Lectures on random nodal portraits

Mikhail Sodin
School of Mathematics
Tel Aviv University
Tel Aviv 69978, Israel
sodin@post.tau.ac.il

Abstract

These are lecture notes for a mini-course given at the St. Petersburg Summer School in Probability and Statistical Physics (June, 2012). Their theme was statistics of the number of connected components of the zero sets of random functions of several real variables.

The results presented in these lectures were obtained in joint works with Fedor Nazarov.

Introduction

Statistics of the number of connected components of the zero sets of random functions of several real variables is an area with a wealth of challenging and difficult questions and with very few advances. The principal difficulty in studying the number of connected components of a random set is a "non-locality" of that number, in contrast to, say, the volume and the Euler characteristics.

One of the reasons for the recent interest in this area is a remarkable bond percolation model proposed by Bogomolny and Schmit [3] for the description of the zero sets of smooth random functions of two variables that satisfy the Helmholtz equation $\Delta F + \kappa^2 F = 0$. Their model is very far from being rigorous.

Another reason for the recent interest comes from the fact that the question that we are studying can be viewed as a statistical version of the first part of Hilbert's 16th problem, see a letter of Sarnak [19] and recent works of Gayet and Welschinger [7] and of Lerario and Lundberg [12].

These lectures are based on results obtained in recent joint works with Fedor Nazarov [16, 17]. These results have two versions. The first one treats zeroes of translation-invariant smooth Gaussian functions on the Euclidean space restricted to domains of large volume. The second one deals with various ensembles of real-valued algebraic and trigonometric polynomials of large degree on the sphere and on the torus, and more generally, with ensembles of smooth Gaussian functions on Riemannian manifolds. The Euclidean version is free from many technical details related to geometry. For this reason, it is easier to formulate and to prove. It is also one of the main ingredients in our approach to the more complicated Riemannian version.

Acknowledgments

For several years, I have been enjoying collaboration with Fedya Nazarov on the topics of these lectures. I am very grateful to him for this collaboration. On many occasions, Boris Tsirelson helped us providing information concerning Gaussian measures. I have had useful discussions with Dima Belyaev, Andrei Okounkov, Leonid Polterovich, Zeév Rudnick, Peter Sarnak, Steve Zelditch, Jean-Yves Welschinger. Alex Barnett and Maria Nastasescu provided me with beautiful and inspiring simulations of nodal portraits. Andrei Iacob helped me to copy-edit these notes. I thank all of them.

This work was supported by grants no. 2006136 of the United States - Israel Binational Science Foundation and no. 166/11 of the Israel Science Foundation of the Israel Academy of Sciences and Humanities.

1 The Euclidean case

1.1 Gaussian functions with translation-invariant distribution

Suppose $F\colon \mathbb{R}^m \to \mathbb{R}^1$ is a smooth Gaussian random function with translation-invariant distribution. Translation invariance means that for any $k \in \mathbb{N}$, any $u_1, ..., u_k \in \mathbb{R}^m$, and any $v \in \mathbb{R}^m$, the random vectors $\big(F(u_1), ..., F(u_k)\big)$

and $\big(F(u_1+v), ..., F(u_k+v)\big)$ have the same multivariate normal distribution. Then the covariance kernel of F depends only on the difference of the variables, i.e., there exists a function $k\colon \mathbb{R}^m \to \mathbb{R}^1$ such that

$$\mathcal{E}\,\{F(u)F(v)\} = k(u-v)\,;$$

here and everywhere below, $\mathcal{E}$ denotes the expectation. Since the function k is Hermitian positive definite and real valued, it is represented by the Fourier integral

$$k(u) = \int_{\mathbb{R}^m} e^{2\pi \mathrm{i} u\cdot\lambda}\,\mathrm{d}\rho(\lambda)\,,$$

where ρ is a positive finite measure symmetric with respect to the origin is called *the spectral measure* of the function F. In principle, the spectral measure contains all information about the random function F, and it is often convenient to parameterize smooth translation-invariant Gaussian functions by their spectral measures.

Usually, we tacitly assume that the function F is *normalized*, that is, $k(0) = \mathcal{E}|F(u)|^2 = 1$. Then the measure ρ is a probability measure.

1.2 A result

For a smooth Gaussian function F, we denote by $N(R;F)$ the number of connected components of the zero set $Z(F) = F^{-1}\{0\}$ that are contained in the open ball $B(R) = \{x\colon |x| < R\}$. We are interested in the asymptotic behaviour of the random variable $N(R;F)$ as $R \to \infty$.

We say that a finite complex-valued measure μ on $\mathbb{R}^m$ is *Hermitian* if for each bounded Borel set $E \subset \mathbb{R}^m$, we have $\mu(-E) = \overline{\mu(E)}$. By $\widehat{\mu}$ we denote the Fourier integral of the measure μ, and by $\mathrm{spt}(\mu)$ we denote the (closed) support of μ.

The following theorem gives a version of the Law of Large Numbers for the random variable $N(R;F)$.

Theorem 1. *Suppose that the spectral measure ρ satisfies the following conditions:*

(ρ1) *ρ has no atoms;*

(ρ2) *for some $p > 4$,*

$$\int_{\mathbb{R}^m} |\lambda|^p\,\mathrm{d}\rho(\lambda) < \infty\,;$$

(ρ3) spt(ρ) *does not lie in a linear hyperplane.*
Then there exists a constant $\nu(\rho) \geqslant 0$ *such that*

$$\lim_{R\to\infty} \frac{N(R;F)}{\operatorname{vol} B(R)} = \nu(\rho) \quad \text{a.s. } \textit{and} \text{ in mean.} \tag{1}$$

Furthermore, the limiting constant $\nu(\rho)$ *is positive provided that*

(ρ4) *there exist a finite compactly supported Hermitian measure* μ *with* $\operatorname{spt}(\mu) \subset \operatorname{spt}(\rho)$ *and a bounded domain* $D \subset \mathbb{R}^m$ *such that* $\widehat{\mu}\big|_{\partial D} < 0$ *and* $\widehat{\mu}(u_0) > 0$ *for some* $u_0 \in D$.

1.2.1 Role of conditions (ρ1) – (ρ3)

Condition (ρ1) yields ergodicity of the action of $\mathbb{R}^m$ by translations on $C^2(\mathbb{R}^m)$ endowed with the probability measure generated by F, which, in turn, implies that the limit in (1) is non-random. Condition (ρ2) guarantees C^2-smoothness of the function F. At last, condition (ρ3) yields the non-degeneracy of the distribution of the gradient ∇F.

1.2.2 How to check condition (ρ4)?

There are two simple and crude sufficient conditions, which hold in many examples:

(ρ5*a*) spt(ρ) *has a non-empty interior.*

(ρ5*b*) spt(ρ) *contains a sphere centered at the origin.*

In the first case, using a duality argument, we see that finite exponential sums

$$\sum_{\lambda\in\operatorname{spt}(\rho)} c_\lambda e^{2\pi \mathrm{i}\lambda\cdot x}, \quad c_{-\lambda} = \overline{c_\lambda}\,,$$

span the space $C(\bar{B})$ for any ball $B \subset \mathbb{R}^m$. Then one can find a finite linear combination of point masses on spt(ρ), which satisfies condition (ρ4). In the second case, we can take the Lebesgue measure on the sphere; its Fourier integral is radially symmetric and vanishes on concentric spheres with radii tending to infinity.

Combining these two ideas, one can show that condition

(ρ5*c*) spt(ρ) *contains an open subset of a sphere centered at the origin*

also ensures condition (ρ4).

1.2.3 What can be said about the constant $\nu(\rho)$?

Unfortunately, our proof of Theorem 1 does not specify the value of the constant $\nu(\rho)$, and there is a huge discrepancy between the lower bounds that can be extracted from the "barrier method" introduced in [16], and the upper bounds obtained by computing the mean number of critical points, cf. Nastasescu's undergraduate thesis [15].

According to the Bogomolny and Schmit prediction [3], in the case when $m = 2$ and ρ is the restriction of the Lebesgue measure to the unit circle,

$$\nu = \frac{3\sqrt{3} - 5}{\pi} \approx 0.0624\,.$$

Recent Konrad's numerical thesis [10] gives a smaller value 0.0596 within 1% of accuracy.

We do not have any clue to an answer to the following intriguing question:

Question 1. *What can be said about the function $\rho \mapsto \nu(\rho)$ in the case when the spectral measure ρ is radial?*

1.2.4 Malevich's work

We are aware of one rigorous result that is directly related to Theorem 1. This is a pioneering work of Malevich [13]. She considered a C^2-smooth translation-invariant Gaussian random function F on $\mathbb{R}^2$ with *positive* covariance function with certain decay at infinity. She proved that $\mathcal{E}N(R; F)/R^2$ is bounded from below and from above by two positive constants. Her proof uses Slepian's inequality and probably cannot be immediately extended to models with covariance functions that change their signs.

1.3 Proof of Theorem 1

First, assuming conditions $(\rho 1) - (\rho 3)$, we show that the random variable $N(R; F)/\operatorname{vol} B(R)$ has a non-random limit $\nu(\rho) \geqslant 0$ both a.s. and in the mean. After that, we show that condition $(\rho 4)$ yields positivity of the limiting value $\nu(\rho)$.

1.3.1 Integral-geometric sandwich

As we already mentioned, the number of connected components is not a "local characteristics". Nevertheless, integral geometry is still helpful. For a closed set $\Gamma \subset \mathbb{R}^m$, we denote by $N(x, r; \Gamma)$ the number of connected components of Γ that are contained in the open ball $B(x, r)$, and by $N^*(x, r; \Gamma)$ the number of components of Γ that intersect the closed ball $\bar{B}(x, r)$. For $x = 0$, we denote the corresponding quantities simply by $N(r; \Gamma)$ and $N^*(r; \Gamma)$.

Lemma 1. *For* $0 < r < R$,

$$\int_{B(R-r)} \frac{N(u, r; \Gamma)}{\operatorname{vol} B(r)} \, \mathrm{d}u \leqslant N(R; \Gamma) \leqslant \int_{B(R+r)} \frac{N^*(u, r; \Gamma)}{\operatorname{vol} B(r)} \, \mathrm{d}u \,.$$

Proof of Lemma 1: Let $\gamma \subset B(R)$ be a connected component of Γ. Put

$$G_*(\gamma) = \bigcap_{v \in \gamma} B(v, r) = \big\{u \colon \gamma \subset B(u, r)\big\},$$
$$G^*(\gamma) = \bigcup_{v \in \gamma} \bar{B}(v, r) = \big\{u \colon \gamma \cap \bar{B}(u, r) \neq \varnothing\big\}.$$

Therefore, $\operatorname{vol} G_*(\gamma) \leqslant \operatorname{vol} B(r) \leqslant \operatorname{vol} G^*(\gamma)$. Summing over all components γ in $B(R)$, we get

$$\sum_\gamma \operatorname{vol} G_*(\gamma) \leqslant N(R; \Gamma) \operatorname{vol} B(r) \leqslant \sum_\gamma \operatorname{vol} G^*(\gamma) \,.$$

Changing the order of the sums and of the integrals representing the volumes, and then dividing by $\operatorname{vol} B(r)$, we get the result. □

1.3.2 Elaborating the sandwich estimate

Applying Lemma 1 to the zero set of F, we get

$$\int_{B(R-r)} \frac{N(u, r; F)}{\operatorname{vol} B(r)} \, \mathrm{d}u \leqslant N(R; F) \leqslant \int_{B(R+r)} \frac{N^*(u, r; F)}{\operatorname{vol} B(r)} \, \mathrm{d}u \,.$$

We will use this two-sided estimate in the double limit when $R \to \infty$ and then $r \to \infty$.

Denote by $\mathfrak{N}(u,r;F)$ the number of critical points of the restriction $F\big|_{\partial B(u,r)}$ of the function F to the sphere $\partial B(u,r)$. Then

$$N^*(u,r;F) - N(u,r;F) \leqslant \mathfrak{N}(u,r;F)\,.$$

In dimension two, this is obvious since $N^*(u,r;F) - N(u,r;F)$ does not exceed the number of zeroes of the restriction of F to the circle $\partial B(u,r)$, which, in turn, does not exceed the number of critical points of this restriction. In dimensions three and higher, a similar argument also works (though requires some basic algebraic topology).

