^{\vec{r}_0'}(\mu) = (H_m^1)^{\vec{r}_0}(\mu \restriction \mathrm{dom}(f_m^{\vec{r}_0}))$, and $H_m^2($
$nu) = (H_m^2)^{\vec{r}_0'}(\nu) = (H_m^2)^{\vec{r}_0}(\nu)$, see more details at (A4) of Lemma 6. Set $q^0 = \langle f'', A_0'', (H_0^0)'', (H_0^1)'', (H_0^2)''\rangle^\frown \vec{r}_0'$. This finishes the construction of q^0 and a^0.

Construction of q^{m+1}: Suppose q^m and a^m are constructed. Define

$$\mathbb{Q}_{m+1} = \{(f,\vec{r}) \in (\mathcal{A}(\kappa_{m+1},\lambda)/f_{m+1}^{q^m}), \leq) \times (\mathbb{P}/q^m \setminus (m+2), \leq^*) : \\ \mathrm{dom}(f) \text{ is a subset of } \mathrm{dom}(f_{m+2}^r)\}.$$

Here note that f_{m+2}^r is the first Cohen part of $\vec{r}$. $\mathbb{Q}_{m+1}$ is κ_{m+1}^+-closed. Let θ be a sufficiently large regular cardinal. Let $N_{m+1} \prec H_\theta$ be such that $|N_{m+1}| = \kappa_{m+1}$, ${}^{<\kappa_{m+1}}N_{m+1} \subseteq N_{m+1}$, $q^m, a^m, \mathbb{P}, \mathbb{A} \in N_{m+1}$ (so $\mathbb{Q}_{m+1} \in N_{m+1}$).

Enumerate the dense open sets in $(\mathbb{Q}_{m+1} \times \mathbb{A}) \cap N_{m+1}$ as $\{D_i : i < \kappa_{m+1}\}$. Build a $\mathbb{Q}_{m+1} \times \mathbb{A}$-decreasing sequence $\{(f_{m+1,i}', \vec{r}_{m+1,i}, a_i^m) : i < \kappa_{m+1}\}$, each in N_{m+1}, where $f_{m+1,0}' = f_{m+1}^{q^m}$, $\vec{r}_{m+1,0} = q^m \setminus (m+2)$, $a_0^m = a^m$, and for all i, $(f_{m+1,i+1}', \vec{r}_{m+1,i+1}, a_{i+1}^m) \in D_i$. Let $(f_{m+1}', \vec{r}_{m+1}, a^{m+1})$ be the greatest lower bound of the sequence $\{(f_{m+1,i}', \vec{r}_{m+1,i}, a_i^m) : i < \kappa_{m+1}\}$. $d_{m+1}' = \mathrm{dom}(f_{m+1}')$, which is $N_{m+1} \cap \lambda$. Let $A_{m+1}' \in E_{m+1}(d_{m+1}')$ be such that $A_{m+1}' \subseteq A_{m+1}(d_{m+1}')$, and A_{m+1}' projects down to a subset of $A_{m+1}^{q^m}$. As usual, $A_{m+1}' \subseteq N_{m+1}$.

Let $\nu \in A_{m+1}'(\kappa_{m+1})$. By Lemma 2, let $\{t_j, \mu_j, h_j^0, h_j^1\}_{j<\lambda_{m+1}(\nu)}$ be an enumeration in N_{m+1} of the quadruples t, μ, h^0, h^1 where $t \in \mathbb{P}_{\langle E_n \restriction \lambda_{m+1}(\nu) : n \leq m\rangle}$, $\mu \in A_{m+1}'$ with $\mu(\kappa_{m+1}) = \nu$, $h^0 \in \mathrm{Col}(\kappa_m^+, <\nu)$, and $h^1 \in \mathrm{Col}(\nu, \rho_{m+1}(\nu)^+)$, respectively.

Define D_ν as the collection of $(g, \vec{r}, x) \in \mathbb{Q}_{m+1} \times \mathbb{A}$ such that there is an $h \in \mathrm{Col}(\lambda_{m+1}(\nu)^+, <\kappa_{m+1})$ with $h \leq (H_{m+1}^2)^{q^m}(\nu)$ meeting the following requirements:

(1) For all $j < \lambda_{m+1}(\nu)$, $\mathrm{dom}(\mu_j) \subset \mathrm{dom}(g)$.
(2) For all $j < \lambda_{m+1}(\nu)$,
  - EITHER $(t^\frown\langle g \oplus \mu_j, \lambda_{m+1}(\nu), h_j^0, h_j^1, h\rangle^\frown \vec{r}, x)$ decides b,
  - OR there is no $g' \leq g$, $h' \leq h$, $\vec{r}' \leq^* \vec{r}$, and $x' \leq x$ such that $\mathrm{dom}(g')$ is a subset of the Cohen part of $\vec{r}'$, and the condition $(t^\frown\langle g' \oplus \mu_j, \lambda_{m+1}(\nu), h_j^0, h_j^1, h'\rangle^\frown \vec{r}', x')$ decides b.

Similar to the other cases, the set D_ν is a dense, open subset of $\mathbb{Q}_{m+1} \times \mathbb{A}$, and $D_\nu \in N_{m+1}$. Hence we have $(f_{m+1}', \vec{r}_{m+1}, a^{m+1}) \in D_\nu$ with a witness $h \in \mathrm{Col}(\lambda_{m+1}(\nu)^+, <\kappa_{m+1})$. Define $(H_{m+1}^2)'(\nu) = h$.

We record some properties of $(H_{m+1}^2)'$:

($\star_{m+1}$) For each $\mu \in A'_{m+1}$ with $\nu = \mu(\kappa_{m+1})$, we have that for all $t \in \mathbb{P}_{\langle E_n \restriction \lambda_{m+1}(\nu):n \leq m\rangle}$, $h^0 \in \mathrm{Col}(\kappa_m^+, <\nu)$, and $h^1 \in \mathrm{Col}(\nu, \rho_{m+1}(\nu)^+)$,

- EITHER $(t^\frown\langle f'_{m+1} \oplus \mu, \lambda_{m+1}(\nu), h^0, h^1, (H^2_{m+1})'(\mu)\rangle^\frown \vec{r}_{m+1}, a^{m+1})$ decides b,
- OR there is no $g \leq f'_{m+1}, h \leq (H^2_{m+1})'(\nu)$, $\vec{r} \leq^* \vec{r}_{m+1}$, and $x \leq a^{m+1}$ such that $(t^\frown\langle g \oplus \mu, \lambda_{m+1}(\nu), h^0, h^1, h\rangle^\frown \vec{r}, x)$ decides b.

Fix $\mu \in A'_1$. Let $\dot{G}$ be a canonical name for a generic object for $\mathbb{P}_{\langle E_n \restriction \lambda_{m+1}(\mu):n \leq m\rangle} \times \mathrm{Col}(\kappa_m^+, <\mu(\kappa_{m+1})) \times \mathrm{Col}(\mu(\kappa_{m+1}), \rho_{m+1}(\mu)^+)$. Let $(\vec{q}^{\,m}_{\leq m})_\mu = \langle((q^m)_n)_\mu : n \leq m)$, which is just the first $m+1$ coordinates of q^m squished by μ. Since the product of collapses $\mathrm{Col}(\kappa_m^+, <\mu(\kappa_{m+1})) \times \mathrm{Col}(\mu(\kappa_{m+1}), \rho_{m+1}(\mu)^+)$ is κ_m^+-closed, we apply the induction hypothesis for the Prikry property for the m-extender case to the condition

$$((\vec{q}^{\,m}_{\leq m})_\mu, (H^0_{m+1})^{q^m}(\mu \restriction \mathrm{dom}(f^{q^m}_{m+1})), (H^1_{m+1})^{q^m}(\mu \restriction \mathrm{dom}(f^{q^m}_{m+1}))).$$

