JUSTIFIABLE PREFERENCES

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ABSTRACT. *Justifiability* is a name for a variety of well–documented behavioral phenomena that violate rational choices. For instance, preferences and choices frequently depend on payoff–irrelevant factors, and not only on the collection of alternatives. Motivated by these findings we present a generalized ambiguity model that involves a collection of multiple priors. We characterize Knightian (Bewley (2002)) preferences that accommodate justifiability. An act f is (weakly) preferred to act g if, according to at least one set of multiple priors (which is derived by a payoff–irrelevant factor) in the collection, f is unanimously preferred to g. These kind of preferences are called *justifiable preferences.* In addition, a particular model of justifiable preferences that are generated by partially–specified probabilities is axiomatized.

Keywords: Justifiable preferences, multiple priors, multiple–multiple priors, partially–specified probabilities.

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1. Introduction

Most theoretical models that study preferences or choice problems assume that they are determined solely by the set of alternatives. However, in real–life this assumption is frequently violated. A body of data has been developed within the social psychology literature suggesting various violations of "rational" decision theory. In many cases preferences depend on payoff–irrelevant variables.

A partial list of phenomena, which will later be referred to as *justifications*, that yield such behavior includes priming (e.g., anchoring and framing), reasoning and emotions.[1] These effects and many others have an important role in molding choices. Quoting Mellers, Schwartz and Cooke (1998),

> *"Framing effects, stimulus contexts, environments, and response modes might seem innocuous, but they can profoundly shape decisions. Preferences can reverse depending on each of these factors. These effects have important implications for policy making, market decisions, and pollsters."*

Estrada, Isen and Young (1994), for example, show that emotions affect doctors' efficiency in integrating information and anchoring on earlier diagnoses. Wright and Bower (1992) and Nygren, Isen, Taylor and Dulin (1996) argue that positive feelings promote overestimation of the likelihood of favorable events, and underestimation of the likelihood of unfavorable events. Thus, depending on his emotions, a doctor would end up having different preferences when facing similar problems, each can be justified by the governing "probability" induced by the particular emotional state at that time.

There exist several strategic choice models that accommodate justifications. An extension of Nash equilibrium, *conjectural equilibrium* (Battigalli (1987)),[2] emerged from the game–theoretic literature on learning. In such an equilibrium each player receives partial information regarding the action profile played by others. This information does not reveal precisely what other players play. Rather, it provides a player with a set of possible strategies that the others might play. This set contains all the strategy profiles

[1]For a comprehensive survey of judgment and decision making see Mellers, Schwartz and Cooke (1998) and the references within.

[2]Closely related solution concepts that were invented independently are Fudenberg and Levine's (1993) self–confirming equilibrium and Kalai and Lehrer's (1993a), (1993b) subjective equilibrium. See Battgalli (1992) for an early survey of the literature on learning, and Fudenberg and Levine (1998) for equilibria and learning in games.

that are consistent with a player's knowledge about others' strategies. In equilibrium, each player plays a *best response* to one of these consistent–with–information strategies. In other words, a strategy profile constitutes a conjectural equilibrium, if every player's strategy is *justified* by a belief consistent with his information.

Justifiability phenomena are not exclusive to individuals. Such behavior can be observed also in organizations. Consider a firm that delegates responsibility to its employees. Every employee is competent and trustworthy in the eyes of the management, and considers only the benefit of the firm. All employees share common sources of relevant information, which usually provide just a partial picture. However, each agent typically has his own background, education, knowledge, gut feelings, intuition and instincts. It is quite plausible in such a situation that different agents would justify their decisions by different assessments about the real state; all are consistent with their information. It might therefore happen that several agents would choose f over g, and others would make the opposite choice. This can also occur when choosing between g and h. Furthermore, it is possible that every agent would choose h over f. To an outsider who observes the conduct of the firm (and not the agents), it seems that the firm is indifferent between f and g, and also between g and h. However, it seems that h is strictly preferred to f. In this case the firm's observable preferences appear intransitive and perhaps irrational. However, when confronted, these "preferences" are justifiable: management can justify every particular decision made, and back up its competent employees.

