Distribution of anomalous exponents of natural images

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Abstract

Recent studies of correlations of intensity in data bases of natural images revealed a remarkable property. The measured correlations are not dissimilar to correlations found in magnets at their point of transition or to correlations in self organized systems. The correlations seem to be described by a robust exponent. In the present letter we present our results concerning the statistical meaning of that result. We study many individual images of one of the data bases considered. We find that the same law governing the decay of correlations in the whole database governs also the images we chose randomly from the database, with one essential difference. The exponent characterizing the decay of correlations in each image is specific and differs from the exponent characterizing the decay of the data base correlations. The distribution of single image exponents was measured and found to exhibit a rather heavy tail. Thus, the data base exponent cannot be considered as a statistical representative of the single image exponent.

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Much attention has been recently focused on the statistical properties of natural images [1–4]. There are a number of practical reasons that motivate such studies. The statistical properties of various features in the image, enter as a prior in various image processing and computer vision tasks. The prior is given as a density function (a probability distribution function) that characterize the probability of a given image to be a "real" natural image. Such a characterization is the key of success in various image processing and computer vision tasks such as transmission, compression and de-noising to name just a few. These are related to the information content of the image. Yet, perhaps more remarkable are the observations of correlations of the light intensity that do not seem to be related to information content in the image. Similar correlations exist in ferromagnets at their critical point or in systems of self organized criticality. It has been shown by several authors [1–4] that intensity correlations exhibit scaling and even more remarkable that scaling showed promise of being universal. Ruderman and Bialek [1] have considered a set of images where each image is considered as an array of pixels in which the light intensity is recorded. After replacing the value of the intensity at each pixel, i, by its logarithm, $\phi(i)$, the power spectrum of the whole set of images is taken. The power spectrum is found to be proportional to $q^{-(2-\eta)}$, where q is the absolute value of the "momentum" vector $\mathbf{q}$, corresponding to the appropriate Fourier components of the image. Such a behavior is typical to the Fourier transform of the spin-spin correlation function at the point of phase transition. In that case the exponent η is known as the anomalous dimension, is positive and usually not large [5–7]. Such power law behavior is also characteristic of system exhibiting self organized criticality, such as the KPZ system [8–12] in which the relevant real space correlation corresponds to the typical height variation of a deposited surface over a distance r. The power spectrum corresponds to the correlation $\langle h_{\mathbf{q}} h_{-\mathbf{q}} \rangle$, where $h_{\mathbf{q}}$ is the Fourier transform of the height function defining the two dimensional surface. In that case the exponent η is negative corresponding to the fact that the surface is self affine. Grey-value images can be viewed, of course, as a surface in a three dimensional space consisting of two spatial dimensions and one feature dimension, the intensity. This observation has been recently generalized to represent images with more features, such as color, as the embedding of surfaces in a spatial-feature higher dimensional spaces [13, 14]. A recent review article [4] gives a comprehensive discussion of the phenomenon of scale invariance in natural images, starting from old observations of television engineers [15, 16] to more recent work on natural images starting in the late 80's and continuing ever since [1, 3, 4, 17, 18]. We will concentrate our discussion, however, around the studies of two groups of researchers, because those studies lead, in our mind to results that are most puzzling.

Ruderman and Bialek [1] find $\eta = 0.19$. As pointed out by Ruderman [2], the scaling form of the power spectrum determines the scaling form of the two point difference function,

$$D(\mathbf{r}) = \left\langle [\phi(\mathbf{r}) - \phi(0)]^2 \right\rangle , \tag{1}$$

to be given by

$$D(\mathbf{r}) = D_1 - D_2 r^{-\eta}, \tag{2}$$

where D_1 and D_2 are constants. Indeed direct measurement of the difference function obtained from averaging over a small data base of images taken from wood scenery yields also $\eta = 0.19$.