Now, let us introduce the notation $(\tau_v F)(u) = F(u+v)$ for the shift by $v \in \mathbb{R}^m$, and rewrite the sandwich estimate in the following form:

$$\begin{aligned}\Big(1-\frac{r}{R}\Big)^m \frac{1}{\operatorname{vol} B(R-r)} \int_{B(R-r)} \frac{N(r;\tau_u F)}{\operatorname{vol} B(r)}\,\mathrm{d}u &\leqslant \frac{N(R;F)}{\operatorname{vol} B(R)} \\ \leqslant \Big(1+\frac{r}{R}\Big)^m \frac{1}{\operatorname{vol} B(R+r)} &\int_{B(R+r)} \frac{N(r;\tau_u F) + \mathfrak{N}(r;\tau_u F)}{\operatorname{vol} B(r)}\,\mathrm{d}u\,.\end{aligned}$$

The next idea is fairly straightforward: we let $R \to \infty$ and apply the ergodic theorem to the LHS and RHS of this estimate.

1.3.3 Ergodicity

We will use Wiener's multi-dimensional version of Birkhoff's ergodic theorem [24, Theorem II$''$].

Theorem 2 (Wiener). *Suppose $\big(\Omega, \mathfrak{S}, \mathcal{P}\big)$ is a probability space, on which $\mathbb{R}^m$ acts by measure-preserving transformations $\big\{\tau_v\big\}_{v\in\mathbb{R}^m}$. Suppose that $\Phi \in L^1(\mathcal{P})$, and that the function $(v,\omega) \mapsto \tau_v\Phi$ is measurable on the product space $\mathbb{R}^m \times \Omega$. Then the limit*

$$\lim_{R\to\infty} \frac{1}{\operatorname{vol} B(R)} \int_{B(R)} \Phi(\tau_v\omega)\,\mathrm{d}v = \bar{\Phi}(\omega)$$

exists with probability 1 *and in $L^1(\mathcal{P})$. The limiting random variable $\bar{\Phi}$ is τ-invariant, which means that for each $v \in \mathbb{R}^m$, $\bar{\Phi} \circ \tau_v = \bar{\Phi}$.*

Recall that the action τ of $\mathbb{R}^m$ is called *ergodic* if for each τ-invariant set $A \in \mathfrak{S}$, either $\mathcal{P}(A) = 0$, or $\mathcal{P}(A) = 1$. In this case, the limiting random variable $\bar{\Phi}$ is a constant function. Due to the $L^1(\mathcal{P})$-convergence, the value of this constant equals the expectation of Φ: $\bar{\Phi} = \mathcal{E}\{\Phi\}$.

Now, let F be a Gaussian function satisfying the assumptions of Theorem 1. By the moment assumption ($\rho 2$), F is C^2-smooth with probability 1. Hence, it generates a measure γ_F on $\big(C^2(\mathbb{R}^m), \mathfrak{S}\big)$, where $\mathfrak{S}$ is the Borel σ-algebra generated by the bounded open sets in $C^2(\mathbb{R}^m)$. That is, our probability space is $\big(C^2(\mathbb{R}^m), \mathfrak{S}, \gamma_F\big)$. Furthermore, $\mathbb{R}^m$ acts on $\big(C^2(\mathbb{R}^m), \mathfrak{S}, \gamma_F\big)$ by shifts τ_v. Since the distribution of F is translation invariant, the action is measure-preserving. Then our assumption ($\rho 1$) (that is, continuity of the spectral measure ρ) yields ergodicity. This follows from a theorem proved independently by Fomin, by Grenander, and by Maruyama.

Theorem 3 (Grenander, Fomin, Mauryama)**.** *The action of the shifts on the distribution-invariant continuous Gaussian function F is ergodic provided that the spectral measure ρ has no atoms.*

The proof given in [8, Section 5.10] after minor adjustments also works in the multivariate case.

We conclude that *under the assumptions* $(\rho 1) - (\rho 3)$ *of Theorem 1, for any random variable $\Phi \in L^1(\gamma_F)$ such that the function $(v, \omega) \mapsto \tau_v \Phi$ is measurable,*

$$\lim_{R\to\infty} \frac{1}{\operatorname{vol} B(R)} \int_{B(R)} \Phi(\tau_v \omega)\, \mathrm{d}\operatorname{vol}(v) = \mathcal{E}\{\Phi\}$$

with probability 1*, as well as in* $L^1(\gamma_F)$.

Next, we fix $r > 0$ and apply[1] this conclusion to the functions $\Phi(F) = N(r; F)$ and $\Phi(F) = \mathfrak{N}(r; F)$ in the sandwich estimate given at the very end of the previous section. We see that, *for each* $r > 0$,

$$\frac{\mathcal{E}N(r;F)}{\operatorname{vol} B(r)} \leqslant \varliminf_{R\to\infty} \frac{N(R;F)}{\operatorname{vol} B(R)} \leqslant \varlimsup_{R\to\infty} \frac{N(R;F)}{\operatorname{vol} B(R)} \leqslant \frac{\mathcal{E}N(r;F) + \mathcal{E}\mathfrak{N}(r;F)}{\operatorname{vol} B(r)}$$

almost surely, and

$$\frac{\mathcal{E}N(r;F)}{\operatorname{vol} B(r)} \leqslant \varliminf_{R\to\infty} \frac{\mathcal{E}N(R;F)}{\operatorname{vol} B(R)} \leqslant \varlimsup_{R\to\infty} \frac{\mathcal{E}N(R;F)}{\operatorname{vol} B(R)} \leqslant \frac{\mathcal{E}N(r;F) + \mathcal{E}\mathfrak{N}(r;F)}{\operatorname{vol} B(r)}\,.$$

Our next step is to to get rid of the term $\mathcal{E}\mathfrak{N}(r;F)$ on the RHS. This will yield existence of the limit (1) in Theorem 1.

[1] Here and in what follows, we skip verification of measurability.

1.3.4 Kac-Rice premise

To show that $\mathcal{E}\mathfrak{N}(r;F) = O(r^{m-1})$ as $r \to \infty$, we use a classical tool devised by Kac and Rice [1, Chapter 11], [2, Chapter 6].

For Gaussian vectors X and Y, we denote by $\mathtt{Cov}[X,Y]$ their covariance matrix, that is, $\mathtt{Cov}[X,Y]_{i,j} = \mathcal{E}\{X_i Y_j\}$. For a function $g\colon \bar{B} \to \mathbb{R}^m$, we denote by $n(\bar{B};g)$ the cardinality of its zero set $Z(g) = g^{-1}\{0\}$. If g is a C^1-function, then $|Dg(x)|$ denotes the Hilbert-Schmidt norm of its derivative $Dg(x)$, i.e., $|Dg(x)|^2 = \sum_{i,j=1}^m |\partial_{x_i} g_j(x)|^2$.

Lemma 2. *Suppose that $B \subset \mathbb{R}^m$ is a ball and $g\colon \bar{B} \to \mathbb{R}^m$ is a Gaussian $C^1(\bar{B})$-function. Then*

$$\mathcal{E}\left\{n(\bar{B};g)\right\} \lesssim \sup_{\bar{B}} \frac{(\mathcal{E}|Dg|^2)^{\frac{m}{2}}}{\sqrt{\det \mathtt{Cov}[g,g]}} \cdot \mathrm{vol}(B)\,.$$

Sketch of the proof of Lemma 2: Given a (non-random) C^1-function $g\colon \bar{B} \to \mathbb{R}^m$, and given $\epsilon > 0$ and $\delta > 0$, we put

$$X(\epsilon,\delta;g) = \left\{x \in \bar{B}\colon |g(x)| < \delta(|Dg(x)| + \epsilon)\right\}.$$

It is easy to see that if the function g vanishes at the point z and $\delta < \delta_0$ with $\delta_0(\epsilon, g)$ sufficiently small, then $B(z,\delta) \cap \bar{B} \subset X(\varepsilon,\delta)$. We conclude that if the zero set $Z(g)$ contains n different points, then $n \lesssim \varliminf_{\delta\to 0} \delta^{-m}\,\mathrm{vol}\,X(\varepsilon,\delta;g)$. That is,

$$\begin{aligned}
\mathcal{E}\left\{n(\bar{B};g)\right\} &\lesssim \mathcal{E}\big\{\varliminf_{\delta\to 0} \delta^{-m}\,\mathrm{vol}\,X(\varepsilon,\delta;g)\big\} \\
&\lesssim \varliminf_{\delta\to 0} \delta^{-m}\mathcal{E}\big\{\mathrm{vol}\,X(\varepsilon,\delta;g)\big\} \qquad \text{(by Fatou's lemma)} \\
&\lesssim \Big(\varliminf_{\delta\to 0} \delta^{-m} \sup_{x\in\bar{B}} \mathcal{P}\big\{x\colon |g(x)| < \delta(|Dg(x)|+\epsilon)\big\}\Big) \cdot \mathrm{vol}(B)\,.
\end{aligned}$$

Then we estimate the probability on the RHS using the orthogonal decomposition[2] $Dg = \mathtt{Cov}[Dg,g]\,(\mathtt{Cov}[g,g])^{-1}\,g + h$, where the Gaussian vector h is independent of g. □

Now, denote by $x_r(\theta)$ spherical coordinates on the sphere $S_r = \partial B(r)$, and cover S_r by several closed coordinate patches parameterized by the closed

[2] a.k.a. the Gaussian linear regression and the normal correlation theorem.

unit ball $\bar{\mathbb{B}} \subset \mathbb{R}^{m-1}$. In each of these patches, we put $\widetilde{F}_r(\theta) = F(x_r(\theta))$, and apply the previous lemma to the derivative $D\widetilde{F}_r$. This yields the estimate $\mathcal{E}\mathfrak{N}(r; F) = O(r^{m-1})$ for $r \to \infty$. From this, one easily deduces the existence of the limit (1) in Theorem 1.

1.4 Proof of Theorem 1 (continuation)

It remains to show that condition (ρ4) yields positivity of the limiting constant $\nu(\rho)$. We prove that if assumption (ρ4) holds, then $\mathcal{P}\{N(r_0; F) > 0\} > 0$, and therefore, $\mathcal{E}\{N(r_0; F)\} > 0$, at least when r_0 is sufficiently big. Since we already know that, for each $r_0 > 0$, $\nu(\rho) \geqslant \mathcal{E}\{N(r_0; F)\}/\operatorname{vol} B(r_0)$, this will yield positivity of $\nu(\rho)$.

Speaking somewhat informally, this argument shows that *if bounded components of the zero set $Z(F)$ are possible at all, then they must have certain positive density.* It replaces a more explicit "barrier construction" introduced in [16].