Let $t(\mu) \leq^* (\vec{q}^{\,m}_{\leq m})_\mu$, $h^0(\mu) \leq (H^0_{m+1})^{q^m}(\mu \restriction \mathrm{dom}(f^{q^m}_{m+1}))$, and $h^1(\mu) \leq (H^1_{m+1})^{q^m}(\mu \restriction \mathrm{dom}(f^{q^m}_{m+1}))$ be such that

$$(t(\mu), h^0(\mu), h^1(\mu)) \text{ decides if } \exists(t, (h^0)^*, (h^1)^*) \in \dot{G},$$
$$(t^\frown\langle f'_{m+1}, \lambda_{m+1}(\mu), (h^0)^*, (h^1)^*, (H^2_{m+1})'(\mu(\kappa_{m+1}))\rangle^\frown \vec{r}_{m+1}, a^{m+1}) \parallel b.$$

By strengthening $t(\mu)$ (under $\leq^*$), $h^0(\mu)$, and $h^1(\mu)$ further, we can assume that for each $\mu \in A'_{m+1}$, $(t(\mu), h^0(\mu), h^1(\mu))$ satisfies exactly one out of the following three mutually exclusive properties:

(Q1) $(t(\mu), h^0(\mu), h^1(\mu)) \Vdash \exists(t, (h^0)^*, (h^1)^*) \in \dot{G}$,
$(t^\frown\langle f'_{m+1} \oplus \mu, \lambda_{m+1}(\mu), (h^0)^*, (h^1)^*, (H^2_{m+1})'(\mu(\kappa_{m+1}))\rangle^\frown \vec{r}_{m+1}, a^{m+1}) \Vdash b$.

(Q2) $(t(\mu), h^0(\mu), h^1(\mu)) \Vdash \exists(t, (h^0)^*, (h^1)^*) \in \dot{G}$,
$(t^\frown\langle f'_{m+1} \oplus \mu, \lambda_{m+1}(\mu), (h^0)^*, (h^1)^*, (H^2_{m+1})'(\mu(\kappa_{m+1}))\rangle^\frown \vec{r}_{m+1}, a^{m+1}) \Vdash \neg b$.

(Q3) $(t(\mu), h^0(\mu), h^1(\mu)) \Vdash \nexists(t, (h^0)^*, (h^1)^*) \in \dot{G}$,
$(t^\frown\langle f'_{m+1} \oplus \mu, \lambda_{m+1}(\mu), (h^0)^*, (h^1)^*, (H^2_{m+1})'(\mu(\kappa_{m+1}))\rangle^\frown \vec{r}_{m+1}, a^{m+1}) \parallel b$.

Shrink A'_{m+1} so that every $\mu \in A'_{m+1}$ falls into the same case above.

Use Lemma 6 to find a condition q^{m+1} such that

$$q^{m+1} \leq^* \langle q^m_n : n \leq m\rangle^\frown\langle f'_{m+1}, A'_{m+1}, (H^0_{m+1})^{q^m}(H^1_{m+1})^{q^m}, (H^2_{m+1})'\rangle^\frown \vec{r}_{m+1}$$

satisfying all properties stated in Lemma 6, where $t(\mu), h^0(\mu), h^1(\mu)$ are as described right before introducing the properties (Q1),(Q2), and (Q3), which means that if $\tau \in A^{q^{m+1}}_{m+1}$ and $\mu = \tau \restriction d'_{m+1}$, and $\nu = \mu(\kappa_{m+1})$, then for $k \leq m$, $f_k^{q^{m+1}+\tau} = f_k^{t(\mu)}$, $(q^{m+1}+\tau) \restriction (m+1) \leq^* t(\mu)$, for $l = 0, 1$, $(H^l_{m+1})^{q^{m+1}}(\tau) = h^0(\mu)$, $(H^2_{m+1})^{q^{m+1}}(\nu) = (H^2_{m+1})'(\nu)$, and $f^{q^{m+1}}_{m+1} \leq f'_{m+1}$. We have finished the construction of q^{m+1}, and a^{m+1}.

Recall we take (q, a') as a lower bound of the sequence $\{(q^m, a^m) : m < \omega\}$.

Claim 7.4. *There is a direct extension of (q, a') which decides b.*

If we can find a direct extension of (q, a') deciding b, then the proof is done. Suppose this is not the case. Let $(r, x) \leq (q, a')$ be such that $(r, x) \Vdash b$ and r is not pure. Suppose r is not pure, which has the least possible value of $\max(\mathrm{supp}(r))$. Our proof is divided into 2 cases: $\max(\mathrm{supp}(r)) = 0$, and $\max(\mathrm{supp}(r)) > 0$.

CASE A $\max(\mathrm{supp}(r)) = 0$.

This means $\mathrm{supp}(r) = 1$. Hence $r \leq^* q + \mu'$ for some $\mu' \in A^q_0$.

We refer to all the notations in the construction of q^0. Set $\mu = \mu' \restriction d'_0$, and $\nu = \mu(\kappa_0)$. We have $r \setminus 1 \leq^* q \setminus 1 \leq^* \vec{r}'_0 \leq^* \vec{r}_0$. Similarly, we can trace back to see that $f^r \leq f'_0 \oplus \mu$. Also $(h^2_0)^r \leq (H^0_2)'(\nu)$ and $x \leq a' \leq a^0$. By the property $(\star_0)$, $(\langle f'_0 \oplus \mu, \lambda_0(\nu), (h^0)^r, (h^1)^r, (H^2_0)'(\nu)\rangle^\frown \vec{r}_0, a^0) \Vdash b$. This means $B_{i_0} = B_0$. Thus for every $\widetilde{\mu} \in A^q_0$, we have

$$(\dagger) \qquad (f'_0 \oplus (\widetilde{\mu} \restriction d'_0), \lambda_0(\widetilde{\mu}), h^0_0(\widetilde{\mu}), h^1_0(\widetilde{\mu}, (H^2_0)'(\widetilde{\mu}(\kappa_0))^\frown \vec{r}_0, a^0) \Vdash b.$$

One can check that $(q + \widetilde{\mu}, a')$ is stronger than the condition in $(\dagger)$. Therefore $(q + \widetilde{\mu}, a') \Vdash b$. Since every extension of (q, a') is compatible with $(q + \widetilde{\mu}, a')$ for some $\widetilde{\mu}$, $(q, a') \Vdash b$.

CASE B $\max(\mathrm{supp}(r)) = m + 1$ for some $m < \omega$.

We will refer to all the notations in the construction of q^{m+1}. Suppose part of the extension used some $\mu' \in A^q_{m+1}$. Set $\mu = \mu' \restriction d'_{m+1}$ and $\nu = \mu(\kappa_{m+1})$. By tracing back, and lemma 5, the following properties hold for r and x:

1. $r \restriction (m + 1) \leq t(\mu)$.
2. $f^r_{m+1} \leq f'_{m+1} \oplus \mu$.
3. For $l = 0, 1$, $(h^l_{m+1})^r \leq h^l(\mu)$.
4. $(h^2_{m+1})^r \leq (H^2_{m+1})'(\nu)$.
5. $r \setminus (m + 2) \leq^* \vec{r}_{m+1}$.
6. $x \leq a^{m+1}$.