The work described above, as well as the work of other researchers that found scaling in natural images [19] is based on the study of data bases that are limited to images of a particular type. More surprising, however, is the finding of Huang and Mumford [3]. They studied these correlations over a large archive of 4000 images, provided by van Hateren [20]. Taking pairs of points from all images together, they were able to show that the form given by eq. (2) above, describes the behavior of the difference function over a range of distances between 4 and 32 pixels with the same exponent, $\eta = 0.19$. The archive they have used, however is very diverse and contains scenes of forest, buildings, grass, clouds, roads and rivers all in the same image and images taken from very different angles. The fact that the anomalous dimension observed fits that obtained in refs. [1, 2], in spite of the obvious diversity of the different images in the archive, is remarkable and puzzling indeed. Our experience in the field of continuous phase transitions and of self organized criticality is that various exponents characterizing the system are universal. Namely, those exponents are exactly the same for systems which are quite different from each other as long as they share a small number of some physical attributes such as dimension, symmetry and range of interaction. The puzzlement with the results described above is that they indicate that natural images are universal in some sense. This is extremely surprising and yet very important if true.

The purpose of the present article is mainly to understand whether the robustness of the exponent really indicates universality or not. This is done in two steps. First we try to reproduce the result of Huang and Mumford [3] and then to study its statistical significance. Namely, to ask whether the database exponent is representative of single images in that database. This is achieved by showing that each image separately can indeed be described by a difference function of the form given above (2) and constructing the distribution function of the separate η's. Our purpose here is not to explain the origin of the form of the two point difference function in each image or in the database but to show that they exist and to study the statistical meaning of our finding.

We use the archive of van Hateren [20] that was used by Huang and Mumford [3]. For each image, given as a 1024 x 1536 array of intensities, we calculate the difference function $D(\mathbf{r})$ in the following way. We choose at random five million pairs of points in the image that are separated by distances between 0 and 32 pixels and are oriented in different directions. (The reason for our choosing distances only between o and 32 pixels is our intention to repeat exactly the procedure of Huang and Mumford [3]. It is clear that consideration of larger distances may be of importance but this is postponed to future publications.) The difference function of each image is obtained by averaging $(\phi(\mathbf{r}) - \phi(\mathbf{0}))^{\mathbf{2}}$ for each distance in the range of up to 32 pixels over the pairs corresponding to that distance. The data base difference function is the average over images of the image difference function. We obtain the data base difference function by averaging over the first 1400 images chosen from the archive. We then fit the data by the form given in eq. (2). The observed results are depicted against the best fit in fig. 1. The pluses represents the observed results while the continuous line represents the best fit. The value of η corresponding to the best fit is $\eta = 0.19$. This value varies a little due to averaging over different numbers of images. In our measurements the variation was of the order of $\Delta\eta = \pm 0.01$ (For instance, for the first 1100 images of the archive the value $\eta = 0.18$ is obtained). The result of Huang and Mumford is thus reproduced.

Our next question is whether the image difference functions can be fitted by the same form of behavior as the data base difference function but not necessarily with the same constants. We do expect, however, that the observed image difference function will be noisier than the data base difference function just because the average is on a number of pairs (at each distance) that is by the order of one thousand smaller. We have obtained the fit for all our

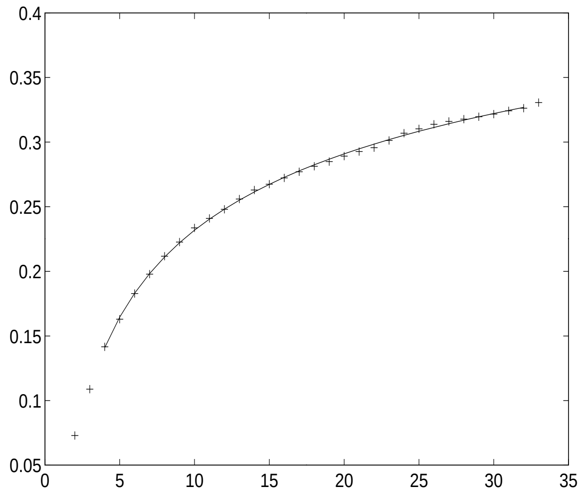

FIG. 1: Averaged difference function. The average is taken over 1400 images. $\eta = 0.19$

images but naturally we can present it only for a small number as depicted in figures 2-3.