The main contribution of this paper, motivated by the psychology literature, is the presentation of the notion of justifiability within classical decision theory. In an Anscombe–Aumann (1963) framework we provide an axiomatization of preference relations that accommodate justifiability.[3] This paper joins the vast literature of non–expected utility attempting to explain behavioral evidence (as discussed above), which violates expected utility maximization.

A prominent illustration of such behavior is provided by Ellsberg (1961), which shows that partially informed decision makers typically do not adopt a unique prior that rationalizes their choices, and therefore, do not adhere to expected utility theory (Savage (1954) and Anscombe–Aumann (1963)). This observation raises the question of how decision makers perceive uncertainty. The literature offers several alternatives for this

[3]In Section 6.2 we point out to alternative frameworks.

issue. One way to model perceived uncertainty is by a convex closed set of probability distributions – the multiple priors model. This set reflects the ambiguity of the decision maker generated by the partial information he obtained.

Bewley's (2002) Knightian decision theory suggests that an agent prefers f to g if the expected utility induced by f is greater than that induced by g, with respect to every prior distribution in a set of priors that describes the agent's perception of ambiguity. Consider, for example, a physician that treats a certain disease. He usually resorts to the traditional treatment f, unless a new optional treatment g, that seems to dominate f with respect to all parameters, comes up. Not switching to g once, implies that the Bewley type of physician will never switch to g. This pattern of behavior does not accommodate the findings of the social psychology literature discussed above. According to the justifiability phenomena, it is plausible that in similar situations, different circumstances would generate different justifications, that in turn induce different sets of prior distributions and therefore different decisions. It is possible that f would be preferred to g in light of one justification, whereas g would be preferred to f when another justification governs the choice problem.

Justifiability introduces a natural extension of the classical ambiguity model of multiple priors. The decision maker is provided with a collection of sets of probability distributions. This is a *multiple–multiple priors* model that reflects hyper–ambiguity: not only that the decision maker is endowed with several priors, he is endowed with several sets of priors.[4]

Our main result axiomatize Knightian preferences that adhere to justifications and incorporate multiple–multiple priors. More formally, we characterize a binary relation $\succeq$ over acts, such that there exist a vN–M utility function u and a *collection of closed and convex sets of probability distributions* $\mathcal{P}$ over the state space, where $f \succeq g$ if and only if there exists $P \in \mathcal{P}$ such that, with respect to every $p \in P$, the expected value of $u(f)$ is at least as high as that of $u(g)$. That is, f is weakly preferred to g if with respect to at least one set of priors (induced by some justification) in $\mathcal{P}$, f dominates g in the Knightian sense. We refer to such preferences as *justifiable preferences*.

This result is a first step towards a new approach to decision making. The incorporation of Knightian decision theory to justifiability should be thought of as an initial

[4]Multiple–multiple priors appear independently in Nascimento and Riella (2009) in a different context.

process similar to that of deletion of strictly dominated strategies in a game–like environment.[5] As in the standard Bewley (2002) model, it is unclear how an agent resolves incomparability when facing a decision problem governed by some justification.

It should be emphasized that our model is by no means normative. We do not purport to advise decision makers what decisions to take or what axioms to adopt. The preference relation we study should not be interpreted as the true preferences of the decision maker (individual or organization), as they lack a comprehensive justification. However, they can be locally justified: any instance of the preference relation can be rationalized by at least one belief. The particular belief that determines the preference between two alternatives might be well affected by payoff–irrelevant factors such as emotions, framing and alike. The axioms we provide might help an outside observer determine whether the collective preference is consistent with justifiable preferences and can be rationalized by multiple–multiple priors.

The second part of the paper is devoted to a particular case of justifiable preferences induced by partially–specified probabilities (see Lehrer (2007)). Here, multiple priors emerge from concrete partial information about the real distribution governing the state space. This partial information induces a vN–M representation of the preference over a set of acts that includes the set of constant acts. There are typically many distributions that can serve as the prior of this vN–M representation. The multiple priors that generate the justifiable preferences consist of all these distributions.

The paper is organized as follows: The next section describes the framework and axioms. Formal definition of multiple–multiple priors and the axiomatization of justifiable preferences are given in Section 3. The close relation between Knightian and justifiable preferences is discussed in Section 4. A particular case of justifiable preferences induced by partially–specified probabilities is axiomatized in Section 5. Lastly, related literature and final remarks are in Section 6.

