Scale-Space Generation via Uncertainty Principles

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Abstract. This study is concerned with the uncertainty principles which are related to the Weyl-Heisenberg, the $SIM(2)$ and the Affine groups. A general theorem which associates an uncertainty principle to a pair of self-adjoint operators was previously used in finding the minimizers of the uncertainty principles related to various groups, e.g., the one and two-dimensional Weyl-Heisenberg groups, the one-dimensional Affine group, and the two-dimensional similitude group of $\mathbb{R}^2$, $SIM(2) = \mathbb{R}^2 \times (\mathbb{R}^+ \times SO(2))$. In this study the relationship between the affine group in two dimensions and the $SIM(2)$ group is investigated in terms of the uncertainty minimizers. Moreover, we present scale space properties of a minimizer of the $SIM(2)$ group.

1 Introduction

The 2D Gabor function and Gabor-Morlet wavelets are commonly used in computer vision. Mostly in relation to texture analysis, synthesis and segmentation. The use of these functions is usually motivated by the fact that the Gaussian window minimizes the uncertainty and attains the maximal possible accuracy in both the spatial and frequency domains. In fact, the Gabor transform is a representation of the Weyl-Heisenberg group while the 2D Gabor-Morlet transform is a representation of the 2D affine group or of subgroups thereof. Since both the 2D Gabor-Morlet wavelet transform and the multi-window Gabor transform involve rotation and scaling (and potentially few more transformations) in addition to the usual translation and frequency modulations, it makes sense to look for a window shape that maximizes the accuracy in all attributes. This study explores this question and shows that this aim can only be partially attained.

The Gaussian function appears as a pivot in scale-space theory as well, where its successive applications to images produce coarser resolution images. It is shown, in fact, that the family of Gaussian functions posses semi-group properties with respect to the width of the Gaussian. This raises the question whether families of functions that minimize the uncertainty for other groups of transformations posses the same characteristic. It is shown in this study that this is in fact true for the cases that we consider. This is an intriguing fact whose full significance is not yet fully understood.

In this study we consider the results obtained for the similitude group [1, 3] and apply them to the affine group in two dimensions. Moreover, we explore the scale-space nature of the minimizer derived by Ali, Antoine and Gazeau [1] and find that their solution has smoothing and edge detection attributes which can produce scale-space representation of images.

The rest of this paper is organized as follows: First, we review the uncertainty principle theorem for self-adjoint operators and point out related works. We then apply it to the Weyl-Heisenberg group and the affine group in one and two dimensions. We conclude by pointing out the scale-space properties of the minimizers obtained.

2 Background and Related Work

The uncertainty principle is a fundamental concept in quantum mechanics as well as in signal and information theory. In quantum mechanics, the Heisenberg uncertainty principle states that the position and momentum of a particle cannot be simultaneously known. In signal and information theory, Gabor [5] showed that there exists a trade off between time resolution and frequency resolution for one-dimensional signals, and that there is a lower bound on their product. These results were extended to consideration of images [9].

A special attention has been given to the functions which attain the lower bound of the inequality defined by the uncertainty principle. It is used to define the canonical coherent states for quantum systems in physics. In signal processing it was discussed, inter alia, by Gabor. He showed that Gaussian-modulated complex exponentials provide the best trade-off for time resolution and frequency resolution. These are equivalent to a family of canonical coherent states generated by the Weyl-Heisenberg group.

A general theorem which is well known in quantum mechanics and harmonic analysis [4] relates an uncertainty principle to any two self-adjoint operators and provides a mechanism for deriving a minimizing function for the uncertainty equation: two self-adjoint operators, A and B obey the uncertainty relation:

$$\Delta A_f \Delta B_f \ \geq \frac{1}{2}|\langle [A,B]\rangle| \quad \forall f, \tag{1}$$

where $\Delta A_f, \Delta B_f$ denote the variances of A and B respectively with respect to the signal f. The triangular parnthesis mean an average over the signal i.e. $\langle X\rangle = \int f^* X f$. The mean of the action of an operator P on a function f is denoted as: $\mu_P = \langle P\rangle$ and the commutator $[A,B]$ is given by: $[A,B] := AB - BA$.