1.4.1 A Gaussian lemma that yields positivity of $\nu(\rho)$

To see that $\mathcal{P}\{N(r_0; F) > 0\} > 0$ for some $r_0 > 0$, we use the following

Lemma 3. *Let μ be a compactly supported Hermitian measure with* $\operatorname{spt}(\mu) \subset \operatorname{spt}(\rho)$*. Then for each ball $B \subset \mathbb{R}^m$ and for each $\varepsilon > 0$,*

$$\mathcal{P}\big\{\|F - \widehat{\mu}\|_{C(\bar{B})} < \varepsilon\big\} > 0\,.$$

Proof of Lemma 3: We will use an equivalent description of the translation-invariant Gaussian function F with a given spectral measure. Consider the reproducing kernel Hilbert space $\mathcal{H}(\rho) = \mathcal{F}L^2_{\mathrm{H}}(\rho)$, which consists of Fourier integrals $\widehat{\mu}$ of measures $\mu = h\,\mathrm{d}\rho$, with a Hermitian density $h \in L^2_{\mathrm{H}}(\rho)$, $h(-x) = \overline{h(-x)}$. The space $\mathcal{H}(\rho)$ is equipped with the scalar product transferred from $L^2(\rho)$: $\langle \widehat{\mu}_1, \widehat{\mu}_2 \rangle_{\mathcal{H}(\rho)} = \langle h_1, h_2 \rangle_{L^2(\rho)}$. Take any orthonormal basis $\big\{e_k\big\}$ in $\mathcal{H}(\rho)$. Then F is represented by series

$$F(u) = \sum_k \xi_k e_k(u)$$

where the ξ_k are independent identically distributed Gaussian random variables, and the series converges in $L^2(\gamma_F)$. By a classical result of Kolmogorov[3], with probability 1, the series also converges locally uniformly in $\mathbb{R}^m$. This yields a special case of Lemma 3 for measures of the form $\mathrm{d}\mu = h\,\mathrm{d}\rho$ with $h \in L^2_{\mathrm{H}}(\rho)$,

In the general case, we approximate the measure μ in the weak topology by measures of the form $h\,\mathrm{d}\rho$ with compactly supported $h \in L^2_{\mathrm{H}}(\rho)$, and recall that for compactly supported measures, the weak convergence yields locally uniform convergence of their Fourier transforms with all derivatives. Hence, the lemma. □

Applying Lemma 3 to a measure μ from condition (ρ4), we see that, for some $r_0 > 0$, $\mathcal{P}\{N(r_0; F) > 0\} > 0$.

1.5 Some questions

We close this lecture with several questions related to Theorem 1.

Question 2. *Find the asymptotics of the variance of $N(R; F)$ as $R \to \infty$.*

It is likely that under some assumptions, similar to those of Theorem 1, the variance of the random variable $N(R; F)$ also grows as $\operatorname{vol} B(R)$.

The proof of Theorem 1 yields that under assumptions (ρ1)−(ρ4) connected components of the zero set $Z(F)$ with large diameter have zero density. This is the only thing we know about statistics of the components with large diameter, and we would like to know more. For instance, given $0 < \alpha < 1$, denote by $N_\alpha(R; F)$ the number of connected components of the zero set $Z(F)$ of diameter comparable to R^α that are contained in the ball $B(R)$.

Question 3. *Find the asymptotics of the mean $\mathcal{E}\big\{N_\alpha(R; F)\big\}$ as $R \to \infty$.*

2 The Riemannian case

Now, let (f_L) be a random parametric ensemble of smooth Gaussian functions on a smooth compact m-dimensional Riemannian manifold X without boundary, and let L be a large scaling parameter. By $N(f_L)$ we denote the number of connected components of the zero set of the function f_L. We aim to understand the asymptotic behaviour of the random variable $N(f_L)$ as

[3] Kolmogorov's theorem can be found in many textbooks on advanced probability, for instance, in M. Hairer's lecture notes [9, Theorem 3.17]

$L \to \infty$. The idea of our approach is rather simple; as in many situations, the devil is in the details. We fix an arbitrary point $x \in X$, blow up local coordinates at the point x at L times, and denote by $f_{x,L}$ the scaled random Gaussian functions. Our standing assumption is that there exists a Gaussian function F_x on $\mathbb{R}^m$ with translation-invariant distribution such that the covariance of F_x approximates well the covariance of $f_{x,L}$ when $L \to \infty$. To make things more transparent, suppose for the moment that the distribution of F_x does not depend on the point $x \in X$. If the limiting Gaussian function F satisfies assumptions of Theorem 1, we may hope that, for large enough L and R,

$$\frac{N(f_L)}{L^m \operatorname{vol}(X)} \approx \frac{N(R;F)}{\operatorname{vol} B(R)} \approx \nu(\rho)\,,$$

where ρ is the spectral measure of F and $\nu(\rho)$ is the limiting constant from Theorem 1.

If the limiting functions F_x depend on the point x, then, in a similar way, we expect that

$$\frac{N(f_L)}{L^m} \approx \int_X \bar{\nu}\, \mathrm{d}\operatorname{vol}_X\,,$$

where $\bar{\nu}(x) = \nu(\rho_x)$, and ρ_x is the spectral measure of F_x.

Note that *the limiting measure $\bar{\nu}\, \mathrm{d}\operatorname{vol}_X$ does not depend on the choice of the Riemannian metric on X, only the smooth structure on X matters.*

2.1 Setup

A convenient way to define the Gaussian ensemble (f_L) is to start with a family $\mathcal{H}_L$ of reproducing kernel Hilbert spaces of smooth real-valued functions on X. In all examples we have in mind, the spaces $\mathcal{H}_L$ are finite-dimensional and $\dim \mathcal{H}_L \to \infty$ as $L \to \infty$. By $K_L(x,y)$ we denote the reproducing kernel of the space $\mathcal{H}_L$, that is,

$$f(y) = \langle f(\,\cdot\,), K_L(\,\cdot\,,y)\rangle_{\mathcal{H}_L}, \qquad f \in \mathcal{H}_L,\ y \in X.$$

In what follows, we assume that the function $x \mapsto K_L(x,x)$ does not vanish on X, that is, there is no point $x \in X$ at which all functions in $\mathcal{H}_L$ vanish. The Hilbert space $\mathcal{H}_L$ generates a random Gaussian function

$$f_L(x) = \sum \xi_k e_k(x), \qquad x \in X,$$

where $\{e_k\}$ is an orthonormal basis in $\mathcal{H}_L$ and ξ_k are independent standard Gaussian random variables. The covariance of the Gaussian function f_L equals

$$\mathcal{E}\{f_L(x)f_L(y)\} = \sum e_k(x)e_k(y) = K_L(x,y)$$

and does not depend on the choice of the orthonormal basis $\{e_k\}$ in $\mathcal{H}_L$. Hence, the distribution of F also does not depend on the choice of the orthonormal basis.

We say that the functions f_L are *normalized* if everywhere on X, $\mathcal{E}f_L^2(x) = K_L(x,x) = 1$. Later on, we will always assume that the functions f_L are normalized. Note that in the most basic examples, including the ones we consider below, the function $x \mapsto K_L(x,x)$ is constant, so the normalization boils down to dividing by that constant.

2.2 Translation-invariant local limits

First, we transplant the functions f_L together with the kernels K_L to the Euclidean space and then blow up the local coordinates at L times. Put

$$\Phi_x = \exp_x \circ I_x \colon \mathbb{R}^m \to X\,, \qquad \Phi_x(0) = x,$$

where $\exp_x \colon T_xX \to X$ is the exponential map, and $I_x \colon \mathbb{R}^m \to T_x(X)$ is a linear Euclidean isometry. The particular choice of the isometry I_x is irrelevant for us.[4] We define the scaled covariance kernel $K_{x,L}$ at a point $x \in X$ by

$$K_{x,L}(u,v) = K_L\left(\Phi_x(L^{-1}u), \Phi_x(L^{-1}v)\right).$$

This is the covariance kernel of the scaled Gaussian functions

$$f_{x,L}(u) \stackrel{\text{def}}{=} f_L\big(\Phi_x(L^{-1}u)\big), \qquad u \in \mathbb{R}^m\,;$$

i.e., $K_{x,L}(u,v) = \mathcal{E}\{f_{x,L}(u)f_{x,L}(v)\}$.

Definition 1. *A Gaussian ensemble* (f_L) *has* translation-invariant local limits *as* $L \to \infty$*, if for a.e.* $x \in X$*, there exists a positive definite continuous even function* $k_x \colon \mathbb{R}^m \to \mathbb{R}^1$*, such that for each* $R < \infty$*,*

$$\lim_{L\to\infty} \sup_{|u|,|v|\leqslant R} |K_{x,L}(u,v) - k_x(u-v)| = 0\,.$$

[4] Moreover, the choice of the exponential mapping is not essential either. It suffices to take any smooth diffeomorphism Φ_x of a neighbourhood of the origin in $\mathbb{R}^m$ onto a neighbourhood of the point x such that $\Phi_x(0) = x$ and the differential $\mathrm{d}\Phi_x(0)$ is a linear isometry.

The limiting kernels $k_x(u-v)$ are covariance kernels of translation-invariant Gaussian functions $F_x\colon \mathbb{R}^m \to \mathbb{R}^1$. Furthermore, $k_x = \widehat{\rho}_x$ where ρ_x are probability measures on $\mathbb{R}^m$, symmetric with respect to the origin. We call the function F_x *the local limiting function* and the measure ρ_x *the local limiting spectral measure* of the family f_L at the point x.

Next, we introduce two conditions which guarantee that every limiting spectral measure ρ_x satisfies conditions (ρ2) and (ρ3) imposed in Theorem 1 that dealt with the Euclidean case.

2.3 Smoothness and non-degeneracy

Definition 2 (Separate C^3-smoothness)**.** The Gaussian ensemble (f_L) is C^3-*smooth* if, for every $R<\infty$,

$$\varlimsup_{L\to\infty} \sup\Big\{ \big|\big(\partial_u^i\partial_v^j K_{x,L}\big)(u,v)\big| \colon x\in X,\ |u|,|v|\leqslant R,\ 0\leqslant i,j\leqslant 3\Big\} < \infty\,. \quad (2)$$

Several remarks are in order:

• Note that $\partial_u^i\partial_v^j K_{x,L}(u,v) = \mathcal{E}\left\{\partial_u^i f_{x,L}(u)\partial_v^j f_{x,L}(v)\right\}$. Therefore, using the Cauchy-Schwarz inequality, we see that it suffices to verify the smoothness condition on the "diagonal" $u=v$ and $i=j$.

• By condition (2), for every $R<\infty$,

$$\varlimsup_{L\to\infty} \sup_{x\in X} \mathcal{E}\big\{\|f_{x,L}\|^2_{C^2(\bar{B}(R))}\big\} < \infty\,. \quad (3)$$

• Suppose that the kernel K_L has translation-invariant local limits at some $x\in X$. Then, by condition (2), $K_{x,L}(u,v)$ converge to $k_x(u-v)$ in the $C^{2+\alpha}$-norm for any $\alpha<1$. Hence, the second partial derivatives of k_x are α-Hölder functions for any $\alpha<1$, which, in turn yields that all limiting spectral measures ρ_x satisfy the smoothness assumption (ρ2) with any $p<6$, and the corresponding moment is controlled by the upper limit in the smoothness condition (2).

Next, we turn to non-degeneracy. Introduce the matrix $C_{x,L}$ with the entries

$$C_{x,L}(i,j) = \partial_{u_i}\partial_{v_j} K_{x,L}(u,v)\big|_{v=u}\,, \qquad 1\leqslant i,j\leqslant m\,.$$

Definition 3 (Non-degeneracy)**.** The Gaussian ensemble (f_L) is *non-degenerate* if, for every $R < \infty$,

$$\varliminf_{L\to\infty} \inf\bigl\{ |\det C_{x,L}(u)| \colon |u| \leqslant R, x \in X \bigr\} > 0\,. \tag{4}$$

Equivalently,

$$\varliminf_{L\to\infty} \inf\Bigl\{ \mathcal{E}\bigl|\langle \nabla f_{x,L}(u), \xi\rangle\bigr|^2 \colon \xi \in \mathbb{S}^{m-1},\ |u| \leqslant R,\ x \in X \Bigr\} > 0\,. \tag{5}$$

Note that if the ensemble (f_L) is C^3-smooth and has a translation-invariant local limit at some $x \in X$, then the matrix $C_{x,L}$ converges to the matrix c_x with the entries

$$c_x(i,j) = -\bigl(\partial_{u_i}\partial_{u_j} k_x\bigr)(0) = \int_{\mathbb{R}^m} \lambda_i \lambda_j \, \mathrm{d}\rho_x(\lambda)\,.$$

Therefore, in this case, the limiting spectral measures ρ_x satisfy the non-degeneracy condition ($\rho 3$) uniformly in x. That is,

$$\operatorname*{ess\,inf}_{x\in X} \inf_{\xi \in \mathbb{S}^m} \int_{\mathbb{R}^m} \bigl|\langle \lambda, \xi\rangle\bigr|^2 \, \mathrm{d}\rho_x(\lambda) > 0\,.$$

It is useful to note that these smoothness and non-degeneracy conditions hold automatically whenever the limiting spectral measure ρ does not depend on the point $x \in X$ and satisfies conditions ($\rho 2$) and ($\rho 3$), and the scaled kernel $K_{x,L}(u,v)$ converges to $k(u-v)$ together will all partial derivatives in u and v up to the third order, locally uniformly in $u, v \in \mathbb{R}^m$ and uniformly in $x \in X$. This is what we will encounter in all the examples considered below.