[5]While it is commonly acceptable to consider iterated deletion of strictly dominated strategies when analyzing the behavior of agents in a game–like environment, there are several different approaches regarding their following strategic behavior.

2. Framework and Axioms

Consider a decision making model in an Anscombe–Aumann (1963) setting. Let X be a non–empty finite set of *outcomes*, and let $Y = \Delta(X)$ be the set of all *lotteries*,[6] that is, probability distributions over X. Let S be a finite non–empty set of *states of nature*. Now, consider the collection $L = Y^S$ of all functions from states of nature to lotteries. Such functions are referred to as *acts*. Endow this set with the product topology, where the topology on Y is the relative topology inherited from $[0,1]^X$. We denote by L_c the collection of all constant acts. Abusing notation, for an act $f \in L$ and a state $s \in S$, we denote by $f(s)$ the constant act that assigns a lottery $f(s)$ to every state of nature.

Mixtures (convex combinations) of lotteries and acts are performed pointwise. In particular, if $f, g \in L$ and $\alpha \in [0,1]$, then $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha) f(s)$ for every $s \in S$.

The primitive of such a decision model is a binary relation $\succeq$ over L, which represents the preferences of a decision maker (DM) over all acts. $\succ$ is the asymmetric part of the relation, that is $f \succ g$ if $f \succeq g$ but it is not true that $g \succeq f$. $\sim$ is the symmetric part, that is $f \sim g$ if $f \succeq g$ and $g \succeq f$. The binary relation $\succeq$ is *reflexive* if $f \sim f$ for every act f. $\succeq$ is *complete* over $K \subseteq L$ if for every $f, g \in K$, either $f \succeq g$ or $g \succeq f$. $\succeq$ is complete if it is complete over L. It is *transitive* over $K \subseteq L$ if for $f, g, h \in K$, $f \succeq g$ and $g \succeq h$ imply $f \succeq h$. Lastly, the relation is *non–trivial* if there are two acts f and g such that $f \succ g$.

The following is a list of assumptions (axioms) about a binary preference relation $\succeq$ over acts.

A1 *Relation.* $\succeq$ is reflexive, complete over L_c and non–trivial.

A1 is a structural assumption. Completeness is assumed over constants.

For two acts $f, g \in L$, we denote $f \succeq^S g$ and $f \succ^S g$, if respectively, $f(s) \succeq g(s)$ for every $s \in S$, and $f(s) \succ g(s)$ for every $s \in S$

A2 *Unambiguous transitivity.* (i) $f \succeq g$ and $g \succeq^S h$ imply $f \succeq h$; and (ii) $f \succeq g$ and $h \succeq^S f$ imply $h \succeq g$.

A2 combines two preferences. The first, $f \succeq g$, suggests that the DM managed to decide between f and g in spite of the ambiguity. This is the original preference relation. The second, $g \succeq^S h$, reflects a domination of g over h beyond uncertainty, and takes

[6]Given a finite set A, $\Delta(A)$ denotes the collection of all probability distributions over A.

into consideration only the DM's taste. **A2** requires transitivity in the following sense: if $f \succeq g$, and g is beyond any doubt as good as h, then $f \succeq h$.

Note that **A2** is quite natural when discussing justifiability. Assume that there is a justification φ that justifies the preference of f over g. Also assume that g dominates h, thus every justification, including φ, justifies the preference of g over h. Thus, φ justifies the preference of f over h.

The following lemma is immediate and will be useful in obtaining the representation of justifiable preferences. The proof is omitted.

Lemma 1. *Assume that* $\succeq$ *satisfies* ***A2****. Then,*
(1) $\succeq$ *is transitive over* L_c*; and*
(2) $\succeq$ *satisfies* monotonicity*, that is,* $f \succeq^S g$ *implies that* $f \succeq g$.

A3 *Strict monotonicity.* $f \succ^S g$ implies that $f \succ g$.

A3 is a strict monotonicity assumption. It states that a strictly preferred lottery in every state of nature yields a strictly preferred act. This is not the common monotonicity assumption, however, it is necessary for our representation.