A function f is said to have minimal uncertainty if the inequality turns into an equality. This happens if and only if there exists a $\lambda \in i\mathbb{R}$ such that

$$(A - \mu_A)f = \lambda(B - \mu_B)f. \tag{2}$$

Thus, any two self-adjoint operators, whose commutator does not vanish, lead to an uncertainty principle. Moreover, the constraint for equality, together with a realization of the operators in differential form, lead to a set of partial differential equations. The solution is the function which minimizes the uncertainty for the relevant operators.

Both windowed Fourier and wavelet transforms are related to group theory, as both can be derived from square integrable group representations [6]. The windowed Fourier transform is related to the Weyl-Heisenberg group, and the wavelet transform is related to the affine group. The general uncertainty theorem [4] stated above provides a tool for obtaining uncertainty principles using the infinitesimal generators of the groups' representations. In the case of the Weyl-Heisenberg group, the canonical functions which minimize the corresponding uncertainty relation are Gaussian functions. The canonical functions which minimize the uncertainty relations for the affine group in one dimension and for the similitude group in two dimensions were the subject of previous studies, among them is the study of Dhalke and Maass [3] and that of Ali, Antoine and Gazeau [1].

Dahlke and Maass [3], as well as Ali, Antoine and Gazeau [1] studied the uncertainty principle for a sub-group of the affine group, the similitude group of $\mathbb{R}^2$, $SIM(2) = \mathbb{R}^2 \times (\mathbb{R}^+ \times SO(2))$, which is related to the wavelet transform. Dahlke and Maass [3] have included commutators with elements of the enveloping algebra, i.e. polynomials in the generators of the algebra, and managed to find the $2D$ isotropic Mexican hat. Ali, Antoine and Gazeau [1] derived a possible minimizer in the frequency domain for some fixed direction. Their solution is a real wavelet which is confined to some convex cone in the positive half plane of the frequency space and is exponentially decreasing inside.

3 The Weyl-Heisenberg Group

The uncertainty principle related to the Weyl-Heisenberg group has a tremendous importance in two main fields; in quantum mechanics, the uncertainty principle prohibits the observer from exactly knowing the location and momentum of a particle. In signal processing, the uncertainty principle provides a limit on the localization of the signal in both time (spatial) and frequency domains.

Let G be the Weyl-Heisenberg group,

$$G := \{(\omega, b, \tau) | b, \omega \in \mathbb{R}, \tau \in \mathbb{C}, |\tau| = 1\} \tag{3}$$

with group law

$$(\omega, b, \tau) \circ (\omega', b', \tau') = (\omega + \omega', b + b', \tau\tau' e^{i\frac{(\omega b' - \omega' b)}{2}}). \tag{4}$$

Let π be a representation of the group's action on $L^2(\mathbb{R})$; then, the coefficients generated by $\langle f, \pi(x)\psi\rangle$ are known as the windowed Fourier transform of the function f, with ψ being the window function. The windowed Fourier transform is defined by:

$$\langle f, \pi(x)\psi\rangle = (G_\psi f)(\omega, b) = \int f(x)\psi(x-b)e^{-i\omega x}dx \tag{5}$$

The Fourier transform is a profound tool in signal processing. The Gaussian window function $\psi(x) = e^{-\frac{x^2}{2}}$ has an important role in the windowed Fourier analysis as it minimizes the Weyl-Heisenberg uncertainty principle. Next, we review the derivation of the uncertainty principles for the Weyl-Heisenberg group in one and two dimensions using the uncertainty principle theorem. The reader may find the classical proofs of the uncertainty principle for the Weyl-Heisenberg group in the work of Gabor [5] for one-dimensional signals and in the work of Daugman [2] for two-dimensional signals.