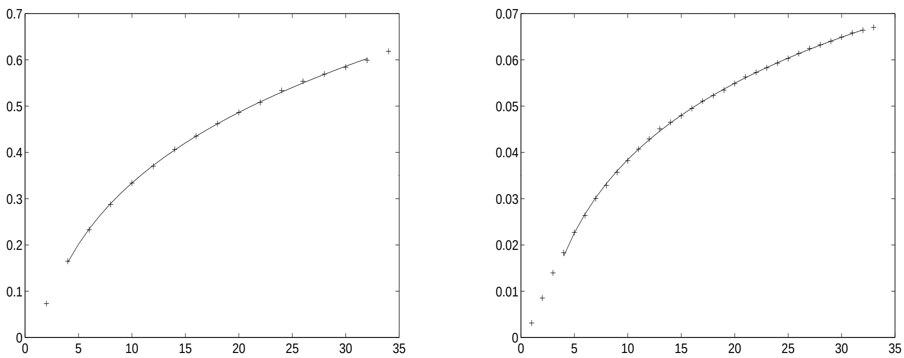

FIG. 2: Left: Image 1401, $eta = -0.2$, Right: Image 262, $eta = -0.05$.

The obtained fit is good. It is characteristic of the images with a relatively large exponent to fluctuate a bit around the fitted line (fig. 3 right). Note that we perform here a non-linear fit that is not necessarily convex and may suffer from local minima. Since the exponents that describe the individual images are not of a definite sign care should be taken when fitting the data. Search procedures starting, say, from a positive exponent, in a case where the exponent is actually negative, may end in a small positive exponent and in a fit which is less than satisfactory. We obviously see that the value of anomalous dimension differs from image to image. Note that the results we give for the anomalous dimensions for different images is based on measurements in the range of 0-32 pixels in accordance with the measurements

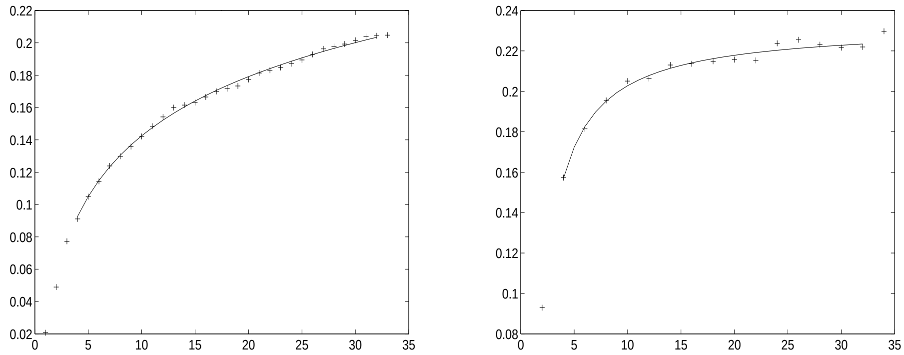

FIG. 3: Left: Image 59, $eta = 0.04$, Right: Image 712, $eta = 1.01$.