2.4 Main result

As above, by $\nu(\rho)$ we denote the limiting constant from Theorem 1. Put $\bar{\nu}(x) = \nu(\rho_x)$.

Theorem 4. *Suppose that (f_L) is a C^3-smooth non-degenerate Gaussian ensemble on X that has translation-invariant local limits. Suppose that the local limiting spectral measures ρ_x have no atoms. Then $\bar{\nu} \in L^\infty(X)$ and*

$$\lim_{L\to\infty} \mathcal{E}\Bigl\{ \Bigl| L^{-m} N(f_L) - \int_X \bar{\nu} \, \mathrm{d}\,\mathrm{vol}_X \Bigr| \Bigr\} = 0\,. \tag{6}$$

2.4.1 A local version of Theorem 4

Theorem 4 has a "local version", which says that the limiting constant $\nu(\rho_x)$ can be recovered by a double limit.

Theorem 5. *Under assumptions of Theorem 4, for almost every $x \in X$ and for every $\varepsilon > 0$,*

$$\lim_{R\to\infty} \lim_{L\to\infty} \mathcal{P}\Big\{ \Big| \frac{1}{\mathrm{vol}\, B(R)} N\big(x, \tfrac{R}{L}; f_L\big) - \bar{\nu}(x) \Big| > \varepsilon \Big\} = 0\,, \tag{7}$$

where $N\big(x, \frac{R}{L}; f_L\big)$ is the number of connected components of the zero set $Z(f_L)$ contained in the open ball in X centered at x and of radius R/L.

Theorem 4 can be viewed as "an integrated version" of Theorem 5.

2.5 Examples

We start with four examples illustrating Theorem 4.

2.5.1 The trigonometric ensemble

Here, $\mathcal{H}_n$ is the subspace of $L^2(\mathbb{T}^m)$ that consists of real-valued trigonometric polynomials in m variables of degree $\leqslant n$ in each of the variables:

$$\mathrm{Re}\Big[\sum_{\nu\in\mathbb{Z}^m\colon\, |\nu|_\infty \leqslant n} c_\nu e^{2\pi \mathrm{i}(\nu\cdot x)} \Big].$$

A straightforward computation shows that the covariance of this ensemble coincides with the product of m Dirichlet's kernels:

$$K_n(x,y) = \prod_{j=1}^{m} \frac{\sin\left[\pi(2n+1)(x_j-y_j)\right]}{(2n+1)\sin\left[\pi(x_j-y_j)\right]}\,.$$

We fix a point $x \in \mathbb{T}$ and put $f_{x,n}(u) = f_n(x+n^{-1}u)$ (that is, the scaling parameter L equals the degree n). Then the scaled kernel $K_n\big(x+n^{-1}u, x+n^{-1}v\big)$ converges locally uniformly in u and v, together with partial derivatives of any order, to the limiting kernel $k(u-v)$, where

$$k(u) = \prod_{j=1}^{m} \frac{\sin 2\pi u_j}{2\pi u_j}\,, \qquad u \in \mathbb{R}^m$$

is the reproducing kernel in the m-dimensional Paley-Wiener space. The limiting spectral measure is the normalized Lebesgue measure σ_m on the cube $[-1,1]^m \subset \mathbb{R}^m$. This measure obviously satisfies assumptions $(\rho 1)-(\rho 4)$ of Theorem 1. Then Theorem 4 yields convergence of $N(f_n)/n^m$ to $\nu(\sigma_m)$, both in mean and with probability one.

2.5.2 Ensemble of spherical harmonics

Here, $\mathcal{H}_n$ is the subspace of $L^2(\mathbb{S}^m)$, which consists of m-dimensional real-valued spherical harmonics of degree n, that is, of restrictions to the unit sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ of homogeneous harmonic polynomials of degree n in $m+1$ variables. The reproducing kernel for this space is well known:

$$K_n(x,y) = Q_n^m(\cos\Theta(x,y))\,,$$

where $\Theta(x,y)$ is the angle between the vectors $x, y \in \mathbb{S}^m$ (that is, $\cos\Theta(x,y) = x\cdot y$), and Q_n^m are the Gegenbauer polynomials[5] that are orthogonal on $[-1,1]$ with the weight $(1-t^2)^{\frac{m-2}{2}}$ and normalized by $Q_n^m(1)=1$. For $m=2$, these polynomials coincide with the standard Legendre polynomials.

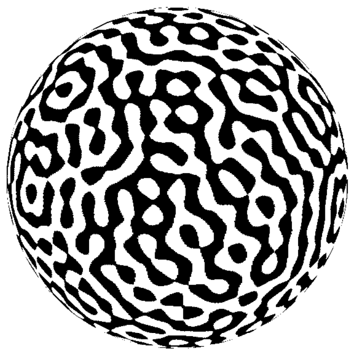

Fig. 1: Nodal portrait of the Gaussian spherical harmonic of degree 40 (figure by A. Barnett). "Elliptic regularity in action": nice boundaries, no small nodal domains, cf. the next figure.

To scale the random spherical harmonic f_n, we fix a point $x \in \mathbb{S}^m$, fix its neighbourhood $\mathcal{O}_x \subset \mathbb{S}^m$ and a neighbourhood $\mathcal{U} \subset \mathbb{R}^m$ of the origin, and

[5] a.k.a. ultraspherical polynomials.

put $f_{x,n}(u) = (f \circ \Phi_x)(n^{-1}u)$. Then the scaled covariance kernel equals

$$K_{x,n}(u,v) = Q_n^m(\cos\Theta(\Phi_x(n^{-1}u), \Phi_x(n^{-1}v)))\,.$$

Now, we observe that the angle $\Theta(\Phi_x(n^{-1}u), \Phi_x(n^{-1}v))$ is close to $n^{-1}|u-v|$ and that by the classical Mehler-Heine-type asymptotics [21, Theorem 8.1.1],

$$\lim_{n\to\infty} Q_n^m\left(\cos\frac{z}{n}\right) = c_m z^{-\frac{m-2}{2}} J_{\frac{m-2}{2}}(z)\,,$$

where J_ℓ is the Bessel function (of the first kind) of index ℓ, and the convergence is locally uniform in $\mathbb{C}$. Keeping these observations in mind, it is not difficult to show that the scaled kernel $K_{x,n}(u,v)$ converges locally uniformly in u and v, together with partial derivatives of any order, to

$$c_m|u-v|^{-\frac{m-2}{2}} J_{\frac{m-2}{2}}(|u-v|)\,,$$

which is the Fourier integral of the normalized Lebesgue measure ω_m on the sphere $\mathbb{S}^m$. Thus, ω_m is the limiting spectral measure, and conditions $(\rho1)$–$(\rho4)$ obviously hold. Then Theorem 4 yields convergence of $N(f_n)/n^m$ to $\nu(\widehat{\omega}_m)$, a.s. and with probability one.

Actually, for this ensemble we can say much more: the probability that $N(f_n)/n^m$ deviates from $\nu(\widehat{\omega}_m)$ by an arbitrary ε is exponentially small when n is large. In Section 4, we will prove this for $m=2$.

2.5.3 Another spherical ensemble

Here, $\mathcal{H}_n$ is the subspace of $L^2(\mathbb{S}^m)$ spanned by *all* polynomials in $m+1$ variables of total degree $\leqslant n$, restricted to $\mathbb{S}^m$. A known computation [14, 18] based on the Christoffel-Darboux formula shows that the reproducing kernel in $\mathcal{H}_n$ equals

$$K_n(x,y) = P_n^{(\frac{m}{2},\frac{m}{2}-1)}(\cos\Theta(x,y))\,, \qquad x,y\in\mathbb{S}^m\,,$$

where $P_n^{(\alpha,\beta)}$ denote Jacobi polynomials of degree n and of index (α,β) (i.e., polynomials orthogonal on $[-1,1]$ with the weight $(1-x)^\alpha(1+x)^\beta$), normalized by $P_n^{(\alpha,\beta)}(1)=1$. For this ensemble, the scaling is the same as in 2.5.2. Now, the Mehler-Heine-type asymptotics [21, Theorem 8.1.1] gives us

$$\lim_{n\to\infty} P_n^{(\frac{m}{2},\frac{m}{2}-1)}\left(\cos\frac{z}{n}\right) = c_m z^{-\frac{m}{2}} J_{\frac{m}{2}}(z)\,,$$

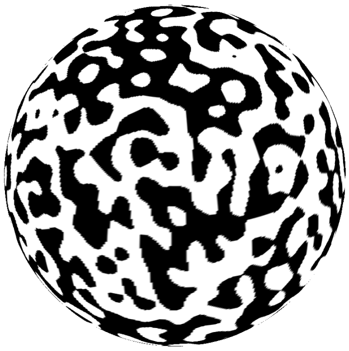

Fig. 2: Nodal portrait of Gaussian linear combination of spherical harmonic of degrees $\leqslant 40$ (figure by A. Barnett). Note some small nodal domains, cf. the previous figure.

with locally uniform convergence in $\mathbb{C}$, and the scaled kernel $K_{x,n}(u,v)$ converges locally uniformly in u and v, together with partial derivatives of any order, to

$$c_m|u-v|^{-\frac{m}{2}}J_{\frac{m}{2}}(|u-v|).$$

This is the Fourier integral of the normalized Lebesgue measure σ_m on the unit ball $\mathbb{B}^m \subset \mathbb{R}^m$. Therefore, the limiting spectral measure is σ_m, and conditions $(\rho1)$–$(\rho4)$ obviously hold. Once again, Theorem 4 yields convergence of $N(f_n)/n^m$ to $\nu(\sigma_m)$, both in mean and a.s..

This example can be extended in different directions. For instance, given a Riemannian manifold X, one can consider finite-dimensional subspaces of $L^2(X)$ spanned by the eigenfunctions of the (minus) Laplacian on X corresponding to the eigenvalues $\leqslant \lambda^2$, cf. [18]. More generally, one can consider subspaces of $L^2(X)$ spanned by the eigenfunctions corresponding to a preassigned window for eigenvalues.

2.5.4 Kostlan's ensemble

We start with a Gaussian ensemble of homogeneous polynomials of degree n in $m+1$ variables. The zero sets of these polynomials are viewed as hypersurfaces in $\mathbb{S}^m$. The corresponding Hilbert space $\mathcal{H}_n$ is endowed with the scalar product

$$\langle f, g\rangle = \sum_{|J|=n} \binom{n}{J}^{-1} f_J g_J\,, \tag{8}$$

where

$$f(X) = \sum_{|J|=n} f_J X^J, \quad g(X) = \sum_{|J|=n} g_J X^J, \qquad X^J = x_0^{j_0} x_1^{j_1} x_2^{j_2} \dots x_m^{j_m}\,,$$

and

$$J = (j_0, j_1, j_2, \dots, j_m), \quad |J| = j_0 + j_1 + j_2 + \dots + j_m, \quad \binom{n}{J} = \frac{n!}{j_0! j_1! j_2! \dots j_m!}\,.$$

The form of the scalar product (8) comes from the complexification: after the continuation of the homogeneous polynomials f and g to $\mathbb{C}^{m+1}$, up to a factor depending on n and m, it coincides with the scalar product in the Fock-Bargmann space,

$$\langle f, g\rangle = c_{n,m} \int_{\mathbb{C}^{m+1}} f(Z)\overline{g(Z)} e^{-|Z|^2}\, \mathrm{d}\,\mathrm{vol}(Z)\,.$$

It is known that the complexified Kostlan ensemble is *the only unitarily invariant* Gaussian ensemble of homogeneous polynomials. On the other hand, there are many other orthogonally invariant Gaussian ensembles, all of them having been classified by Kostlan [11].