3.1 The one dimensional case

The unitary irreducible representation of the Weyl-Heisenberg group in $L^2(R)$ can be defined by: $[U(\omega, b)f](x) := e^{i\omega x}f(x-b)$. The following infinitesimal generators of the group are then given by:

$$(T_\omega f)(x) := i\frac{\partial}{\partial\omega}[U(\omega, b)f](x)|_{\omega=0,b=0} = -xf \tag{6}$$

$$(T_b f)(x) := i\frac{\partial}{\partial b}[U(\omega, b)f](x)|_{\omega=0,b=0} = -i\frac{d}{dx}f \tag{7}$$

The one-dimensional uncertainty principle for the Weyl-Heisenberg group can be derived using the general uncertainty principle.
Corollary:[4] Let $A = T_\omega$ and $B = T_b$ be the infinitesimal operators of the Weyl-Heisenberg group: $A = -x$, $B = -i\frac{\partial}{\partial x}$. If $f \in L^2(R)$ and $a = \mu_A, b = \mu_B \in R$ we have: $\|(A-a)f\|_2\|(B-b)f\|_2 \geq \frac{1}{4}\|f\|_2$, with equality being obtained iff

$$f(x) = ce^{2\pi ibx}e^{-\pi r(x-a)^2} \tag{8}$$

for some $c \in \mathbb{C}$, $r \in \mathbb{R}_+$.

3.2 The two-dimensional case

The unitary irreducible representation of the Weyl-Heisenberg group in $L^2(R^2)$ in two dimensions is given by: $[\widetilde{U}(\omega_1, \omega_2, b1, b2, \tau)f](x,y) = \tau e^{i(\omega_1 x + i\omega_2 y)}f(\overrightarrow{u} - \overrightarrow{b})$, where $\overrightarrow{u} = (x,y)$, $\overrightarrow{b} = (b_1, b2)$. The following infinitesimal generators of the group can be defined as:

$$(T_{\overrightarrow{\omega}} f)(\overrightarrow{u}) := i\frac{\partial}{\partial\overrightarrow{\omega}}[Uf](\overrightarrow{u})|_{\overrightarrow{\omega}=0,\overrightarrow{b}=0} = -\overrightarrow{u}f \tag{9}$$

$$(T_{\overrightarrow{b}} f)(\overrightarrow{u}) := i\frac{\partial}{\partial\overrightarrow{b}}[Uf](\overrightarrow{u})|_{\overrightarrow{\omega}=0,\overrightarrow{b}=0} = -i\nabla f \tag{10}$$

The only non-vanishing commutators of these four operators are:

$$[T_{w_k}, T_{b_k}] = -i \quad , \quad k = 1, 2 \ . \tag{11}$$

Thus, an uncertainty principle can be obtained for translations in the space and frequency domains. This can be solved for each dimension separately. It is interesting to note that using the Weyl-Heisenberg group, there is no coupling between the x and y components. Thus attaining a certain accuracy in the x component does not affect the degree of accuracy of the y component. If we derive the minimization equation, we simply get the same result for the one-dimensional analysis for both x and y. The separability of the Weyl-Heisenberg group results in separable gaussian functions as the minimizers of the combined uncertainty. This is, in fact, an inherent property of the Gaussian functions.

4 The Affine Group

Let G be the affine group, and let U be its canonical left action on $L^2(R)$; the coefficients generated by $\langle f, U(x)\psi\rangle$ are known, in the one-dimensional case, as the wavelet transform of a function f, with ψ as a mother wavelet, or template. The wavelet transform is defined by:

$$(W_\psi f)(a,b) = \int_R f(x)|a|^{-\frac{1}{2}} \overline{\psi(\frac{x-b}{a})} dx \tag{12}$$

4.1 The one-dimensional case

Let A be the affine group,

$$A := \left\{(a,b) | (a,b) \in R^2, a \neq 0\right\} \tag{13}$$

with group law

$$(a,b) \circ (a', b') = (aa', ab' + b). \tag{14}$$

A unitary group representation obtained by the action of $U(A)$ on $f(x)$ is given by:

$$[U(a,b)f](x) = |a|^{-\frac{1}{2}} f\left(\frac{x-b}{a}\right) \tag{15}$$

In preparation for our extension to two dimensions and other groups, we quote the main results presented in the work of Dahlke and Maass [3] for the one dimensional affine group. First, the self-adjoint infinitesimal operators are calculated by computing the derivatives of the representation at the identity element:

$$\begin{aligned} T_a &= -i(\frac{1}{2} - x\frac{\partial}{\partial x}) \\ T_b &= -i\frac{\partial}{\partial x}. \end{aligned} \tag{16}$$