By (2),(4),(5), and (6), and by $(\star_{m+1})$, we have

$$(r \restriction (m+1)^\frown \langle f'_{m+1} \oplus \mu, \lambda_{m+1}(\mu), (h^0_{m+1})^r, (h^1_{m+1})^r, (H^2_{m+1})'(\nu)\rangle^\frown \vec{r}_{m+1}, a^{m+1}) \Vdash b.$$

By (1) and (3), $(t(\mu), h^0(\mu), h^1(\mu))$ has the property (Q1). Set

$$r^\mu := t(\mu)^\frown \langle f'_{m+1} \oplus \mu, \lambda_{m+1}(\nu), h^0(\mu), h^1(\mu), (H^2_{m+1})'(\mu(\kappa_{m+1}))\rangle^\frown \vec{r}_{m+1}$$

We claim that already $(r^\mu, a^{m+1}) \Vdash b$. Suppose not. Find $r' \leq r, x' \leq a^{m+1}$ such that $(r', x') \Vdash \neg b$. let G be a generic extension of $\mathbb{P}_{\langle E_n \restriction \lambda_{m+1}(\nu) : n \leq m\rangle}$ containing $(r' \restriction (m+1), (h^0_{m+1})^{r'}, (h^1_{m+1})^{r'})$, hence containing $(t(\mu), h^0(\mu), h^1(\mu))$. By the property (Q1), we can find $(t, (h^0)^*, (h^1)^*) \in G$ below $(r' \restriction (m+1), (h^0_{m+1})^{r'}, (h^1_{m+1})^{r'})$ such that

$$(\dagger\dagger) \quad (t^\frown \langle f'_{m+1} \oplus \mu, \lambda_{m+1}(\mu), (h^0)^*, (h^1)^*, (H^2_{m+1})'(\mu(\kappa_{m+1}))\rangle^\frown \vec{r}_{m+1}, a^{m+1}) \Vdash b.$$

Since

1. $t \leq r' \restriction (m + 1)$.
2. $f^{r'}_{m+1} \leq f'_{m+1} \oplus \mu$.
3. for $l = 0, 1$, $(h^*)^l \leq (h^l_{m+1})^{r'}$,
4. $(h^2_{m+1})^{r'} \leq (H^2_{m+1})'(\mu(\kappa_{m+1}))$.
5. $r' \setminus (m + 2) \leq \vec{r}_{m+1}$, and
6. $x' \leq a^{m+1}$.

combining the fact that $(r', x') \Vdash \neg b$, and $(\dagger\dagger)$, we have

$$(t^\frown \langle f^{r'}_{m+1}, \lambda_{m+1}(\nu), (h^0)^*, (h^1)^*, (h^2_{m+1})^{r'}\rangle^\frown r' \setminus (m + 2), x') \Vdash b, \neg b.$$

which is a contradiction.

To show that (q, a') forces b, note that every extension of (q, a') is compatible with $(q+\widetilde{\mu}, a')$ for some $\widetilde{\mu}' \in A^{q^m}_{m+1}$. Hence it is enough to show that every extension of (q, a') by $\widetilde{\mu}' \in A^{q^m}_{m+1}$ forces b.

Let $\widetilde{\mu}' \in A^{q^m}_{m+1}$. Let $\widetilde{\mu} = \widetilde{\mu}' \restriction d'_{m+1}$ and $\widetilde{\nu} = \widetilde{\mu}(\kappa_{m+1})$. By the way we shrank A'_{m+1}, $(t(\widetilde{\mu}), h^0(\widetilde{\mu}), h^1(\widetilde{\mu}))$ has the property (Q1). Similar proof as above shows

$$(t(\widetilde{\mu})^\frown \langle f'_{m+1} \oplus \widetilde{\mu}, \lambda_{m+1}(\widetilde{\nu}), h^0(\widetilde{\mu}), h^1(\widetilde{\mu}), (H^2_{m+1})'(\widetilde{\nu})\rangle^\frown \vec{r}_{m+1}, a^{m+1}) \Vdash b.$$

Hence already $(q + \widetilde{\mu}', a') \Vdash b$. This completes the proof of the Prikry property. □

A similar proof shows the following statement, which is known as the "strong Prikry property".

Lemma 7. *Let $p \in \mathbb{P}$, and D be a dense open subset of $\mathbb{P}$. Then there is $p' \leq^* p$, and a finite set $I \subseteq \eta$ (can be empty) such that $I \cap \operatorname{supp}(p') = \emptyset$, and for each $\vec{\mu} \in \prod_{\alpha \in I} A^{p'}_\alpha$, each μ_i addable, $p' + \vec{\mu} \in D$.*

Proof. (Sketch) As the proof of the Prikry property, we induct on the a stronger statement: by induction on η, if $\mathbb{A}$ is $\overline{\kappa}^+_\eta$- closed, then for each $(p, a) \in \mathbb{P} \times \mathbb{A}$, and a dense open set $D \subseteq \mathbb{P} \times \mathbb{A}$, there is a condition $(p', a') \leq^* (p, a)$ and a finite set $I \subseteq \eta$ (can be empty) such that $I \cap \operatorname{supp}(p) = \emptyset$, and for each $\vec{\mu} \in \prod_{\alpha \in I} A^{p'}_\alpha$, each μ_i is addable, and $(p' + \vec{\mu}, a') \in D$. We assume for simplicity that p is pure. The elements of the proof for the case $\eta = \omega$ contain all the elements from the other cases. We will show only the case $\eta = \omega$. The proof has the same style as the proof of the Prikry property, we assume $\mathbb{A}$ is trivial, and remove $\mathbb{A}$ from the proof to make the proof more readable.

We only emphasise the key different ingredients from the proof of the Prikry property. For more details, look at the proof of the Prikry property.

Assume $p = \langle\langle f^p_n, A^p_n, (H^0_n)^p, (H^1_n)^p, (H^2_n)^p\rangle\rangle$ is a pure condition. We will build a $\leq^*$-decreasing sequence $\langle q_m : m < \omega\rangle$, it will then be routine to check that a lower bound of the sequence $\langle q_m : m < \omega\rangle$ will satisfy the condition for the strong Prikry property.

Construction of q^0: Let

$$\mathbb{Q}_0 = \{(f, \vec{r}) \in (\mathcal{A}(\kappa_0, \lambda)/f^p_0, \leq) \times ((\mathbb{P}/p) \setminus 1, \leq^*) : \operatorname{dom}(f) \text{ is a subset of } \operatorname{dom}(f^r_1)\ \}.$$

Note that f^r_1 is the first Cohen part of $\vec{r}$. Fix a sufficiently large regular cardinal θ. Build an elementary submodel $N_0 \prec H_\theta$ of size κ_0 closed under $< \kappa_0$-sequences containing enough information. Let $(f'_0, \vec{r}_0)$ be $(N_0, \mathbb{Q}_0)$-generic. Let $d'_0 = \operatorname{dom}(f'_0)$ and $A'_0 \in E_0(d'_0)$ be such that $A'_0 \subseteq A_0(d'_0)$ and A'_0 projects down to a subset of A^p_0.

Let $\nu \in A'_0(\kappa_0)$. As usual, let $\{\mu_j, h^0_j, h^1_j\}_{j<\lambda_0(\nu)}$ be an enumeration in N_0 of the triples μ, h^0, h^1 where $\mu \in A'_0$, $\mu(\kappa_0) = \nu$, $h^0 \in \operatorname{Col}(\omega_1, < \nu)$, and $h^1 \in \operatorname{Col}(\nu, \rho_0(\nu)^+)$, respectively.

Define D_ν as the collection of $(g, \vec{r}) \in \mathbb{Q}_0$ such that there is an $h \in \operatorname{Col}(\lambda_0(\nu)^+, < \kappa_0)$ with $h \leq (H^2_0)^p(\nu)$ meeting the following requirements:

1. For all $j < \lambda_0(\nu)$, $\operatorname{dom}(\mu_j) \subset \operatorname{dom}(g)$.
2. For all $j < \lambda_0(\nu)$,
    - EITHER $(\langle g \oplus \mu_j, \lambda_0(\nu), h^0_j, h^1_j, h\rangle^\frown \vec{r}) \in D$,

- OR there is no $g' \leq g$, $h' \leq h$, and $\vec{r}' \leq^* \vec{r}$ such that $\mathrm{dom}(g')$ is a subset of the Cohen part of $\vec{r}'$, and the condition $(\langle g' \oplus \mu_j, \lambda_0(\nu), h_j^0, h_j^1, h' \rangle^\frown \vec{r}') \in D$.

Then D_ν is a dense open subset of $\mathbb{Q}_0$ and is in N_0. Hence $(f_0', \vec{r}_0) \in D_\nu$, with a witness $h \in \mathrm{Col}(\lambda_0(\nu)^+, < \kappa_0)$. Define $(H_0^2)'(\nu) = h$.