The normalized covariance kernel of Kostlan's ensemble equals

$$\left(\frac{X \cdot Y}{|X|\,|Y|}\right)^n = (x \cdot y)^n = \cos^n \Theta(x, y)\,.$$

This is a Hermitian positive definite kernel on $\mathbb{S}^m$, and it is not difficult to check that it is the reproducing kernel in the Hilbert space

$$\widetilde{\mathcal{H}_n} = \left\{\widetilde{f}\colon \widetilde{f}(X) = |X|^{-n} f(X),\ f \in \mathcal{H}_n\right\}$$

with the scalar product borrowed from $\mathcal{H}_n$.

For Kostlan's ensemble, the choice of the scaling parameter is different from the one used in the previous examples: it is the square root of the degree, not the degree itself. We put

$$K_{x,n}(u, v) = K\big(\Phi_x(n^{-1/2}u), \Phi_x(n^{-1/2}v)\big) = \cos^n \Theta\big(\Phi_x(n^{-1/2}u), \Phi_x(n^{-1/2}v)\big).$$

Noting that $\Theta\big(\Phi_x(n^{-1/2}u), \Phi_x(n^{-1/2}v)\big)$ is close to $n^{-1/2}|u-v|$ as $n \to \infty$, we find that the scaled kernel converges to the kernel $e^{-\frac{1}{2}|u-v|^2}$ locally uniformly, together with partial derivatives of any order. Thus, the limiting spectral measure is the Gaussian measure γ_m on $\mathbb{R}^m$ with the density $\exp\big[-\frac{1}{2}|\lambda|^2\big]$, and Theorem 4 yields convergence of $n^{-\frac{m}{2}}N(f_n)$ to $\nu(\gamma_m)$ both in mean and a.s..

An interesting feature of Kostlan's ensemble is the very rapid off-diagonal decay of its covariance.

3 The Riemannian case: the proofs

First, we explain the main steps in the proof of the local theorem, and then turn to the proof of Theorem 4, which is based on the local version.

3.1 Proof of Theorem 5

We fix the point $x \in X$, and denote by F_x the corresponding limiting Gaussian function. We also fix the following parameters:

- an arbitrarily small parameter δ, which will control the probabilities of the events we discard;
- $R \geqslant 1$, which will be sent to infinity only at the very last step of the proof;
- a sufficiently big M, which controls the C^2-norms: $\mathcal{E}\|f_{x,L}\|_{C^2(\bar{B}(2R))} \leqslant M$ and $\mathcal{E}\|F_x\|_{C^2(\bar{B}(2R))} \leqslant M$.

3.1.1 Coupling

The Gaussian functions $f_{x,L}$ and F_x are defined on different probability spaces, and we only know that their covariances are close. First of all, we need *to couple* them, i.e., to find Gaussian functions $\widetilde{f}_{x,L}$ and $\widetilde{F}_x$ defined on *the same probability space*, equidistributed with $f_{x,L}$ and F_x correspondingly, and with high probability close to each other in $C^1(\bar{B}(2R))$. The C^1-error of coupling is controlled by a small parameter $\beta(\delta, R)$, whose value will be fixed later.

Lemma 4. *Given $\alpha > 0$, there exist Gaussian functions $\widetilde{f}_{x,L}$ and $\widetilde{F}_x$ defined on the same probability space and equidistributed with $f_{x,L}$ and F_x, such that, for $L \geqslant L_0(\alpha, \delta, R)$,*

$$\mathcal{E}\|\widetilde{f}_{x,L} - \widetilde{F}_x\|_{C^1(\bar{B}(2R))} < \alpha\,.$$

Sketch of the proof of Lemma 4: We fix a finite η-net in $\bar{B}(2R)$ with sufficiently small η. First, using a simple finite-dimensional linear algebra argument, we couple the restrictions of $f_{x,L}$ and F_x to this net and get the functions $\widetilde{f}_{x,L}$ and $\widetilde{F}_x$, which are close to each other on the net. Then, using once again a simple linear algebra argument, but this time an infinite-dimensional one, we extend the coupled random functions from the net to the whole ball $\bar{B}(2R)$. At last, using a priori estimates of the C^2-norm of these functions and the classical Hadamard-Landau inequality, we conclude that they close to each other in $C^1(\bar{B}(2R))$. □

Given $\beta > 0$, we introduce the event

$$\Omega_1 = \{\|f_{x,L} - F_x\|_{C^1(\bar{B}(2R))} \geqslant \beta\}\,.$$

Applying Lemma 4, we assume that the scaling parameter L is so big that $\mathcal{P}(\Omega_1) < \delta$. Then we consider the events

$$\Omega_2 = \Big\{\|f_{x,L}\|_{C^2(\bar{B}(2R))} \geqslant \delta^{-1}M\Big\}, \qquad \Omega_3 = \Big\{\|F_x\|_{C^2(\bar{B}(2R))} \geqslant \delta^{-1}M\Big\}\,.$$

Each of them has probability at most δ. Discarding these events, we assume that

$$\|f_{x,L}\|_{C^2(\bar{B}(2R))} < \delta^{-1}M\,, \qquad \|F_x\|_{C^2(\bar{B}(2R))} < \delta^{-1}M\,.$$

3.1.2 Stability

Discarding the events Ω_1, Ω_2, and Ω_3, we get smooth functions $f_{x,L}$ and F_x defined on the same probability space that are C^1-close to each other. We wish to conclude that the numbers of connected components of their zero sets are also close. Generally speaking, a C^1-perturbation of a smooth function can drastically change the topology of its zero set. The good news is that this does not happen if the function and its gradient are not simultaneously small. We call such functions *stable*. The next step is to show that, with high probability, the functions $f_{x,L}$ and F_x are stable.

Consider the event

$$\Omega_4 = \Big\{\min_{u\in\bar{B}(2R)} \max\{|f_{x,L}(u)|, |\nabla f_{x,L}(u)|\} \leqslant 2\beta\Big\}\,.$$

Lemma 5. *Given $\delta > 0$ and $R > 0$, there exist $\beta = \beta(\delta, R, M)$ and $L_0 = L_0(\delta, R)$, such that $\mathcal{P}\big(\Omega_4\big) < \delta$ provided that $L \geqslant L_0$.*

Note that discarding the event Ω_4 we also have

$$\min_{\bar{B}(2R)} \max\{|F_x|, |\nabla F_x|\} \geqslant \beta\,.$$

The proof of Lemma 5 is based on the following simple and useful observation:

- *if a C^1-Gaussian function g has a constant variance, then the random variables $g(u)$ and $\nabla g(u)$ are independent at each point u.*

Sketch of the proof of Lemma 5: We choose a finite η-net $\{y_j\}$ in $\bar{B}(2R)$ with sufficiently small $\eta = \eta(\delta, R)$. Suppose that, for some $y \in \bar{B}(2R)$, both $|f_{x,L}(y)|$ and $|\nabla f_{x,L}(y)|$ are less or equal than 2β. Then, for some point y_i of the net,

$$|f_{x,L}(y_i)| \lesssim \beta + \delta^{-1} M \eta^2\,, \qquad |\nabla f_{x,L}(y_i)| \lesssim \beta + \delta^{-1} M \eta\,.$$

Using the aforementioned independence (and non-degeneracy of the distribution of the Gaussian vector $\nabla f_{x,L}$), we estimate the probability that both events occur simultaneously, and take the union bound over the net. $\square$

3.1.3 Stable components of the zero set

Discarding events Ω_i, $1 \leqslant i \leqslant 4$, we have, for sufficiently large L,

$$\|f_{x,L} - F_x\|_{C^1(\bar{B}(2R))} < \beta\,,$$

while

$$\min_{\bar{B}(2R)} \max\{|f_{x,L}|, |\nabla f_{x,L}|\} \geqslant \beta\,, \quad \min_{\bar{B}(2R)} \max\{|F_x|, |\nabla F_x|\} \geqslant \beta\,.$$

We claim that this yields

$$N(R-1; F_x) \leqslant N(R, f_{x,L}) \leqslant N(R+1; F_x)\,. \tag{9}$$

Combined with Theorem 1, this yields Theorem 5.

To prove (9), we will use a lemma from multivariable calculus. Denote by V_{+t} an open t-neighbourhood of the set $V \subset \mathbb{R}^m$.

Lemma 6. *Fix positive α and β. Let f be a C^1-smooth function on an open ball $B \subset \mathbb{R}^m$ such that at every point $u \in B$, either $|f(u)| > \alpha$, or $|\nabla f(u)| > \beta$. Then each component γ of the zero set $Z(f)$ with* $\operatorname{dist}(\gamma, \partial B) >$

α/β is contained in an open "annulus" $A_\gamma \subset \gamma_{+\alpha/\beta}$ bounded by two smooth connected hypersufaces such that $f = +\alpha$ on one boundary component of A_γ, and $f = -\alpha$ on the other. Furthermore, "the annuli" A_γ are pairwise disjoint.

Proof of Lemma 6: With no loss of generality, we assume that $\alpha = \beta = 1$ (otherwise, we replace the function f by $\lambda_1 f(\lambda_2 u)$ with appropriate λ_1 and λ_2). We fix the component γ, as in the assumptions. Given $t \in (0,1]$, we denote by γ_t a connected component of the sublevel set $\{|f| < t\}$ that contains γ, and look at the evolution of γ_t as t grows from 0 to 1. During this evolution $|\nabla f| > 1$, therefore, the component γ_t neither merges with other components of the set $\{|f| < t\}$, nor shrinks.

Using once again that $|\nabla f| > 1$ everywhere on the component γ_t, we see that $\gamma_t \subset \gamma_{+1}$, and therefore, during the evolution γ_t cannot reach the boundary ∂B. □

As an immediate corollary, we get the needed stability of components of the zero set:

Lemma 7. *Let the function f meet assumptions of Lemma 6 and let g be any $C(B)$-function with $\sup|g| < \alpha$. Then each component γ of $Z(f)$ with $\operatorname{dist}(\gamma, \partial B) > \alpha/\beta$ generates a component $\widetilde{\gamma}$ of the zero set $Z(f+g)$ such that $\widetilde{\gamma} \subset \gamma_{+\alpha/\beta}$. Different components $\gamma_1 \neq \gamma_2$ of f generate different components $\widetilde{\gamma}_1 \neq \widetilde{\gamma}_2$ of $Z(f+g)$.*

Now, applying Lemma 7 to the functions $f = f_{x,L}$ and $g = F_x - f_{x,L}$ with $\alpha = \beta$, and then, once again, to the functions $f = F_x$ and $g = f_{x,L} - F_x$, we get (9). This completes the proof of Theorem 5. □

3.2 Proof of Theorem 4

For a.e. $x \in X$, denote by F_x the corresponding local limiting function. Then

$$\bar{\nu}(x) = \lim_{R\to\infty} \frac{\mathcal{E}\big\{N(R; F_x)\big\}}{\operatorname{vol} B(R)}\,.$$

Since $N(R; F_x)$ does not exceed the number of critical points of F_x in the ball $B(R)$, by the Kac-Rice upper bound (Lemma 2), $\bar{\nu}$ is a bounded function on X.

We need to show that

$$\lim_{L\to\infty} \mathcal{E}\left| L^{-m} N(f_L) - \int_X \bar{\nu}\, \mathrm{d}\,\mathrm{vol}_X \right| = 0\,.$$

Below, we will explain how we prove the upper bound

$$\lim_{L\to\infty} \mathcal{E}\left[L^{-m} N(f_L) - \int_X \bar{\nu}\, \mathrm{d}\,\mathrm{vol}_X \right]_+ = 0\,.$$

The proof of the lower bound is similar, but simpler, since it does not require a separation of small and long components, which we will describe below.

3.2.1 Discarding long components

Long components are the ones whose diameter is much bigger than $1/L$. They cannot be captured by our local approximation of the zero set $Z(f_L)$ by $Z(F_x)$, so we need to discard them.