Using these operators, the affine uncertainty principle is given, and the following differential equation can be solved to obtain the uncertainty minimizer:

$$(T_a - \mu_a)f = \lambda(T_b - \mu_b)f, \tag{17}$$

which reads:

$$-\frac{1}{2}if - ixf' - \mu_a f = -i\lambda f' - \lambda\mu_b f. \tag{18}$$

The solution to this equation is: $f(x) = c(x-\lambda)^{\alpha}$, where $\alpha = -\frac{1}{2} - i\lambda\mu_a + i\mu_b$. Dahlke and Maass [3] provide constraints on α, so that the obtained solution is in $L^2(R)$.

4.2 The two-dimensional case

In the studies of Dahlke and Maass [3], and of Ali, Antoine and Gazeau [1], the uncertainty principle is derived for a subgroup of the affine group which includes translations, rotations and a uniform scaling in the x and y directions. Let us begin by briefly quoting their main findings before extending them to the affine group itself.

The $2D$ similitude group of $\mathbb{R}^2$, $SIM(2) = \mathbb{R}^2 \times (\mathbb{R}^+ \times SO(2))$ Consider the group $B = \mathbb{R}^+ \times \mathbb{R}^2 \times SO(2)$ with group law $(a, b, \tau_\theta) \circ (a', b', \tau_{\theta'}) = (aa', b + a\tau_\theta b', \tau_{\theta+\theta'})$. The unitary representation of B in $L^2(\mathbb{R}^2)$ is given by:

$$[U(a,b,\theta)f](x,y) = \frac{1}{a} f\left(\tau_{-\theta}\left(\frac{x-b_1}{a}, \frac{y-b_2}{a}\right)\right), \tag{19}$$

where the rotation $\tau_\theta \in SO(2)$ acts on a vector (x, y) in the following way:

$$\tau_\theta(x,y) = (xcos(\theta) - ysin(\theta), xsin(\theta) + ycos(\theta)), \tag{20}$$

and $\theta \in [0, 2\pi)$. The self-adjoint infinitesimal operators are given by:

$$T_\theta = i(\overrightarrow{u}^{\perp})^t \cdot \nabla, \quad T_a = -i(1 + \overrightarrow{u}^t \cdot \nabla),$$
$$T_{\overrightarrow{b}} = -i\nabla.$$

where $(\overrightarrow{u}^{\perp})^t = (-y, x)$ The only non-vanishing commutation relations are:

$$[T_a, T_{b_k}] = -iT_{b_k},\ [T_\theta, T_{b_k}] = i\epsilon_{3kl}T_{b_l}\ ,$$

where ϵ_{ijk} is the full anti-symmetric tensor and summation is implied on repeated indices. These four non-zero uncertainty relations lead to a set of four partial differential equations:

$$\begin{aligned}
i\frac{\partial f}{\partial x}y - i\frac{\partial f}{\partial y}x - \mu_\theta f &= -i\lambda_1\frac{\partial f}{\partial x} - \lambda_1\mu_{b_1}f \\
i\frac{\partial f}{\partial x}y - i\frac{\partial f}{\partial y}x - \mu_\theta f &= -i\lambda_2\frac{\partial f}{\partial y} - \lambda_2\mu_{b_2}f \\
-if - i\frac{\partial f}{\partial x}x - i\frac{\partial f}{\partial y}y - \mu_a f &= -i\lambda_3\frac{\partial f}{\partial x} - \lambda_3\mu_{b_1}f \\
-if - i\frac{\partial f}{\partial x}x - i\frac{\partial f}{\partial y}y - \mu_a f &= -i\lambda_4\frac{\partial f}{\partial y} - \lambda_4\mu_{b_2}f
\end{aligned} \tag{21}$$