Now fix $\mu \in A_0'$. Find $h^0(\mu) \leq (H_0^0)^p(\mu \restriction d_0^p)$ and $h^1(\mu) \leq H^1(\mu \restriction d_0^p)$ (if exist) such that there are $f \leq f_0'$, $h \leq (H_0^2)'(\mu(\kappa_0))$, and $\vec{r} \leq^* \vec{r}_0$,

$$\langle f' \oplus \mu, \lambda_0(\mu), h^0(\mu), h^1(\mu), h \rangle^\frown \vec{r} \in D.$$

Hence

$$\langle f_0' \oplus \mu, \lambda_0(\mu), h^0(\mu), h^1(\mu), (H_0^2)'(\mu(\kappa_0)) \rangle^\frown \vec{r}_0 \in D.$$

Otherwise, set $h^0(\mu) = (H_0^0)^p(\mu \restriction d_0^p)$, $h^1(\mu) = (H_0^2)^p(\mu \restriction d_0^p)$. Define $(H_0^0)'(\mu) = h^0(\mu)$ and $(H_0^1)'(\mu) = h^1(\mu)$. Shrink A_0' to A_0'' so that

(R1) EITHER for every $\mu \in A_0''$, there are $f' \leq f_0'$, $h \leq (H_0^2)'(\mu(\kappa_0))$, and $\vec{r} \leq^* \vec{r}_0$ such that $\langle f' \oplus \mu, \lambda_0(\mu), h^0(\mu), h^1(\mu), h \rangle^\frown \vec{r} \in D$,

(R2) OR for every $\mu \in A_0''$, for every $f' \leq f_0'$, $h \leq (H_0^2)'(\mu(\kappa_0))$, and $\vec{r} \leq^* \vec{r}_0$ such that $\langle f' \oplus \mu, \lambda_0(\mu), h^0(\mu), h^1(\mu), h \rangle^\frown \vec{r} \notin D$.

Finally, define $q_0 = \langle f_0', A_0'', (H_0^0)' \restriction A_0'', (H_0^1) \restriction A_0'', (H_0^2)' \restriction A_0''(\kappa_0) \rangle^\frown \vec{r}_0$. Here is the property of q^0: if q' is an extension of q^0 with $\mathrm{supp}(q') = \{0\}$ and $q' \in D$, then for every $\tau \in A_0^{q^0}$, $q^0 + \tau \in D$.

Construction of q^{m+1}: Suppose q^m is constructed. Define

$$\mathbb{Q}_{m+1} = \{(f, \vec{r}) \in (\mathcal{A}(\kappa_{m+1}, \lambda)/f_{m+1}^{q^m}), \leq) \times (\mathbb{P}/q^m \setminus (m+2), \leq^*) : \\ \mathrm{dom}(f) \text{ is a subset of } \mathrm{dom}(f_{m+2}^r)\}.$$

Here note that f_{m+2}^r is the first Cohen part of $\vec{r}$. Fix a sufficiently large regular cardinal θ. Build an elementary submodel $N_{m+1} \prec H_\theta$ of size κ_{m+1} closed under $< \kappa_{m+1}$-sequences and containing enough information. Let $(f_{m+1}', \vec{r}_{m+1})$ be $(N_{m+1}, \mathbb{Q}_{m+1})$-generic. Let $d_{m+1}' = \mathrm{dom}(f_{m+1}')$ and $A_{m+1}' \in E_{m+1}(d_{m+1}')$ be such that $A_{m+1}' \subseteq A_{m+1}(d_{m+1}')$ and A_{m+1}' projects down to a subset of $A_{m+1}^{q^m}$.

Let $\nu \in A_{m+1}'(\kappa_{m+1})$. As usual, let $\{t_j, \mu_j, h_j^0, h_j^1\}_{j < \lambda_{m+1}(\nu)}$ be an enumeration in N_{m+1} of the quadruples t, μ, h^0, h^1 where $t \in \mathbb{P}_{\langle E_n \restriction \lambda_{m+1}(\nu) : n \leq m \rangle}$, $\mu \in A_{m+1}'$ with $\mu(\kappa_{m+1}) = \nu$, $h^0 \in \mathrm{Col}(\kappa_m^+, < \nu)$, and $h^1 \in \mathrm{Col}(\nu, \rho_{m+1}(\nu)^+)$, respectively.

Define D_ν as the collection of $(g, \vec{r}) \in \mathbb{Q}_{m+1}$ such that there is an $h \in \mathrm{Col}(\lambda_{m+1}(\nu)^+, < \kappa_{m+1})$ with $h \leq (H_{m+1}^2)^{q^m}(\nu)$ meeting the following requirements:

(1) For all $j < \lambda_{m+1}(\nu)$, $\mathrm{dom}(\mu_j) \subset \mathrm{dom}(g)$.

(2) For all $j < \lambda_{m+1}(\nu)$,
- EITHER $t_j^\frown \langle g \oplus \mu_j, \lambda_{m+1}(\nu), h_j^0, h_j^1, h \rangle^\frown \vec{r} \in D$,
- OR there is no $g' \leq g$, $h' \leq h$, and $\vec{r}' \leq^* \vec{r}$, such that $\mathrm{dom}(g')$ is a subset of the Cohen part of $\vec{r}'$, and the condition $t_j^\frown \langle g' \oplus \mu_j, \lambda_{m+1}(\nu), h_j^0, h_j^1, h' \rangle^\frown \vec{r} \in D$.

The set D_ν is a dense, open subset of $\mathbb{Q}_{m+1}$, and $D_\nu \in N_{m+1}$. Hence we have $(f_{m+1}', \vec{r}_{m+1}) \in D_\nu$ with a witness $h \in \mathrm{Col}(\lambda_{m+1}(\nu)^+, < \kappa_{m+1})$. Define $(H_{m+1}^2)'(\nu) = h$.

Now outside of N_{m+1}, let E_μ be the collection of $(t, h^0, h^1) \in \mathbb{P}_{\langle E_n \restriction \lambda_{m+1}(\nu) : n \leq m \rangle} \times \mathrm{Col}(\kappa_m^+, < \nu) \times \mathrm{Col}(\nu, \rho_{m+1}(\nu)^+)$ such that EITHER

$$t^\frown\langle f'_{m+1} \oplus \mu, \lambda_{m+1}(\nu), h^0, h^1, (H^2_{m+1})'(\nu)\rangle^\frown \vec{r}_{m+1} \in D,$$

OR for all $g \leq f'_{m+1} \oplus \mu$, $h^2 \leq (H^2_{m+1})'(\mu)$, and $\vec{r'} \leq^* \vec{r}_{m+1}$,

$$t^\frown\langle f'_{m+1} \oplus \mu, \lambda_{m+1}(\nu), h^0, h^1, (H^2_{m+1})'(\nu)\rangle^\frown \vec{r}_{m+1} \notin D.$$

We can use the property of $D_{\mu(\kappa_{m+1})}$ to show that E_μ is open dense. Use the induction hypothesis to find $t(\mu) \leq^* (q^m \restriction (m+1))_\mu$, $h^0(\mu) \leq (H^0_{m+1})^{q^m}(\mu \restriction d^{q^m}_{m+1})$, and $h^1(\mu) \leq (H^1_{m+1})^{q^m}(\mu \restriction d^{q^m}_{m+1})$ with the least $[OR]^{<\omega}$-element $\vec{\alpha}^\mu = \alpha^\mu_0 < \cdots < \alpha^\mu_{k(\mu)-1}$, in the lexicographic order, such that for every $\vec{\tau} \in \prod_{\alpha \in \vec{\alpha}^\mu} A^{t(\mu)}_\alpha$, $h^0 \leq h^0(\mu)$, and $h^1 \leq h^1(\mu)$, we have $(t(\mu) + \vec{\tau}, h^0, h^1) \in E_\mu$.