Definition 4. *We call a connected component of the zero set $Z(f_L)$ D-long if $\mathrm{diam}(\gamma) > D/L$, and denote by $N_{\mathtt{D-long}}(f_L)$ the number of D-long components of the zero set $Z(f_L)$.*

Lemma 8. *For $D \geqslant 1$, we have*

$$\overline{\lim_{L\to\infty}}\, L^{-m}\, \mathcal{E} N_{\mathtt{D-long}}(f_L) \lesssim 1/D\,.$$

Sketch of the proof: We fix a $D/(4L)$-net on X of cardinality $\simeq (L/D)^m$, cover X by balls B_j of radius $D/(2L)$ centered at the points of this net, and bound the number of D-long components by the total number of critical points of the restrictions $f_L\big|_{\partial B_j}$. To estimate the mean number of critical points of these restrictions, we apply the Kac-Rice upper bound. □

3.2.2 Discarding small components

Small components are boundary components of nodal domains[6] of f_L whose volume is much smaller than L^{-m}. Ignoring small components, we will be able to control the number of components by the volume of the set they are contained in.

[6] Nodal domains of f_L are connected components of the set $\{f_L \neq 0\}$.

Definition 5. *We call a nodal domain G of the function f_L δ-small if $\mathrm{vol}_X(G) < \delta L^{-m}$. We call a connected component γ of the zero set $Z(f_L)$ δ-small if it is a boundary component of a δ-small nodal domain. We denote by $N_{\delta-\mathtt{small}}(f_L)$ the number of δ-small components of $Z(f_L)$.*

Lemma 9. *There exists a constant $c > 0$ such that*

$$\overline{\lim_{L\to\infty}}\, L^{-m}\, \mathcal{E} N_{\delta-\mathtt{small}}(f_L) \lesssim \delta^c \,.$$

To prove this lemma, we use a chain of four lemmas on (non-random) smooth functions. The starting point is Morrey's version of Poincaré's inequality [6, Section 4.5.3]:

Lemma 10. *Suppose H is a smooth function in the unit ball $\mathbb{B} \subset \mathbb{R}^m$. Then, for $q > m$,*

$$\sup_{\mathbb{B}} |H - H(0)| \leqslant C(m, q) \|DH\|_{L^q(\mathbb{B})}.$$

As a corollary, we get

Lemma 11. *Let $B \subset X$ be a ball centered at c of a sufficiently small radius. Suppose that $f \in C^2(\bar{B})$, $Df(c) = 0$, and f vanishes somewhere on the boundary ∂B. Then, for $q > m$,*

$$\|Df\|_{C(\bar{B})} \lesssim \big(\mathrm{vol}_X(B)\big)^{\frac{1}{m}-\frac{1}{q}} \|D^2 f\|_{L^q(B)}$$

and

$$\|f\|_{C(\bar{B})} \lesssim \big(\mathrm{vol}_X(B)\big)^{\frac{2}{m}-\frac{1}{q}} \|D^2 f\|_{L^q(B)} \,.$$

Next, it is convenient to introduce the notation

$$I(B) = \int_B |D^2 f|^q \,\mathrm{d}\,\mathrm{vol}_X \ .$$

The previous lemma yields the following one:

Lemma 12. *Under the assumptions of the last lemma, we have*

$$I(B)^{-s} \lesssim \big(\mathrm{vol}_X(B)\big)^t \int_B |f|^{-(1-\varepsilon)} |Df|^{-(m-\varepsilon)} \,\mathrm{d}\,\mathrm{vol}_X \,,$$

where $\varepsilon > 0$ is so small and $q > m$ is so large that the parameters

$$s = \frac{m+1-2\varepsilon}{q}\,, \quad t = (1-\varepsilon)\Big(\frac{2}{m} - \frac{1}{q}\Big) + (m-\varepsilon)\Big(\frac{1}{m} - \frac{1}{q}\Big) - 1$$

are positive.

For a smooth function f on X, we denote by $N = N_f(\delta)$ the number of nodal domains of f of volume less than δ. Note that if f has N nodal domains of volume less than δ, then we can find N disjoint balls B_j so that

- the gradient of f vanishes at the center of each ball B_j;
- the function f vanishes somewhere on the boundary of each ball B_j;
- the volume of each ball B_j is less than δ.

Then Lemma 12 allows us to estimate from above the number of these balls:

Lemma 13. *Given a positive integer m, there exist parameters $q > m$, $\varepsilon > 0$, $c > 0$, $s > 0$, such that, for any $f \in C^2(X)$,*

$$N_f(\delta) \lesssim \delta^c \Big(\int_X |D^2 f|^q \, \mathrm{d}\,\mathrm{vol}_X \Big)^{\frac{s}{s+1}} \Big(\int_X |f|^{-(1-\varepsilon)} |Df|^{-(m-\varepsilon)} \, \mathrm{d}\,\mathrm{vol}_X \Big)^{\frac{1}{s+1}}.$$

We apply this estimate to the random function f_L with δL^{-m} instead of δ. Taking the expectation, applying Hölder's inequality, and using the smoothness and non-degeneracy of the ensemble (f_L), as well as the independence of $f_L(x)$ and $Df_L(x)$, we get Lemma 9.

3.2.3 Integral-geometric estimate for normal components

We fix a small parameter δ and a large parameter D.

Definition 6. *We call the connected component γ of $Z(f_L)$ normal if it is neither δ-small, nor D-long. We denote by $N_{\mathtt{norm}}(f_L)$ the number of normal components of the zero set of f_L.*

Keeping in mind Lemma 8 and Lemma 9, it suffices to show that

$$\lim_{L\to\infty} \mathcal{E}\Big[L^{-m} N_{\mathtt{norm}}(f_L) - \int_X \bar{\nu} \, \mathrm{dvol}_X \Big]_+ = 0\,.$$

We start with a Riemannian version of the integral-geometric estimate 1.3.1. Denote by $N_{\mathtt{norm}}(x, r; f_L)$ the number of normal components of the zero set $Z(f_L)$ contained in the geodesic ball $B(x; r)$ and by $N^*_{\mathtt{norm}}(x, r; f_L)$ the number of normal components of the zero set $Z(f_L)$ that intersect the geodesic ball $B(x; r)$.

Lemma 14. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $r < \delta$,*

$$(1-\varepsilon)\int_X \frac{N_{\texttt{norm}}(x,r;f_L)}{\operatorname{vol} B(R)}\,\mathrm{d}\operatorname{vol}_X(x) \leqslant N_{\texttt{norm}}(f_L) \leqslant (1+\varepsilon)\int_X \frac{N^*_{\texttt{norm}}(x,r;f_L)}{\operatorname{vol} B(R)}\,\mathrm{d}\operatorname{vol}_X(x)\,. \quad (10)$$

Here $\operatorname{vol} B(r)$ is the Euclidean volume of the m-dimensional ball of radius r. The proof of this lemma is very close to that of Lemma 1 and we skip it.

Since the diameters of normal components do not exceed D/L, we have

$$N^*_{\texttt{norm}}(x,r;f_L) \leqslant N_{\texttt{norm}}(x,r+L^{-1}D;f_L)\,.$$

Next, we fix a sufficiently big R, put $D=\sqrt{R}$, and use the right half of estimate (10) with $r=R/L$, where L is big enough. We get

$$\frac{N_{\texttt{norm}}(f_L)}{L^m} \leqslant (1+2\varepsilon)\int_X \frac{N_{\texttt{norm}}(x,(R+D)/L;f_L)}{\operatorname{vol} B(R+D)}\,\mathrm{d}\operatorname{vol}_X(x)\,.$$

Using the left half of estimate (10), we see that the integral

$$\int_X \frac{N_{\texttt{norm}}(x,(R+D)/L;f_L)}{\operatorname{vol} B(R+D)}\,\mathrm{d}\operatorname{vol}_X(x)$$

is majorized by the total number of critical points of the function f_L. In turn, by the smoothness assumption and the Kac-Rice estimate, the expectation of the latter number is $\lesssim L^m \operatorname{vol}_X(X)$. Therefore,

$$\mathcal{E}\Big[L^{-m}N_{\texttt{norm}}(f_L) - \int_X \bar\nu\,\mathrm{d}\operatorname{vol}_X\Big]_+ \leqslant \int_\Omega\int_X \Big[\frac{N_{\texttt{norm}}(x,(R+D)/L;f_L)}{\operatorname{vol} B(R+D)} - \bar\nu(x)\Big]_+ \mathrm{d}\operatorname{vol}_X(x)\,\mathrm{d}\mathcal{P}(\omega) + O(\varepsilon)\,.$$

Thus, it remains to estimate the double integral on the RHS.

3.2.4 Completing the proof of Theorem 4

To simplify notation, we assume that $\operatorname{vol}_X(X)=1$. Put

$$\Omega_{x,R,L}(\varepsilon) = \Big\{\Big|\frac{N(x,R/L;f_L)}{\operatorname{vol} B(R)} - \bar\nu(x)\Big| > \varepsilon\Big\}\,.$$

By Theorem 5, for a.e. $x \in X$,

$$\lim_{R\to\infty}\lim_{L\to\infty}\mathcal{P}\bigl\{\Omega_{x,R+D,L}(\varepsilon)\bigr\} = 0\,.$$

Applying Egorov's theorem to this double limit, we see that, *given* $\eta > 0$, *there exists a set* $X_\eta \subset X$ *with* $\mathrm{vol}(X_\eta) > 1-\eta$, *such that the convergence is uniform on* X_η, *that is,*

$$\lim_{R\to\infty}\lim_{L\to\infty}\sup_{x\in X_\eta}\mathcal{P}\bigl\{\Omega_{x,R,L}(\varepsilon)\bigr\} = 0\,.$$

Since we have discarded δ-small components (and since $D=\sqrt{R}$),

$$N_{\texttt{norm}}(x,(R+D)/L;f_L) \lesssim \delta^{-1}\,\mathrm{vol}\,B(R+D)\,.$$

Therefore, uniformly in $\omega\in\Omega$,

$$\int_{X\setminus X_\eta}\frac{N_{\texttt{norm}}(x,(R+D)/L;f_L)}{\mathrm{vol}\,B(R+D)}\,\mathrm{d}\,\mathrm{vol}_X(x) \lesssim \eta\delta^{-1}\,,$$

and uniformly in $x\in X$,

$$\int_{\Omega_{x,R+D,L}(\varepsilon)}\frac{N_{\texttt{norm}}(x,(R+D)/L;f_L)}{\mathrm{vol}\,B(R+D)}\,\mathrm{d}\mathcal{P}(\omega) \lesssim \delta^{-1}\mathcal{P}\bigl\{\Omega_{x,R+D,L}(\varepsilon)\bigr\}.$$

The remaining integral is small by the very definition of the set $\Omega_{x,R+D,L}(\varepsilon)$:

$$\int_{\Omega\setminus\Omega_{x,R+D,L}(\varepsilon)}\int_{X_\eta}\Bigl[\frac{N_{\texttt{norm}}(x,(R+D)/L;f_L)}{\mathrm{vol}\,B(R+D)} - \bar\nu(x)\Bigr]_+\,\mathrm{d}\,\mathrm{vol}_X(x)\,\mathrm{d}\mathcal{P}(\omega) \leqslant \varepsilon\,.$$

It remains to let $L\to\infty$, $R\to\infty$, $\eta\to 0$, $\delta\to 0$, and then $\varepsilon\to 0$. $\square$

4 Random monochromatic waves

In this lecture, we will discuss the nodal portraits of two-dimensional random functions f satisfying the Helmholtz equation $\Delta f + \kappa^2 f = 0$. We consider two instances closely related to each other:

• The random ensemble of two-dimensional spherical harmonics, which we have already discussed in 2.5.2.

• The Gaussian Helmholtz waves. These are translation-invariant Gaussian functions on $\mathbb{R}^2$ whose spectral measure is the Lebesgue measure on the unit circle $\mathbb{S}^1$.

Recall that the Gaussian Helmholtz wave is the limiting function for the ensemble of two-dimensional spherical harmonics.