It turns out that there does not exist a non-zero solution to this system of PDEs. It is not clear wether the theoretical bounds given by the uncertainty theorem are tight in the sense that they are the infimum value over the L^2 functional space or that better bounds are possible. Research on these questions is ongoing. Here we try to modify our quest in two different manners. One approach is to find a subset of generators which have mutually minimized relations. The generators span a linear space, the Lie algebra. We look for the possibly maximal subspace for which a non-trivial function minimizes the related uncertainties. This is the approach taken by Ali, Antoine and Gazeau [1]. They observe that the relationships between T_a and T_{b_1}, and T_θ and T_{b_2} can be transformed into the relationships between T_a and T_{b_2}, and T_θ and T_{b_1} by a $\frac{\pi}{2}$ rotation. Thus, they define a new translation operator $T_b = T_{b_1} cos(\gamma) + T_{b_2} sin(\gamma)$, so that a minimizing function can be obtained for this new operator as well as for T_a and T_θ with respect to a fixed direction $\boldsymbol{\gamma}$. The minimizer they obtain in the frequency space k_x, k_y is a function which vanishes outside some convex cone in the half-plane $k_x > 0$ and is exponentially decreasing inside:

$$\hat{\psi(\hat{k})} = c|\boldsymbol{k}|^s e^{-\lambda k_x}, \tag{22}$$

where $s > 0$ and $\lambda > 0$.

Another approach is to replace few of the generators by elements of the universal enveloping algebra. Dahlke and Maass [3] followed this path. The solution they find is a minimizer to the operators: T_a, T_θ and $T_b := T_{b1}^2 + T_{b2}^2$. A possible solution is the Mexican hat function: $\psi(x,y) = [2 - 2\beta r^2]e^{-\beta r^2}$, where $r := \sqrt{x^2 + y^2}$.

Note that in the first approach the subspace chosen is not a sub-algebra. It is closed under summation but not under the multiplication (defined as commutation relation). The latter operation can take an element in the subspace of the Lie algebra out of it. In the second approach we build a minimizer for a full algebra. Here we simply changed the underline symmetry, namely we do not allow uncorrelated translations in the x and y directions.

The Affine Group in 2D Let us explore the most straight forwards representation of the Affine group. Let define $s = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$, $D = s_{11}s_{22} - s_{21}s_{12}$, $\overrightarrow{b} = (b_1, b_2)$ and $\overrightarrow{x} = (x, y)$. We restrict our discussion to the case $D \geq 0$. A similar derivation can be obtained for the case $D \leq 0$. The representation corresponding to the action of the Affine group is accordingly given by:

$$[U(s, \overrightarrow{b})f](\overrightarrow{x}) = \sqrt{D} f\left(s\left(\overrightarrow{x} - \overrightarrow{b}\right)\right). \tag{23}$$

Let us calculate the infinitesimal operators associated with: $s_{11}, s_{12}, s_{21}, s_{22}, b_1, b_2$:

$$T_{s_{11}}(x,y) = i(\frac{1}{2} + x\frac{\partial}{\partial x}), \quad T_{s_{22}}(x,y) = i(\frac{1}{2} + y\frac{\partial}{\partial y}),$$

$$T_{s_{12}}(x,y) = iy\frac{\partial}{\partial x}, \quad T_{s_{21}}(x,y) = ix\frac{\partial}{\partial y},$$
$$T_{b_1}(x,y) = -i\frac{\partial}{\partial x}, \quad T_{b_2}(x,y) = -i\frac{\partial}{\partial y}. \tag{24}$$

As these operators were derived from a unitary representation, they are self-adjoint. The non-vanishing commutation relations are:

$$\begin{array}{lll}
[T_{s_{11}}, T_{s_{12}}] = iT_{s_{12}}, & [T_{s_{11}}, T_{s_{21}}] = -iT_{s_{21}}, & [T_{s_{11}}, T_{b_1}] = iT_{b_1} \\
[T_{s_{12}}, T_{s_{22}}] = iT_{s_{12}}, & [T_{s_{12}}, T_{b_2}] = iT_{b_1}, & [T_{s_{21}}, T_{s_{22}}] = -iT_{s_{21}} \\
[T_{s_{21}}, T_{b_1}] = iT_{b_2}, & [T_{s_{22}}, T_{b_2}] = iT_{b_2}, & [T_{s_{12}}, T_{s_{21}}] = -i(T_{s_{11}} - T_{s_{22}})
\end{array}$$