For each $\mu \in A'_{m+1}$, define $F_\mu : A^{t(\mu)}_{\alpha^\mu_0} \times \dots A^{t(\mu)}_{\alpha^\mu_{k(\mu)-1}} \to 2$ by $F_\mu(\tau_0, \dots, \tau_{k(\mu)-1}) =$ 1 if and only if

$$(t(\mu) + \langle \tau_0, \dots, \tau_{k(\mu)-1}\rangle)^\frown$$
$$\langle f'_{m+1} \oplus \mu, \mu(\kappa_{m+1}), h^0(\mu), h^1(\mu), (H^2_{m+1})'(\mu(\kappa_{m+1}))\rangle^\frown \vec{r}_{m+1} \in D.$$

By Lemma 1, we have a measure one set $B^{t(\mu)}_{\alpha^\mu_i} \subseteq A^{t(\mu)}_{\alpha^\mu_i}$ for all $i < k(\mu)$ such that $F \restriction B^{t(\mu)}_{\alpha^\mu_0} \times \dots B^{t(\mu)}_{\alpha^\mu_{k(\mu)-1}}$ is constant. Shrink the measure one sets $A^{t(\mu)}_{\alpha^\mu_0}, \dots, A^{t(\mu)}_{\alpha^\mu_{k(\mu)-1}}$ inside $t(\mu)$ to $B^{t(\mu)}_{\alpha^\mu_0}, \dots, B^{t(\mu)}_{\alpha^\mu_{k(\mu)-1}}$, respectively, that F_μ is constant on the product of those measure one sets. Restrict the collapses based on the measure one sets we just shrank. Call the resulting condition $t^*(\mu)$. By the shrinking of measure one sets in $t(\mu)$, we arranged that

(S1) EITHER $t^*(\mu)+\langle \tau_0, \dots, \tau_{k(\mu)-1}\rangle^\frown\langle f'_{m+1}\oplus\mu, h^0(\mu), h^1(\mu), (H^2_{m+1})'(\mu(\kappa_{m+1})\rangle^\frown \vec{r}_{m+1} \in D$ for all $\vec{\tau}$,

(S2) OR for all τ, there are no $g \leq f'_{m+1} \oplus \mu$, $h^2 \leq (H^2_{m+1})'(\mu(\kappa_{m+1}))$, and $\vec{r'} \leq^* \vec{r}_{m+1}$ such that $t^*(\mu) + \langle \tau_0, \dots, \tau_{k(\mu)-1}\rangle \frown \langle g, h^0(\mu), h^1(\mu), h^2\rangle^\frown \vec{r'} \in D$.

Shrink A'_{m+1} further so that every μ satisfies (S1), or every μ satisfies (S2). If every μ satisfies (S1), shrink further so that there is a sequence $\vec{\alpha}_{m+1}$ such that for every $\mu \in A'_{m+1}$, $\vec{\alpha}^\mu = \vec{\alpha}_{m+1}$.

Observe that $t^*(\mu) \leq^* (q^m \restriction (m+1))_\mu$, $h^0(\mu) \leq (H^0_{m+1})^{q^m}(\mu \restriction d^{q^m}_{m+1})$, and $h^1(\mu) \leq (H^1_{m+1})^{q^m}(\mu \restriction d^{q^m}_{m+1})$. Use Lemma 6 to integrate these components together to form a condition q^{m+1}. Hence for $\tau \in A^{q^{m+1}}_{m+1}$ with $\mu = \tau \restriction d^{q^m}_{m+1}$, and $\nu = \mu(\kappa_{m+1}) = \tau(\kappa_{m+1})$, we have $(q^{m+1} + \tau) \restriction (m+1) \leq^* t^*(\mu)$, for $n \leq m$, $f^{q+\tau}_n = f^{t^*(\mu)}_n$, $(H^0_{m+1})^{q^{m+1}}(\tau) = h^0(\mu), (H^1_{m+1})^{q^{m+1}}(\tau) = h^1(\mu)$, and $(H^2_{m+1})^{q^{m+1}}(\nu) = h^2(\nu)$. This completes the construction of q^{m+1}. Here is what we have: if q' is an extension of q^{m+1} such that $\text{supp}(q')$ is the least in the lexicographic order in $[OR]^{<\omega}$, $\max(\text{supp}(q')) = m+1$, and $q' \in D$, then every extension q'' of q^{m+1} with $\text{supp}(q'') = \text{supp}(q')$ is in D. Now we have $q \leq^* q^m$ for all m.

Claim 7.5. *q satisfies the strong Prikry property.*

Proof: (sketch) Let $q' \leq q$ with $q' \in D$. Assume q' is not pure with the least $\text{supp}(q')$ in the lexicographic order in $[OR]^{<\omega}$. Enumerate $\text{supp}(q')$ in increasing order as $\alpha_0 < \cdots < \alpha_{k-1}$. If $\alpha_{k-1} = 0$, then the proof is easy. Assume $\alpha_{k-1} = m+1$. Using the notations from the construction of q^{m+1}, we have that for every $\tau \in A^{q'}_{m+1}$,

$\tau \restriction d'_{m+1}$ satisfies the property (S1), and $\vec{\alpha}_{m+1} = \langle \alpha_0, \ldots, \alpha_{k-1} \rangle$. By the way we shrank A'_{m+1}, for every $\vec{\tau} \in \prod_{\alpha \in \vec{\alpha}_{m+1}} A^q_\alpha$, we have $q + \vec{\tau} \in D$. ■

□

5. CARDINAL PRESERVATION

This section is dedicated to showing some cardinal preservation results. We analyze the generic extension. From now, assume the length of the extender sequence is limit η. Also assume $\rho = \bar{\kappa}_\eta$ and $\overline{\kappa}_0 = \max\{\omega, \eta\}$. Let G be $\mathbb{P}$-generic over V.

Lemma 8. *For each $\alpha < \eta$, the cardinal κ_α is preserved in $V[G]$, $(2^{\kappa_\alpha})^{V[G]} \leq \kappa_{\alpha+1}$. As a consequence, for any limit ordinal $\beta \leq \eta$, $\overline{\kappa}_\beta$ is a singular strong limit cardinal in $V[G]$, and $\overline{\kappa}_\beta = \aleph_\beta^{V[G]}$.*

Proof. Let $\alpha < \eta$. By a density argument, we find $p \in G$ with $\alpha, \alpha+1 \in \operatorname{supp}(p)$. Then G/p is factored into $G_0 := G/p \restriction \mathbb{P}_{\langle E_\gamma \restriction \lambda_\alpha : \gamma < \alpha \rangle}$, $G_1 := G/p \restriction \{\alpha\}$ and $G_2 := G/p \restriction (\mathbb{P} \setminus (\alpha+1))$, for some $\lambda_\alpha < \kappa_\alpha$. G_0 comes from a forcing that has λ_α-c.c. Thus κ_α is preserved in the extension by G_0. G_1 comes from a forcing which is equivalent to $\mathcal{A}(\kappa_\alpha, \lambda_{\alpha+1}) \times \operatorname{Col}(\bar{\kappa}_\alpha, <\nu) \times \operatorname{Col}(\nu, s_\alpha(\nu)^+) \times \operatorname{Col}(s_\alpha(\nu)^{+3}, <\kappa_\alpha)$ for some inaccessible ν. We see that κ_α is still preserved, and in the extension by G_0 and G_1, $2^{\kappa_\alpha} \leq 2^{\kappa_\alpha^+} = \lambda_{\alpha+1} < \kappa_{\alpha+1}$. Finally G_2 comes from a forcing whose $\leq^*$ is κ_α^+-closed. By the Prikry property, it does not add new κ_α-sequences. Hence in $V[G]$, κ_α is preserved, and $(2^{\kappa_\alpha})^{V[G]} \leq \kappa_{\alpha+1}$.