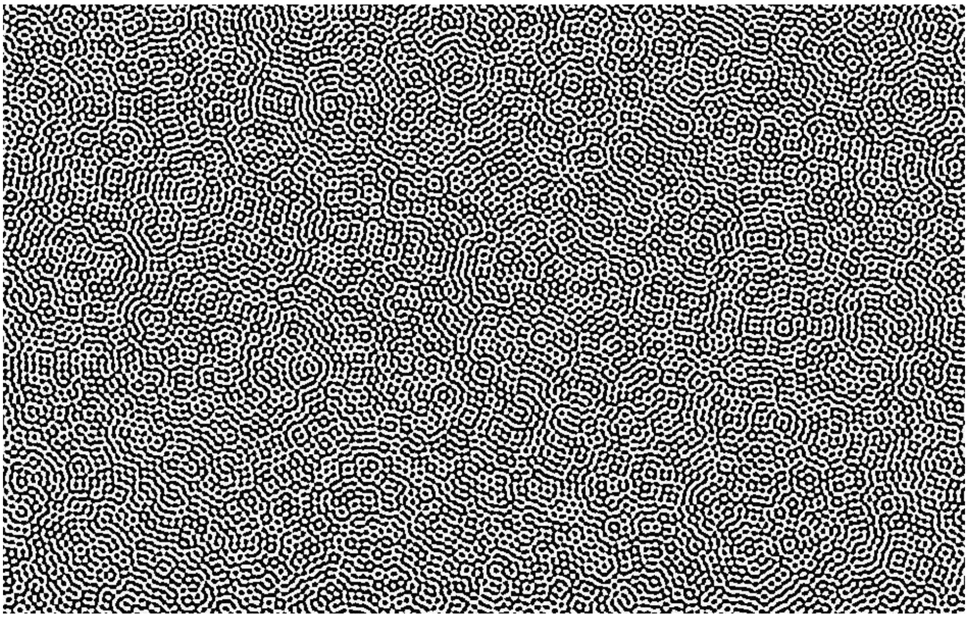

Fig. 3: Nodal portrait of the Gaussian Helmholtz wave (figure by A. Barnett)

4.1 The Bogomolny-Schmit bond percolation model

Bogomolny and Schmit studied nodal portraits of the Gaussian Helmholtz waves. Their *hypothesis* is that the distribution of nodal domains is roughly the same as in the critical bond percolation model on a square lattice. Their starting point is the square lattice in $\mathbb{R}^2$ whose length per unit area coincides with the mean length of the zero set $Z(F)$ per unit area (the latter can be readily computed). The cells of the lattice represent local maxima and minima, while the sites are saddle points with zero saddle heights. Though this picture is not realistic (a.s., the function F and its gradient ∇F cannot vanish simultaneously), nevertheless, it takes into account two important features of nodal portrait of F:

• all local maxima of the function F are positive and all local minima are negative (this follows from the Helmholtz equation);

• in large disks, the number of local maxima plus the number of local minima approximately equals the number of saddle points.

To make the model more realistic, Bogomolny and Schmit suggest to change at each site the line crossing to one of the two equiprobable avoided

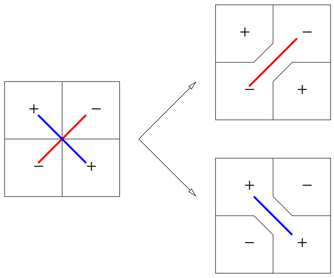

Fig. 4: Avoided nodal crossings in the Bogomolny-Schmit model

crossings, as shown in Fig 4.1. At different sites, the changes are independent.

Then Bogomolny and Schmit introduce two dual square lattices: the 'blue' one, with vertices at the cells of the grid where the function is positive, and the 'red' one, with vertices at the cells of the grid where the function is negative. Each realization of the random choice of avoided crossings generates two graphs, the blue one, whose vertices are the blue lattice points and the red one, whose vertices are the red lattice points. Two vertices are connected by an edge if the corresponding cells of the grid belong to the same nodal domain of the random function. Each of these graphs uniquely determines the topology of the whole nodal portrait (so it suffices to consider only one of them), and each of them represents the critical bond percolation on the corresponding square lattice. Then using some heuristics coming from statistical mechanics, Bogomolny and Schmit computed the limits of the mean [7],

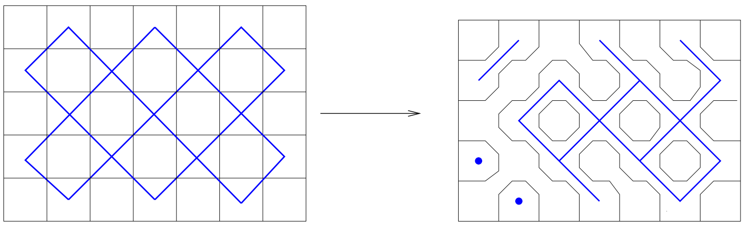

Fig. 5: Bond percolation on the 'blue' lattice

$$\lim_{R\to\infty} \frac{\mathcal{E}N(R;F)}{R^2} = \frac{3\sqrt{3}-5}{\pi} \approx 0.0624\,.$$

[7] as we have already mentioned, Konrad's thesis [10] puts some doubt about this exact value

They also argued that the growth of the variance of $N(R;F)$ is proportional to R^2 and that fluctuations of the random variable $N(R;F)$ are asymptotically Gaussian when $R \to \infty$. They concluded their work with a remarkable prediction of the power distribution law for the areas of nodal domains, based on percolation theory.

The major problem with this model is that it completely ignores correlations between values of the function F, which decay only as the distance to the power $-\frac{1}{2}$. The 'minor' problem is that there is still no rigorous mathematical treatment of the critical bond percolation on the square lattice.

Question 4. *Reveal "a hidden universality law" that provides a rigorous foundation for the Bogomolnny-Schmit work.*

It seems that, at present, we are very far from understanding this universality. We do not have answers to the following, much more basic questions:

Question 5. *Show that with probability one the zero set $Z(F)$ has no infinite component.*

Question 6. *Show that for each $\varepsilon > 0$, the probability that the set $\{x\colon F(x) > \varepsilon, |x| < R\}$ has a component of diameter bigger than εR tends to zero as $R \to \infty$.*

In one aspect, we went beyond the Bogomolny-Schmit predictions. Namely, for random Gaussian monochromatic waves, we can rigorously prove the exponential concentration of the number of connected components around its mean value. Contrary to the Bogomolny-Schmit model, this result is not particularly two-dimensional. To simplify the exposition, we restrict ourselves to the ensemble of two-dimensional spherical harmonics.

4.2 Exponential concentration

Here, $\mathcal{H}_n$ is the $2n+1$-dimensional subspace of $L^2(\mathbb{S}^2)$ consisting of the real-valued spherical harmonics of degree n on $\mathbb{S}^2$. This is the space of eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^2$ with the n-th eigenvalue $\lambda_n = n(n+1)$. An alternative definition says that the elements of this space are restrictions of harmonic homogeneous polynomials in $\mathbb{R}^3$ to the sphere $\mathbb{S}^2$. By (f_n) we denote the ensemble of random Gaussian spherical harmonics of degree n built on the space $\mathcal{H}_n$. Recall that the limiting spectral measure for this ensemble is the Lebesgue measure ω on the unit circumference

$\mathbb{S}^1 \subset \mathbb{R}^2$ and that the limiting translation-invariant function is the Gaussian Helmholtz wave. According to Theorem 4, $N(f_n)/n^2$ converges to a positive constant $\nu(\omega)$ both in mean and a.s., and according to the Bogomolny-Schmit prediction, $\nu(\omega) = (3\sqrt{3} - 5)/\pi$.

Theorem 6. *For every $\varepsilon > 0$, there exist positive constants $c(\varepsilon)$ and $C(\varepsilon)$ such that*

$$\mathcal{P}\Big\{\Big|\frac{N(f_n)}{n^2} - \nu(\omega)\Big| > \varepsilon\Big\} \leqslant C(\varepsilon)e^{-c(\varepsilon)n}\,.$$

The exponential concentration in Theorem 6 can be viewed as another manifestation of Levy's concentration of measure principle. It is particularly interesting, since the covariance of the ensemble of Gaussian spherical harmonics has a very slow off-diagonal decay.

The proof of Theorem 6 is based on the Gaussian isoperimetric inequality, which was independently found by Sudakov and Tsirelson [20] and Borell [4].

4.2.1 Gaussian isoperimetry

Put $d = 2n + 1$. It is convenient to view the Hilbert space $\mathcal{H}_n$ as a d-dimensional Euclidean space equipped with a standard Gaussian measure γ_d with $\mathcal{E}|x|^2 = 1$. As above, by $V_{+\varrho}$ we denote an open ϱ-neighbourhood of the set $V \subset \mathbb{R}^d$.

Theorem 7 (Borell, Sudakov-Tsirelson)**.** *Suppose $V \subset \mathbb{R}^d$ is a Borel set and $\Pi \subset \mathbb{R}^d$ is an affine half-space such that $\gamma_d(V) = \gamma_d(\Pi)$. Then, for every $\varrho > 0$, $\gamma_d(V_{+\varrho}) \geqslant \gamma_d(\Pi_{+\varrho})$.*

A simple computation shows that if $\gamma_d(\Pi_{+\varrho})$ is not too close to 1, then $\gamma_d(\Pi)$ must be exponentially small in d, like $\exp[-c\varrho^2 d]$. Returning to the space $\mathcal{H}_n$ of spherical harmonics, we get

Corollary 1 (Levy's concentration of Gaussian measure on $\mathcal{H}_n$)**.** *Let $V \subset \mathcal{H}_n$ be any Borel set of spherical harmonics. Suppose that the set $V_{+\varrho}$ satisfies $\mathcal{P}(V_{+\varrho}) \leqslant \frac{3}{4}$. Then $\mathcal{P}(V) \leqslant 2e^{-c\varrho^2 n}$.*

To use the concentration of measure principle, we need to show that the number $N(f)$ doesn't change too much under small perturbation of f in the $L^2(\mathbb{S}^2)$-norm. Certainly, it is not true for all $f \in \mathcal{H}_n$, but we show that the "unstable" spherical harmonics $f \in \mathcal{H}_n$ for which small perturbations can lead to a drastic decrease in the number of components of the zero set are exponentially rare.

4.2.2 Uniform lower semicontinuity of $N(f)/n^2$ outside a small exceptional set

Here is a fundamental lemma. It gives a quantitative version of estimates we were using in the course of the proof of Theorem 5.

Lemma 15 (Uniform lower semi-continuity of $N(f_n)/n^2$)**.** *For every $\epsilon > 0$, there exist $\varrho > 0$ and an exceptional set $E \subset \mathcal{H}_n$ of probability $\mathcal{P}(E) \leqslant C(\varepsilon)e^{-c(\varepsilon)n}$, such that for all $f \in \mathcal{H}_n \setminus E$ and for all $g \in \mathcal{H}_n$ satisfying $\|g\| \leqslant \varrho$, we have*
$$N(f+g) \geqslant N(f) - \varepsilon n^2.$$

A seeming asymmetry in this statement appears since we perturb non-exceptional spherical harmonics by arbitrary ones with small norm.

Theorem 6 readily follows from this lemma combined with the previous results. Indeed, denote by m_n the median of the random variable $N(f)/n^2$. By Theorem 4, we have $m_n \to \nu(\omega)$ as $n \to \infty$. Therefore, it suffices to show that
$$\mathcal{P}\Big\{\Big|\frac{N(f)}{n^2} - m_n\Big| > \varepsilon\Big\} < C(\varepsilon)e^{-c(\varepsilon)n^2}.$$
First, we consider the set $V = \big\{f \in \mathcal{H}_n \colon N(f) > (m_n + \epsilon)n^2\big\}$. Then for $f \in (V \setminus E)_{+\varrho}$ we have $N(f) > m_n n^2$, and therefore $\mathcal{P}\big((V \setminus E)_{+\varrho}\big) \leqslant \frac{1}{2}$. Hence, by the concentration of Gaussian measure, $\mathcal{P}(V \setminus E) \leqslant 2e^{-c\varrho^2 n}$, and finally
$$\mathcal{P}(V) \leqslant \mathcal{P}(V \setminus E) + \mathcal{P}(E) \leqslant 2e^{-c\varrho^2 n} + C(\varepsilon)e^{-c(\varepsilon)n} \leqslant C(\varepsilon)e^{-c(\varepsilon)n}\,.$$
Now, let us look at the set $V = \big\{f \in \mathcal{H}_n \colon N(f) < (m_n - \epsilon)n^2\big\}$. For this set, $V_{+\varrho} \subset \big\{f \in \mathcal{H}_n \colon N(f) < m_n n^2\big\} \cup E$, so that
$$\mathcal{P}(V_{+\varrho}) \leqslant \frac{1}{2} + C(\varepsilon)e^{-c(\varepsilon)n} < \frac{3}{4}$$
for large n, and it follows that $\mathcal{P}(V) \leqslant 2e^{-c\varrho^2 n}$. This proves Theorem 6 modulo Lemma 15. □

The proof of Lemma 15 goes in two steps. First, we single out the exceptional set E of unstable spherical harmonics and estimate its measure. Then we show that the number of connected components of non-exceptional spherical harmonics is stable under small perturbations.