Thus, of the fifteen possible commutation relations we obtain nine uncertainty principles. It is interesting to note that the scaling in the x direction (s_{11}) is not constrained by the scaling in the y direction (s_{22}). The same goes for the x and y translations. Using the uncertainty theorem for self-adjoint operators, we obtain a set of differential equations whose solution is the function which attains the minimal uncertainty relations. A simultaneous solution for all equations necessarily imposes: $f \equiv 0$. No function attains the minimality of uncertainty in L^2 for all the relations. Facing this situation we have several options: We may look for a function that minimizes the uncertainty relations of subgroups of the affine group. We may be satisfied with an algebraic subspace (which is not necessarily an algebra of a subgroup), we may find a subspace of the universal enveloping algebra (the polynomials in the generators), or finally we can limit ourself to a subset of *the non-commuting pairs* of generators. For example, we take the following linear combinations of the generators: $T_\theta = T_{s_{12}} - T_{s_{21}} = i(yf_x - xf_y)$ and $T_{scale} = T_{s_{11}} + T_{s_{22}} = i(f + xf_x + yf_y)$. We may consider these new operators as representing the total orientation and scale changes due to the operation of the affine group. Moreover, these operators, along with the translation operators are identical to those obtained for the $SIM(2)$ group, and thus, we can easily implement the derivations of the minimizer of Ali, Antoine and Gazeau [1] to these operators. Another immediate possibility is to follow the derivation of Dahlke and Maass [3] by using rotation invariant functions which can be presented by: $f(x,y) = g(\sqrt{x^2+y^2})$. These are the minimizers of the following three operators, which are defined as polynomials in the existing six operators:

$$T_\theta = T_{s12} - T_{s12},$$
$$T_{scale} = T_{s11} + T_{s22} = i(1 + r\frac{\partial}{\partial r}),$$
$$T_r = T_{b_1}^2 + T_{b_2}^2 = \frac{1}{r} - \frac{\partial^2}{\partial r^2}$$

The equations to be solved are:

$$(T_\theta - \mu_\theta)g(r) = \lambda_1(T_r - \mu_r)g(r) \tag{25}$$
$$(T_\theta - \mu_\theta)g(r) = \lambda_2(T_{scale} - \mu_{scale})g(r) \tag{26}$$
$$(T_r - \mu_r)g(r) = \lambda_3(T_{scale} - \mu_{scale})g(r). \tag{27}$$

Naturally, the motivation for defining these new operators is the rotation invariance property of T_θ, i.e. $T_\theta g(r) = 0$. Thus, instead of seven equations to be solved we are left with only three. We can simply select $\lambda_1 = \lambda_2 = 0$ to obtain:

$$-g''(r) - \frac{1}{r}g'(r) - \mu_r g = \lambda_3 i(g(r) + rg'(r)) - \lambda_3 \mu_{scale} g. \tag{28}$$

As can be seen, we have obtained the exact equation obtained by Dhalke and Maass for which a Mexican hat solution can be found.

Ali, Antoine and Gazeau have divided the four commutators they obtained for the similitude group generators into two groups which are transformed into each other by $\frac{\pi}{2}$ rotation. We apply this approach to the fifteen commutators obtained. Thus, the set of commutators:

$$[T_{s_{11}}, T_{s_{12}}], [T_{s_{11}}, T_{s_{21}}], [T_{s_{11}}, T_{b_1}], [T_{s_{12}}, T_{s_{21}}], [T_{s_{12}}, T_{b_2}]$$

transforms under rotation of $\frac{\pi}{2}$ into the complementary set of commutators:

$$[T_{s_{22}}, T_{s_{21}}], [T_{s_{22}}, T_{s_{12}}], [T_{s_{22}}, T_{b_2}], [T_{s_{21}}, T_{s_{12}}], [T_{s_{21}}, T_{b_1}].$$