By our analysis, on each interval $(\overline{\kappa}_\alpha, \kappa_\alpha]$, there is a $\nu_\alpha \in (\overline{\kappa}_\alpha, \kappa_\alpha)$ such that $\overline{\kappa}_\alpha^+$, ν_α, $s_\alpha(\nu_\alpha)^{++}$,$s_\alpha(\nu_\alpha)^{+3}$, and κ_α are preserved (see Lemma 9), and ν_0 becomes η^+, so we are done. □

The only reason we designate $\bar{\kappa}_0 = \max\{\omega, \eta\}$ is to make sure we have room to directly extend any given condition η-many times. In practice, we can split forcing into blocks. For instance if $\eta > \omega_1$, we may split $\mathbb{P}$ to $\mathbb{P} \restriction \omega$ and $\mathbb{P} \setminus \omega$. As a result, assuming $\bar{\kappa}_0 = \omega$ does no harm, and we can keep factoring as we need. To avoid worries about cardinal arithmetic up to η, we may assume $\overline{\kappa}_0 = \omega$.

We know for each limit $\beta \leq \eta$, $(\bar{\kappa}_\beta^{++})^V$ is preserved by chain condition. $(\bar{\kappa}_\beta^+)^V$ is also preserved by this forcing:

Lemma 9. *For any limit ordinal $\beta \leq \eta, \bar{\kappa}_\beta^{+V}$ is preserved in $V[G]$.*

Proof. We only show the case $\beta = \eta$ here. The case $\beta < \eta$ is similar, together with the fact that $(\mathbb{P} \setminus \beta, \leq^*)$ is $\bar{\kappa}_\beta^+$-closed.

Suppose not. Then in $V[G]$, let $\xi = \operatorname{cf} \overline{\kappa}_\eta^+ < \overline{\kappa}_\eta$. Choose $\alpha < \eta$ such that $\xi < \kappa_\alpha$. Extend p so that $\alpha \in \operatorname{supp}(p)$. Break p into $p \restriction \alpha$, $p(\alpha)$, and $p \setminus \alpha$. Since $p \restriction \alpha$ and the Collapse parts in $p(\alpha)$ come from forcings which have κ_α-.c.c., and the Cohen part of $p(\alpha)$ comes from a forcing which is κ_α^+-closed. $\overline{\kappa}_\eta$ is collapsed in the forcing in which $p \setminus \alpha$ lives (which is $\mathbb{P} \setminus \alpha$).

In V, let $\{\dot{\gamma}_i : i < \xi\}$ be a sequence of names, forced by $p \setminus \alpha \in \mathbb{P} \setminus \alpha$, to be a cofinal sequence in $\bar{\kappa}_\eta^{+V}$. Build a sequence of conditions $\{p_i : i < \xi\}$ such that $p_0 = p \setminus \alpha$, $\{p_i \setminus : i < \xi\}$ is $\leq^*$-decreasing, and p_{i+1} satisfies Lemma 7 for $D_i = \{q \in \mathbb{P} \setminus \alpha : q$ decides the value of $\dot{\gamma}_i\}$.

Set r to be a $(\leq^*)$- lower bound of $\{p_i : i < \xi\}$ in $\mathbb{P} \setminus (\alpha+1)$. By Lemma 7, for each $i < \xi$, $A_i = \{\beta : \exists r' \leq r, r' \Vdash \dot{\gamma}_i = \check{\beta}\}$ has size at most κ_{α_i} for some $\alpha_i < \eta$.

Set $\beta_i = \sup A_i$, and $\beta = \sup_{i<\xi} \beta_i$. Then $r \Vdash \sup\{\dot{\gamma}_i : i < \xi\} \leq \check{\beta}$ and $\beta < (\bar{\kappa}_\eta^+)^V$, which is a contradiction. □

From the series of lemmas above and some chain condition arguments, in the generic extension $V[G]$, we get that for each limit ordinal $\beta \leq \eta$, $\overline{\kappa}_\beta$, $(\overline{\kappa}_\beta^+)^V$, $(\overline{\kappa}_\beta^{++})^V$ are preserved, $\overline{\kappa}_\beta = \aleph_\beta^{V[G]}$, $(\overline{\kappa}_\beta^+)^V = (\aleph_{\beta+1})^{V[G]}$, $(\overline{\kappa}_\beta^{++})^V = (\aleph_{\beta+2})^{V[G]}$, and if $\beta < \eta$ is limit, then $\lambda_\beta = \aleph_{\beta+3}$, where $\lambda_\beta = \lambda_\beta^p$ for some $p \in G$ with $\beta \in \operatorname{supp}(p)$.

From this point, for limit $\beta < \eta$, let $\lambda_\beta = \lambda_\beta^p$ for some $p \in V[G]$ with $\beta \in \operatorname{supp}(p)$, and $\lambda_\eta = \lambda$. Next, we are going to verify that in $V[G]$, for limit $\beta \leq \eta$, $2^{\overline{\kappa}_\beta} = \lambda_\beta$. On one hand, $2^{\overline{\kappa}_\beta} \leq \lambda_\beta$ by a chain condition argument.

To show $\lambda_\beta \leq 2^{\overline{\kappa}_\beta}$, we will build a scale of length λ_β. We analyse the scales in the next section.

6. SCALE ANALYSIS

Lemma 10. *Let $\beta \leq \eta$ be a limit ordinal, let q be a condition such that $\beta \in \operatorname{supp}(q)$ if $\beta < \eta$, and let $\lambda_\beta = \lambda$ if $\beta = \eta$ and $\lambda_\beta = \lambda_\beta^q$ if $\beta < \eta$. Let $\gamma \in [\overline{\kappa}_\beta, \lambda_\beta)$ and $\alpha < \beta$. Let D be the collection of $p \leq q$ such that*

(1) $\alpha \in \operatorname{supp}(p)$.

(2) If we enumerate $\operatorname{supp}(p) \cap (\beta+1) \setminus \alpha$ in decreasing order as $\alpha_0 > \cdots > \alpha_{k-1} = \alpha$ such that if $\beta < \eta$, $\alpha_0 = \beta$, then $\gamma \in \operatorname{dom}(f_{\alpha_0}^p)$, the sequence of ordinals defined inductively by setting $\gamma_0 = \gamma$ and $\gamma_{i+1} = f_{\alpha_i}^p(\gamma_i)$ for as long as $\gamma_i \in \operatorname{dom}(f_{\alpha_i}^p)$ reaches a stage where γ_{k-1} is defined, and $\gamma_{k-1} \in \operatorname{dom}(f_\alpha^p)$.

Then D is open dense below q.

Fix a limit ordinal $\beta \leq \eta$. In $V[G]$, let $\lambda_\beta = \lambda_\beta^p$ when $p \in G$ with $\beta \in \operatorname{supp}(p)$. Note that by genericity of $V[G]$, λ_β is well-defined.

For $\gamma \in [\overline{\kappa}_\beta, \lambda_\beta)$, define a function $t_\gamma : \beta \to \bar{\kappa}_\beta$ as follows: for $\alpha < \beta$, find $p \in G$ with p lying in the dense set from Lemma 10. Enumerate $\operatorname{supp}(p) \cap (\beta+1) \setminus \alpha$ in decreasing order as $\alpha_0 > \cdots > \alpha_{k-1} = \alpha$. Define $\gamma_0, \ldots \gamma_{k-1}$ as in Lemma 10, and define $t_\gamma(\alpha) = f_\alpha^p(\gamma_{k-1})$.