4.2.3 Several facts about spherical harmonics

We start with a few standard facts about spherical harmonics of degree n, which will be used in the proof of Lemma 15. These facts can be derived either from the fact that they are eigenfunctions of the Laplacian on the sphere corresponding to the eigenvalue $n(n+1)$, or from the fact that they are traces of homogeneous harmonic polynomials in $\mathbb{R}^3$ of degree n on the unit sphere. Everywhere below we assume that $n \geqslant 1$.

Lemma 16 (Mean-value property). *For any $f \in \mathcal{H}_n$ and any point $x \in \mathbb{S}^2$, we have*

$$|f(x)|^2 \lesssim n^2 \int_{D(x,1/n)} f^2 .$$

Here $D(x, 1/n)$ is a spherical disc of radius $1/n$ centered at x.

Lemma 17 (Length estimate). *For any $f \in \mathcal{H}_n$ that is not identically 0, the total length of $Z(f)$ does not exceed Cn.*

The next lemma follows from the classical Faber-Krahn inequality:

Lemma 18 (Area estimate). *For any connected component G of $\mathbb{S}^2 \setminus Z(f)$, we have* $\mathrm{Area}(G) \gtrsim n^{-2}$.

Actually, it is not difficult to show that every nodal domain of f contains a disc of radius c/n.

4.2.4 Spherical harmonics with many unstable disks

Here, we use an idea similar to the one used in the proof of Theorems 5 and 4: the nodal portrait of a spherical harmonic is unstable under small perturbations only if in many different places on the sphere $\mathbb{S}^2$ the function f and its gradient ∇f are simultaneously small.

We fix small positive parameters α and δ and a large positive parameter R; all of them will be some powers of ε. Then we cover the sphere $\mathbb{S}^2$ by $\simeq R^{-2}n^2$ disks D_j of radius R/n in such a way that the concentric disks $4D_j$ with 4 times larger radius cover the sphere with a bounded multiplicity. We denote by $\mathcal{D}$ the collection of the discs D_j.

Definition 7 (unstable disks). *We call the disk D_j* stable*, if for each $x \in 3D_j$ either $|f(x)| \geqslant \alpha$, or $|\nabla f(x)| \geqslant \alpha n$. Otherwise, the disk D_j is* unstable.

Definition 8 (exceptional spherical harmonics)**.** *We call the spherical harmonic* $f \in \mathcal{H}_n$ exceptional*, if the number of unstable disks is at least* δn^2*, and denote by* E *the set of all exceptional spherical harmonics of degree* n*.*

Lemma 19. *Given* $\delta > 0$*, there exist positive small* $\alpha_0(\delta)$ *and* $c(\delta)$*, and a positive large* $C(\delta)$ *such that*

$$\mathcal{P}(E) \leqslant C(\delta) e^{-c(\delta)n}$$

provided that $\alpha \leqslant \alpha_0(\delta)$*.*

We skip the proof of this lemma (it is given in Section 4.2 of [16]), but note that, curiously enough, it uses once again the concentration of measure principle.

4.2.5 Proof of the uniform lower semicontinuity

Fix a "stable" spherical harmonic $f \in \mathcal{H}_n \setminus E$. We need to show that at most ϵn^2 components of the zero set $Z(f)$ can disappear after perturbation of f by another spherical harmonic $g \in \mathcal{H}_n$ with sufficiently small L^2-norm $\|g\| < \varrho$. First, in several steps, we identify possibly 'unstable' connected components of the zero set $Z(f)$ that can disappear after perturbation, show that their number is small compared to n^2, and discard them. Then Lemma 7 will yield that all other connected components of $Z(f)$ do not disappear after the perturbation.

• First, we discard the nodal components γ whose diameter is bigger than R/n. By the length estimate in Lemma 17, their number is $\lesssim R^{-1}n^2$, which is small compared to n^2.

With each remaining component γ of the nodal set $Z(f)$ we associate a disk D_j from the collection $\mathcal{D}$ such that $D_j \cap \gamma \neq \varnothing$. Then $\gamma \subset 2D_j$. By the area estimate in Lemma 18, the number of components γ intersecting D_j (and, thereby, contained in $2D_j$) is bounded.

• Second, we discard the components γ with unstable disks D_j. Since f is not exceptional, and since, by the area estimate, each disk D_j cannot intersect too many components contained in $2D_j$, the number of such components is also small compared to n^2.

• At last, we discard the components γ such that

$$\max_{3D_j} |g| \geqslant \alpha\,.$$

To estimate the number N of such disks, we denote by $D_j^* \subset 4D_j$ the disk of radius $1/n$ centered at the point y_j where $|g|$ attains its maximum in $3D_j$. By Lemma 16,

$$\int_{D_j^*} |g|^2 \gtrsim n^{-2}|g(y_j)| = \alpha^2 n^{-2}\,,$$

whence

$$\varrho^2 \geqslant \|g\|_{L^2(\mathbb{S}^2)} \gtrsim N\alpha^2 n^{-2}\,,$$

that is, $N \lesssim \varrho^2\alpha^{-2}n^2$. As above, using Lemma 18 and juxtaposing the areas, we conclude that the number of components γ affected by this is $\lesssim R^2 N \lesssim R^2\varrho^2\alpha^{-2}n^2$, which is much less than ϵn^2, provided that ϱ^2 is much less than $\epsilon\alpha^2 R^{-2}$.

In the three steps above, we have discarded at most εn^2 connected components of $Z(f)$. Let γ be one of the remaining components. By our construction, there is a disc D_j so that

- $\gamma \subset 2D_j$;
- $\max\big(|f|, n^{-1}|\nabla f|\big) \geqslant \alpha$ everywhere in $3D_j$;
- $|g| < \alpha$ everywhere in $3D_j$.

Then, by Lemma 7, the component γ survives when we perturb the function f by g. This completes the proof of Lemma 15. □

4.3 More questions

Question 7. *Is it possible to extend the exponential concentration result to random functions that do not solve the Helmholtz equation?*

This question is open even in the one-dimensional case, that is, for zero crossings of sufficiently smooth Gaussian stationary processes on $\mathbb{R}^1$, cf. Tsirelson's lecture notes [22]. The principal obstacle are small nodal domains.

Nothing is known about the number of connected components of the zero set for a 'randomly chosen' high-energy Laplace eigenfunction f_λ on an arbitrary compact surface X without boundary endowed with a smooth Riemannian metric g. It is tempting to expect that Theorem 6 models what is happening when X is the two-dimensional sphere $\mathbb{S}^2$ endowed with a generic Riemannian metric g that is sufficiently close (with several derivatives) to the constant one. However, the following more naïve question is widely open:

Question 8. *Suppose X is the two-dimensional sphere $\mathbb{S}^2$ endowed with a generic Riemannian metric g that is sufficiently close (with several derivatives) to the constant one. Denote by $N(f_\lambda)$ the number of connected components of the zero set of the Laplace eigenfunction f_λ on X. Is it true that $\limsup_{\lambda\to\infty} N(f_\lambda) = +\infty$?*

Instead of perturbing the "round metric" on the sphere $\mathbb{S}^2$, one can add a small potential V to the Laplacian on the " round sphere". The question remains just as hard.

References

[1] R. Adler, J. Taylor, *Random fields and geometry.* Springer Verlag, New York, 2007.

[2] J.-M. Azaïs, M. Wschebor, *Level sets and extrema of random processes and fields.* John Wiley & Sons Inc., Hoboken, NJ, 2009.

[3] E. Bogomolny, C. Schmit, *Percolation model for nodal domains of chaotic wave functions.* Phys. Rev. Letters, **88** (2002), 114102. `arXiv:nlin/0110019v1`; *Random wavefunctions and percolation*, J. Phys. A **40** (2007), 14033–14043. `arXiv:0708.4335v1`

[4] C. Borell, *The Brunn-Minkowski inequality in Gauss space.* Invent. Math. **30** (1975), 207–216.

[5] J. Bourgain, Z. Rudnick, *Restriction of toral eigenfunctions to hypersurfaces and nodal sets.* Geom. Funct. Anal. 22 (2012), 878–937. `arXiv:1105.0018`

[6] L. Evans, R. Gariepy, *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[7] D. Gayet and J.-Y. Welschinger, *Exponential rarefaction of real curves with many components.* Inst. Hautes Études Sci. Publ. Math. **113** (2011), 69–96. `arXiv:1005.3228v1`; *What is the total Betti number of a random real hypersurface?* `arXiv:1107.2288v1`; *Betti numbers of random real hypersurfaces and determinants of random symmetric matrices*, `arXiv:1207.1579v1`

[8] U. Grenander, *Stochastic processes and statistical inference*, Ark. Mat. **1** (1950), 195–277.

[9] M. Hairer, *An Introduction to Stochastic PDEs.* `http://www.hairer.org/notes/SPDEs.pdf`

[10] K. Konrad, *Asymptotic Statistics of Nodal Domains of Quantum Chaotic Billiards in the Semiclassical Limit*, Dartmouth College, 2012.

[11] E. Kostlan, *On the expected number of real roots of a system of random polynomial equations*, Foundations of computational mathematics (Hong Kong, 2000), 149-188, World Sci. Publ., River Edge, NJ, 2002.

[12] A. Lerario, E. Lundberg, *Statistics on Hilbert's Sixteenth Problem.* `arXiv:1212.3823`.

[13] T. L. Malevich, *Contours that arise when the zero level is crossed by Gaussian fields.* Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **16** (1972), 20–23.

[14] J. Marzo, *Marcinkiewicz-Zygmund inequalities and interpolation by spherical harmonics.* J. Funct. Anal. **250** (2007), 559–587. `arXiv:math/0611503`

[15] M. Nastasescu, *The number of ovals of a real plane curve.* Senior Thesis, Princeton University, 2011.

[16] F. Nazarov, M. Sodin, *On the number of nodal domains of random spherical harmonics.* Amer. J. Math. **131** (2009), 1337–1357. `arXiv:0706.2409v1`

[17] F. Nazarov, M. Sodin, *On the number of connected components of the zero sets of smooth random functions*, in preparation.

[18] L. Nicolaescu, *The blowup along the diagonal of the spectral function of the Laplacian.* `arXiv:1103.1276`

[19] P. Sarnak, Letter to B. Gross and J. Harris on ovals of random plane curves, May 2011. `http://publications.ias.edu/sarnak/paper/510`

[20] V. N. Sudakov and B. Tsirel'son, *Extremal properties of half-spaces for spherically invariant measures.* Problems in the theory of probability distributions, II. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **41** (1974), 14–24. (Russian)

[21] G. Szegö, *Orthogonal polynomials.* Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975.

[22] B. Tsirelson, *Gaussian measures.* Lecture notes. Tel Aviv University, Fall 2010.
`http://www.tau.ac.il/~tsirel/Courses/Gauss3/main.html`

[23] B. Tsirelson, *Measurability and continuity.* Lecture notes. Tel Aviv University, Fall 2012.
`http://www.tau.ac.il/~tsirel/Courses/MeasCont/main.html`

[24] N. Wiener, *The ergodic theorem*, Duke Math. J. **5** (1939), 1–18.