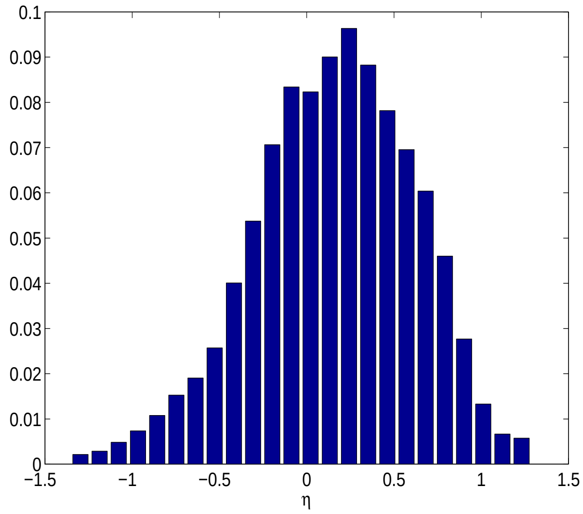

FIG. 4: Distribution of η for 1500 images in the archive

of Huang and Mumford on the full data base. Now, if the value of the exponent was the same for all individual images, the value of the data base anomalous dimension would have been identical to the value of the images. For the data base value to be considered as a representative of the values on the images, the distribution of the image anomalous dimension should be narrow. Figure 5 shows the distribution of values of η as obtained from the first 1500 images in the archive. Two obvious features of the distribution are that it has a maximum in the vicinity of $\eta = 0$ and a rather wide distribution giving significant weight to relatively high values of η. The conclusion, therefore, is that the database η is not representative at all of the image η's. It would be of interest to ask whether there are any specific features in images with high η which are relatively rare compared to images with

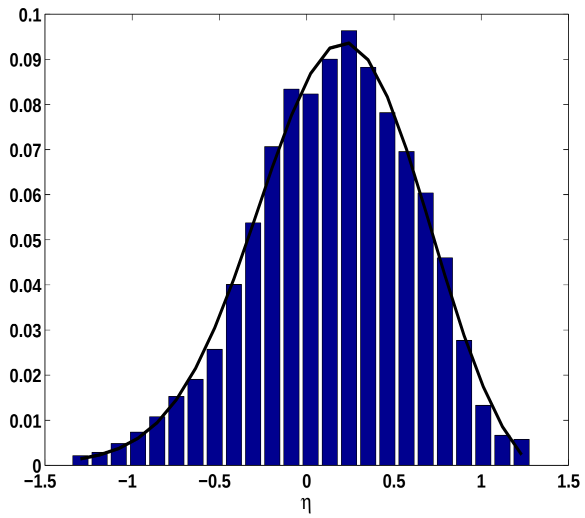

FIG. 5: Distribution of η for 1500 images in the archive. Fit function: $\left(C_1 + C_2(\eta - \langle\eta\rangle) + C_3(\eta - \langle\eta\rangle)^2 + C_4(\eta - \langle\eta\rangle)^3\right)\exp(-C_5(eta \ - \ \langle\eta\rangle)^2)$. Mean=0.155, Variance=0.21, Skewness=-0.28, Kurtosis=2.9, Kurtosis Excess=-0.1

lower η's. For the benefit of the readers we have presented a number of images of high η versus a number of images with small η in the following web site [21]. Our conclusion from direct observation of the images is that a large value of eta is usually associated with an image in which the detail seems less pronounced. As expected, of course, from equation (2) which defines eta. Having said that it is also evident that the difference to the eye between images differing considerably in eta is not very striking, and we have found it quite difficult, in some cases, to guess just by looking at a pair of such images that their difference in η is that big.

We have demonstrated in this study that the distribution of the η's in not Gaussian. Since only for the Gaussian case the mean, median and mode coincide it is natural to wonder which one of them characterizes best the distribution. In fact even if the distribution was narrow, without being Gaussian, it could have sufficed for an approximate characterization of the distribution. Obviously, this is not the case either. Thus, neither of the natural parameters of the distribution described above should be taken as characterizing the distribution. Clearly, higher moments are of interest in such a case, and in the caption of Fig. 5 we give an analytic fit to the distribution of exponent and the values of the mean, variance, skewness and kurtosis of the distribution.

Scaling in natural images remains, as far as we are concerned, an open question that needs further "experimental" study as well as the construction of physical models that can explain the emergence of such correlations. Already at this stage it is quite clear that universality does not exist in natural images although scaling does.

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