To check that t_γ is well-defined, suppose $p, q \in G$ satisfy the conditions in Lemma 10. Find $r \in G$ with $r \leq p, q$. Hence $((\operatorname{supp}(p) \cup \operatorname{supp}(q)) \cap (\beta+1)) \setminus \alpha \subseteq (\operatorname{supp}(r) \cap (\beta+1)) \setminus \alpha$ and $\alpha_0 = \beta$ if $\beta < \eta$. Assume $r \leq^* p + \langle \mu_0, \ldots, \mu_{l-1} \rangle$ and $r \leq^* q + \langle \tau_0, \ldots, \tau_{l-1} \rangle$. For simplicity, assume μ_i is an β_i-object, τ_j is an ζ_j-object, $\alpha < \beta_0 < \cdots < \beta_{l-1}$ and $\alpha < \zeta_0 < \cdots < \zeta_{m-1}$. We will show that p and r compute the same $t_\gamma(\alpha)$-value. A similar argument will show q computes the same $t_\gamma(\alpha)$ as r. Simplify further that $l = 1$, $\mu = \mu_0$, and $\beta = \beta_0$. Enumerate $(\operatorname{supp}(p) \cap (\beta+1)) \setminus \alpha$ in decreasing order as $\alpha_{k-1} > \ldots \alpha_n > \alpha_{n-1} > \cdots > \alpha_0 = \alpha$, where $\alpha_n > \xi > \alpha_{n-1}$. Then

$$\begin{aligned} f_{\alpha_{k-1}}^r \circ \ldots f_{\alpha_n}^r \circ f_\xi^r \circ f_{\alpha_{n-1}}^r \circ \ldots f_{\alpha_0}^r(\gamma) &= f_{\alpha_{k-1}}^p \circ \ldots f_{\alpha_n}^r \circ f_\xi^r \circ f_{\alpha_{n-1}}^p \circ \ldots f_{\alpha_0}^p(\gamma) \\ &= f_{\alpha_{k-1}}^p \circ \ldots f_{\alpha_n}^p \circ \mu^{-1} \circ \mu \circ f_{\alpha_{n-1}}^p \circ \ldots f_{\alpha_0}^p(\gamma) \\ &= f_{\alpha_{k-1}}^p \circ \ldots f_{\alpha_n}^p \circ f_{\alpha_{n-1}}^q \circ \ldots f_{\alpha_0}^p(\gamma). \end{aligned}$$

Thus p and r compute the same $t_\gamma(\alpha)$. Lemmas 11 and 12 are parallel to Lemmas 2.29 and 3.12 in [3],

Lemma 11. *In $V[G]$, $\langle t_\gamma : \gamma \in [\bar{\kappa}_\beta, \lambda_\beta)\rangle$ is $<^*$- increasing, where $t <^* t'$ means there is $\alpha < \eta$ such that for all $\alpha' > \alpha$, $t(\alpha') < t'(\alpha')$.*

Proof. We prove the case $\beta = \eta$. The case $\beta < \eta$ is similar. Let $\gamma < \gamma' \in [\bar{\kappa}_\eta, \lambda)$. We will show the conclusion by a density argument. Let $p \in \mathbb{P}$. We can find $p' \leq p$ such that γ and γ' are in the domains of the Cohen parts of p. Assume γ and γ' belong to $\mathrm{dom}(f^{p'}_{\alpha_0})$ where $\max(\mathrm{supp}(p')) < \alpha_0$

We can also assume that for $\alpha > \alpha_0$, the domain of each object in $A^{p'}_\alpha$ contains γ, and γ'. We will show

$$p' \Vdash \forall \alpha > \alpha_0 (\dot{t}_\gamma(\alpha) < \dot{t}_{\gamma'}(\alpha)).$$

This is true because for each $\alpha > \alpha_0$, we can find $q \leq p'$ with $\alpha \in \mathrm{supp}(q)$ and for $\alpha' \in \mathrm{supp}(q) \setminus \alpha$, we use an addable α'-object whose domain contains γ and γ'. Every addable object is order-preserving, and by a density argument, we are done. $\square$

In particular, we conclude in $V[G]$, $2^{\aleph_\beta} = \aleph_\beta^{+3}$ for $\beta < \eta$, and $2^{\aleph_\eta} = \aleph_\eta^{++}$.

We set λ_α as λ^p_α when $p \in G$ and $\alpha \in \mathrm{supp}(p)$. We have

Lemma 12. *In $V[G]$, $\langle t_\gamma : \gamma \in [\bar{\kappa}_\beta, \lambda_\beta)\rangle$ is a scale in $(\prod_{\alpha<\beta} \lambda_\alpha, <_{bd})$.*

Proof. Again, assume for simplicity that $\beta = \eta$. First, note that for each condition p and $\alpha > \max(\mathrm{supp}(p))$, $\mathrm{ran}(\mathrm{mc}_\alpha(d^p_\alpha)) \subseteq \lambda = j(s_\alpha)(\kappa_\alpha)^{++}$. Hence there is a measure one set of μ such that $\mathrm{ran}(\mu) \subseteq s_\alpha(\mu(\kappa_\alpha))^{++} = \lambda_\alpha(\mu)$. Hence the type is correct.

Let $\dot{h}$ be a name and p be a condition forcing that $\dot{h} \in \prod_{\alpha<\eta} \dot{\lambda}_\alpha$. Suppose now for simplicity that p is pure. For $\alpha < \eta$, let $D_\alpha = \{q : q \text{ decides } \dot{h}(\alpha)\}$. Find $q \leq^* p$ witnessing the strong Prikry property for D_α, with the finite set of coordinates I_α, for all $\alpha < \eta$. Assume further that $\alpha \in I_\alpha$. Define $Y_\alpha(\vec{\mu})$ to be the value that $q + \vec{\mu}$ decides for $\dot{h}(\alpha)$. Note that this is less than $\lambda_\alpha(\mu_\alpha)$, which is a regular cardinal. Now for $\vec{\mu} \in \prod_{\delta \in I_\alpha \setminus \alpha} A^q_\delta$, define $Z_\alpha(\vec{\mu}) := \sup_{\vec{\tau}}(Y(\vec{\tau}^\frown\vec{\mu}))+1$. By a simple counting argument $Z_\alpha(\vec{\mu}) < \lambda_\alpha(\mu_\alpha)$, and for $\vec{\mu} \in \prod_{\delta \in I_\alpha \setminus \alpha} A^q_\delta$, we still have $q + \vec{\mu} \Vdash \dot{h}(\alpha) < Z_\alpha(\vec{\mu})$. We have that for $\vec{\mu} \in \prod_{\delta \in I_\alpha \setminus (\alpha+1)} j_{E_\alpha}(A^q_\delta)$,

$$j_{E_\alpha}(q) + (\langle \mathrm{mc}_\alpha(d^q_\alpha)\rangle^\frown \vec{\mu}) \Vdash j_{E_\alpha}(\dot{h}(\alpha)) < \lambda.$$

Since for $\delta > \alpha$, $j_{E_\alpha}(A^q_\delta)$ comes from a measure which is $j_{E_\alpha}(\kappa_\delta)$-complete, and $j_{E_\alpha}(\kappa_\delta) > j_{E_\alpha}(\kappa_\alpha) \geq \lambda$, there are $\gamma_\alpha < \lambda$ and measure one set B^α_δ for $\delta \in I_\alpha \setminus (\alpha+1)$ such that for $\vec{\mu} \in \prod_{\delta \in I_\alpha \setminus (\alpha+1)} B^\alpha_\delta$,

$$j_{E_\alpha}(q) + (\langle \mathrm{mc}_\alpha(d^q_\alpha)\rangle^\frown \vec{\mu}) \Vdash j_{E_\alpha}(\dot{h}(\alpha)) = \gamma_\alpha.$$

We run through the process as above for all $\alpha < \eta$. Take $\gamma = \sup_{\alpha<\eta} \gamma_\alpha$. Let $r \leq^* q$ with $\gamma \in \mathrm{dom}(f^r_0)$. Hence for $\alpha < \eta$, and $\vec{\mu} \in \prod_{\delta \in I_\alpha \setminus (\alpha+1)} j_{E_\alpha}(A^r_\delta) \cap B^\alpha_\delta$,

$$j_{E_\alpha}(r) + (\langle \mathrm{mc}_\alpha(d^r_\alpha)\rangle^\frown \vec{\mu}) \Vdash j_{E_\alpha}(\dot{h}(\alpha)) < \mathrm{mc}_\alpha(d^r_\alpha)(j_{E_\alpha}(\gamma)).$$

Since the sets of such $\vec{\mu}$ are of measure one, by elementarity we may shrink measure one sets A^r_δ for $\delta \in I_\alpha \setminus \alpha$ so that every extension of r using objects in $\{A^r_\delta : \delta \in I_\alpha\}$ decides that $\dot{h}(\alpha) < \mu(\gamma)$ where μ is the object being used in A^r_α. Repeat the shrinking process for all α and call the resulting condition s.