If the commutator between $T_{s_{21}}$ and $T_{s_{12}}$ is omitted, we may obtain the following set of differential equations:

$$\begin{aligned}
i(\frac{f}{2} + xf_x) - \mu_{11} f &= \lambda_1 (iyf_x - \mu_{12} f) \\
i(\frac{f}{2} + xf_x) - \mu_{11} f &= \lambda_2 (ixf_y - \mu_{21} f) \\
i(\frac{f}{2} + xf_x) - \mu_{11} f &= \lambda_3 (-if_x - \mu_{b_1} f) \\
-if_y - \mu_{b_2} f &= \lambda_4 (iyf_x - \mu_{12} f)
\end{aligned} \tag{29}$$

where $\mu_{ij} = \mu_f(T_{s_{ij}})$. Selecting all λ's to be zeros, a possible solution for this system is: $f(x,y) = x^{-i\mu_{11} - \frac{1}{2}} e^{i\mu_{b_2} y}$. This solution, however, does not belong to L^2. If we allow λ_3 to be non-zero, we may obtain a solution of the form $f(x,y) = (\lambda_3 + x)^{-\frac{1}{2} - i\mu_{11} + i\lambda_3 \mu_{b1}}$. The L^2 constraint can be obtained by selecting: $|\lambda_3| \geq \frac{1}{2\mu_{b_1}}$.

5 Scale-Space Nature of the Uncertainty Principle Minimizers

It is well known that the Gaussian function has an important role in the scale-space framework. When a Gaussian is convolved with an image, the result is a smoother version of the original image. The degree of smoothness is determined by the standard deviation of the Gaussian in either the x, y or both directions. In the latter case, the spread does not have to be identical in both dimensions.

The Gaussian function is also the minimizer of the uncertainty related to the Weyl-Heisenberg group. In fact, we obtain as the minimizer a one-parameter family of functions: The Gaussian with parameter $t = \sigma^2/2$. This one-parameter family is a semi-group with respect to the convolution, i.e. the convolution of two Gaussians with different values of t_1 and t_2 is equivalent to a Gaussian with parameter $t_1 + t_2$. This is all very well known of course. The interesting question is whether the minimizer of the uncertainty relations of other groups depends on parameters such that it forms a semi-group with respect to convolution. We consider here the minimizers of the uncertainties related to the $SIM(2)$ group and to the affine group.

The solution offered by Dahlke and Maass is scale-space by nature. The minimizer that they found is the Mexican hat function: $\psi(x, y) = \beta(1 - \beta r^2)e^{-\beta r^2}$, where $r := \sqrt{x^2 + y^2}$. Its Fourier transform is $\pi^2 k^2 e^{-\frac{\pi^2 k^2}{\beta}}$. Clearly, if we define $\beta = 1/t$ then the semi-group property is trivially satisfied, with t as the semi-group parameter. Note that this is a scale-space of *edge detector* and not of the image as usual. It is in fact an element of the jet-space of the traditional Gaussian scale-space.

The rest of this section is devoted to exploring the scale-space nature of the minimizer given by Ali, Antoine and Gazeau for the uncertainty related to the $SIM(2)$ group [1]. Their solution is given in the frequency space (k_x, k_y). It is a function which vanishes outside some convex cone in the half-plane $k_x > 0$ and is exponentially decreasing inside:

$$\hat{\psi}_{s\lambda}(\boldsymbol{k}) = c|\boldsymbol{k}|^s e^{-\lambda k_x}, \tag{30}$$

where $s > 0$ and $\lambda > 0$. It is quite obvious, from the mere definition of the function, that successive applications of the filters with two values of either s or λ correspond to a single application of an effective parameter: $\hat{\psi}_{s_1\lambda_1}\hat{\psi}_{s_2\lambda_2} = \hat{\psi}_{(s_1+s_2)(\lambda_1+\lambda_2)}$. Moreover, this function has the following properties: The portion $|\boldsymbol{k}|^s = (k_x^2 + k_y^2)^{\frac{s}{2}}$ in frequency space is the transformation (up to a sign) of the Laplacian operator in the spatial space $:\Delta^{\frac{s}{2}}$, and thus can be considered as an edge enhancement operator. The portion $e^{-\lambda k_x}$ can be considered as a directional smoothing operator.

We look first at the one-dimensional equivalent of the solution of Ali, Antoine and Gazeau [1], which is known as the Cauchy wavelets [7,8]: $\hat{\psi}(\xi) = c\xi^s e^{-\lambda\xi}$ for $\xi \geq 0$ and $\hat{\psi}(\xi) = 0$ for $\xi < 0$, and $s > 0$. Their application to a rectangular pulse function (Fig. 1) provides the following results: as s increases, the edges become more evident, thus the edge is enhanced, while as λ increases, the signal becomes smoother (Fig. 2).