We claim that $s \Vdash \dot{h}(\alpha) < \dot{t}_\gamma(\alpha)$ for all α. To show this, fix an α. Let $s' \leq s$ such that s' decides $\dot{h}(\alpha)$. Assume $s' \leq^* s^* + \vec{\mu}$ where $\vec{\mu}$ comes from measure one sets $\{A^r_\delta : \delta \in X\}$ and $X \supseteq I_\alpha$. By the strong Prikry property, $s^{**} := s^* + (\vec{\mu} \restriction I_\alpha)$ decides $\dot{h}(\alpha)$ to be an ordinal less than $\mu_\alpha(\gamma)$. A straightforward calculation tells that s^{**} decides $\dot{h}(\alpha)$ to be an ordinal less than $\dot{t}_\gamma(\alpha)$. Hence, we are done.

□

To investigate the scale further, note that if $p \in G$ and $\beta \in \text{supp}(p)$, $\lambda_\beta = \rho_\beta^{++}$ for some β. The exact same argument shows that $\langle t_\gamma : \gamma \in [\overline{\kappa}_\beta, \rho_\beta^+)\rangle$ is a scale in $(\prod_{\alpha<\beta} \rho_\alpha^+, <_{bd})$. Recall that a scale $\langle h_\alpha : \alpha < \chi^+\rangle$ on $\prod_{\beta<\theta} \theta_\beta$ is *very good* if modulo club filter, every $\alpha < \chi^+$ with $\text{cf}(\alpha) > \theta$ is a *very good point*, meaning there is a club $C \subseteq \alpha$ of type $\text{cf}(\alpha)$ and $\gamma < \theta$ such that for β_0 and β_1 in C with $\beta_0 < \beta_1$ and $\xi > \gamma$, $f_{\beta_0}(\xi) < f_{\beta_1}(\xi)$.

Lemma 13. *$\langle t_\gamma : \gamma \in [\overline{\kappa}_\beta, \rho_\beta^+)\rangle$ is a very good scale.*

Proof. For simplicity, assume $\beta = \eta$. Let $\gamma < \rho^+$, say $\eta < \text{cf}(\gamma) < \kappa_\alpha$ for some $\alpha < \eta$. Let $C \subseteq (\gamma \setminus \rho$ be a club of order type $\text{cf}(\gamma)$. Let $p \in \mathbb{P}$ be such that $\alpha + 1 \in \text{supp}(p)$. Let $\theta = \min(\eta \setminus supp(p))$. Extend p to p' so that $C \subseteq d_\theta^{p'}$. Shrink the measure one set $A^{p'}_{\alpha+1}$ so that the domain of any θ-object in the measure one set contains C. Call the final condition q. It is easy to see that $q \Vdash \forall \beta_0, \beta_1 \in C(\beta_0 < \beta_1 \to \forall \xi > \theta(f_{\beta_0}(\xi) < f_{\beta_1}(\xi)))$. Hence the scale is very good. □

7. LARGE CARDINALS

In this section we clarify how to obtain a pairwise coherent sequence of extenders.

Definition 8. Let κ be a cardinal. Then κ is **weakly compact** if for every transitive set M such that $|M| = \kappa$, M is closed under $< \kappa$-sequences, and satisfies enough set theory, then there is an elementary embedding $j : M \to N$ such that N is transitive, $|N| = \kappa$, N is closed under $< \kappa$-sequences, and $\text{crit}(j) = \kappa$.

Definition 9. Let $\kappa < \chi \leq \delta$ be cardinals, and $A \subseteq V_\delta$. κ is **χ-A-strong** if there is an elementary embedding $j : V \to M$ with M transitive, $\text{crit}(j) = \kappa$, $V_\chi \subseteq M$, and $j(A) \cap V_\chi = A \cap V_\chi$. κ is **$< \delta$-A-strong** if κ is λ-A-strong for $\lambda < \delta$. Finally, δ is a **Woodin cardinal** if for $A \subseteq V_\delta$, there is $\kappa < \delta$ such that κ is $< \delta$-A-strong.

Definition 10. Let δ be a cardinal. Then δ is **superstrong** if there is an elementary embedding $j : V \to M$ such that M is transitive, $\text{crit}(j) = \delta$, and $V_{j(\delta)} \subseteq M$.

Here is a standard fact:

Proposition 11. *If δ is superstrong, then δ is weakly compact and Woodin.*

We will elaborate how we derive the sequence of pairwise coherent extenders from a weakly compact Woodin cardinal. Note that the least Woodin cardinal is not weakly compact. This is because weakly compact cardinals are $\prod_1^1$-reflecting and "δ is Woodin" is captured by a $\prod_1^1$-sentence holding in V_δ.

Lemma 14. *If δ is weakly compact Woodin, then for all $A \subseteq V_\delta$ there is $\kappa < \delta$ which is δ-A-strong.*

Proof. Let $A \subseteq V_\delta$. By Woodinness of δ, let $\kappa < \delta$ be $< \delta$-A-strong. Let $M \prec H_\theta$ be of size δ, closed under $< \delta$-sequences, $trcl(\{V_\delta, A\}) \in M$. By elementarity, $M \models \kappa$ is $< \delta$-A-strong. Let $\bar{M}$ be the transitive collapse of M with the associated isomorphism $i : M \to \bar{M}$. Hence $i(\kappa) = \kappa$, $i(\delta) = \delta$, and $i(A) = A$. Hence $\bar{M} \models \kappa$ is $< \delta$-A-strong. Since δ is weakly compact, let $j : \bar{M} \to N$ with $|N| = \delta$, N is closed under $< \delta$-sequence, $\delta = \text{crit}(j)$, and N is transitive. Thus $N \models \kappa$ is $< j(\delta)$-$j(A)$-strong. Since $\delta < j(\delta)$, in N, κ is δ-$j(A)$-strong. Thus κ is δ-A-strong in N. We can derive an extender in N witnessing κ is δ-A-strong. Since the extender can be coded as a subset of V_δ, κ is δ-A-strong. □

Note that by Woodinness of δ, we can also prove that for each $A \subseteq V_\delta$, the set of $\kappa < \delta$ which is δ-A-strong is unbounded in δ.

Theorem 12. *Suppose that for each $A \subseteq V_\delta$, $\{\kappa : \kappa$ is δ-A-strong$\}$ is unbounded in δ, then there exists a sequence of extenders $\langle E_\alpha : \alpha < \eta \rangle$ such that E_α is a (κ_α, δ)-extender for some κ_α, and the sequence of extenders is pairwise coherent.*

Proof. Pick $\kappa_0 \in (\eta, \delta)$ such that κ_0 is δ-$\emptyset$-strong. Derive a (κ_0, δ)-extender E_0 from the strongness of κ_0. Let $\alpha < \eta$. Suppose $\vec{E}_\alpha := \langle E_\beta : \beta < \alpha \rangle$ have been constructed, we see that $\vec{E}_\alpha$ can be coded as a subset of V_δ. Find $\kappa_\alpha > \sup_{\beta < \alpha} \kappa_\beta$ such that κ_α is δ-$\vec{E}_\alpha$-strong. Set a witness $j_{E_\alpha} : V \to M_\alpha$. Then derive an (κ_α, δ)-extender from j_{E_α}. By the strongness of E_α, we conclude that $j_{E_\alpha}(E_\beta) \cap V_\delta = E_\beta \cap V_\delta = E_\beta$ for all $\beta < \alpha$. □

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