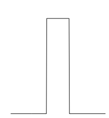

Fig. 1. A one-dimensional rectangular pulse function.

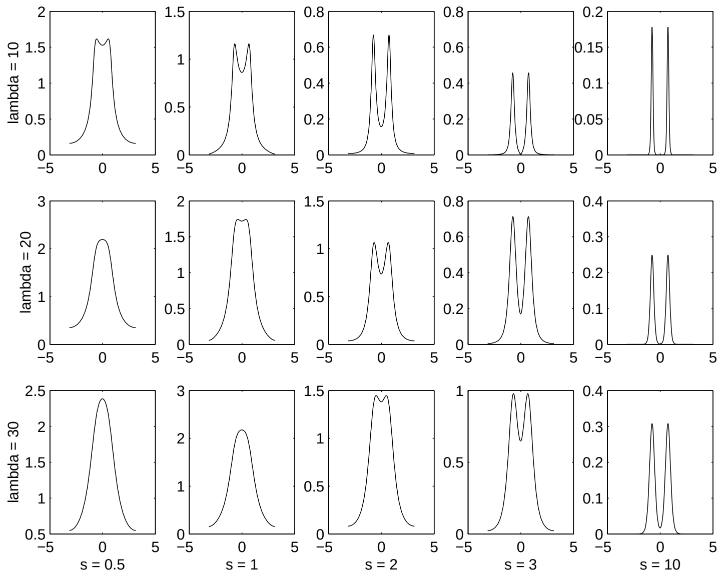

Fig. 2. When the $1D$ Cauchy wavelets are applied to a rectangular pulse, the larger s is the more noticeable the edges are (left to right). The larger λ is the smoother the edges become (up to bottom).

We next apply the two-dimensional minimizer filter to a test image of a clown, symmetrizing the filters as follows: $\hat{\psi}(\hat{k}) = c|\boldsymbol{k}|^s e^{-\lambda|k_x|}$. When the value of λ is kept constant, increasing s results in a progressive edge enhancement (Fig. 3 1st row). When the value of s is kept constant and the value of λ is increased, there is a motion blurring effect in the x-direction (Fig. 3 2nd row).

6 Discussion and Conclusions

In this work we study the possibility of designing a window shape that is optimal with respect to all the possible parameters of the two-dimensional affine transform. The study is based on minimizing the uncertainty relations that are inherent in the non-commutative affine symmetry. We generalized ideas and techniques that were used by Dahlke and Maass [3] and Ali, Antoine and Gazeau [1] for lower dimensional groups.

Our study shows that there is no function that minimizes the uncertainty with respect to all parameters of the affine transformations. We were able to show, though, the existence of an L^2 window that minimizes a subset of the commutation relations.

Moreover, the scale-space properties of the minimizer offered by Ali, Antoine and Gazeau, are considered. We find that the two-parameter minimizer family is

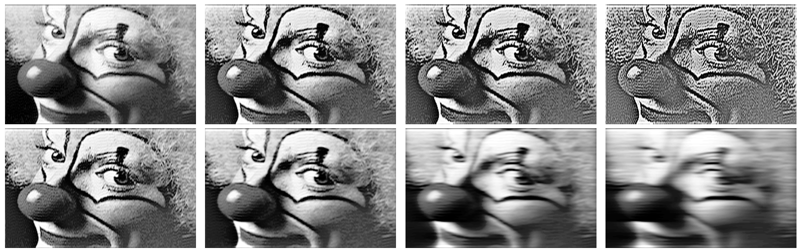

Fig. 3. 1st row: For a constant value of $\lambda = 0.00001$, increasing the value of s, $0.01, 0.2, 0.5, 1$ (left to right), results in edge enhancement. 2nd row: For a constant value of $s = 0.2$, increasing the value of λ is increased: $0.001, 0.01, 0.05, 0.1$ (left to right) results in a effect of motion-blurring in the x-direction.

a semi-group with respect to each parameter and that modifying the function's parameters results in either edge enhancement or motion-like blurring.

Our preliminary results point to the need to further explore the scale-space attributes of uncertainty minimizers.

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