A4 *Continuity.* For any act f, the set $\{g : g \succeq f\}$ is closed.

A5 *Independence.* $f \succeq g$ if and only if $\alpha h + (1-\alpha)f \succeq \alpha h + (1-\alpha)g$ for every $h \in L$ and $\alpha \in [0,1]$.

A5 is the classical independence assumption.

A function $u : Y \to \mathbb{R}$, also referred to as a *utility function*, is *affine* if for every $q \in Y$ it satisfies $u(q) = \sum_{x \in X} q(x)u(x)$. Given such a utility function and an act $f \in L$, we denote $u(f) = (u(f(s)))_{s \in S}$.

3. Main result

3.1. **Multiple–multiple priors.** "Multiple priors" as a concept of perception of ambiguity is considered standard by now. Different axiomatizations derive this concept to obtain a variety of attitudes towards ambiguity. Gilboa and Schmeidler (1989) present the notion of multiple priors to obtain uncertainty averse preferences in a maxmin model. Ghirardato et al. (2004) axiomatize a model termed α–maxmin, differentiating ambiguity attitude from ambiguity. In this model, a DM valuates an act partially by maxmin and partially by maxmax.[7] The variational preferences model, introduced by Maccheroni et al. (2006), suggests that using a particular prior involves a cost, and that agents maximize the net worst–case utility that also takes into account the cost involved. As the maxmin model, variational preferences are ambiguity averse. Recently, Cerreia et al. (2008) established a representation, resorting to multiple priors, for general uncertainty averse preferences. Smooth preferences, presented and axiomatized by Klibanoff et al. (2005), suggest that the decision maker's ambiguity is reflected by multiple priors, however there exists an additional subjective distribution over the set of priors, suggesting the probability that a specific prior is the "correct" one.

Bewley (2002) introduces a different approach towards uncertainty in the multiple priors model, and axiomatizes Knightian preferences. Under Knightian preferences the decision maker prefers f to g if f dominates g in the sense that according to all priors in the multiple prior set, the expected utility induced by f is greater than that induced by g.

None of the models referred to above accommodate justifiability. The notion of justifiability naturally raises the more general concept for perception of ambiguity: *Multiple–multiple priors*. The DM perceived ambiguity is modeled by a collection of sets of distributions over the states, where each justification is represented by a different set (see the elaborated discussion in Section 1).

To illustrate the notion of multiple–multiple priors, recall the example of the firm given in the introduction. The firm's employees are partially informed regarding the true state of nature. They share common sources of information, such as those provided by the firm, however, each of them might obtain private information (depending on business acquaintances and past experiences, for example) different than that obtained by the others. Also, it is natural that, due to uncertainty, different agents would end up with different information, some overlapping but also some contradicting. For example, each agent ends up with the true expectation of a collection of random variables, or the true probability of a partial collection of events. The list (of random variables or events) obtained by one agent need not be exactly the same as that obtained by the others. Typically information is incomplete, and each agent's ambiguity amounts to a set of priors, each of which is consistent with the information obtained. Therefore, the firm is provided with a collection of sets of priors, and has to base its decisions accordingly.

[7]The maxmax preference order induced by a set of priors P and a utility function u is defined by $f \succeq g \quad \Leftrightarrow \quad \max_{p\in P} p \cdot u(f) \geq \max_{p\in P} p \cdot u(g)$.

We now give the formal definition of multiple–multiple priors. Consider the collection of all non–empty, closed and convex subsets of $\Delta(S)$, endowed with the Hausdorff metric. This metric space is compact (Federer (1969)). The closure operator in this space will be denoted by $\mathrm{CL}(\cdot)$.

Definition 1. *A nonempty subset of this space is a* collection of multiple priors. *A collection of multiple priors $\mathcal{P}$ is* loosely closed *if for every $P \in \mathrm{CL}(\mathcal{P})$ there exists $P' \in \mathcal{P}$, such that $P' \subseteq P$.*

3.2. **Representation of justifiable preferences.** We are now ready to formulate the main result of this paper.

Theorem 1. *For a binary relation $\succeq$ over L the following are equivalent:*
*(1) $\succeq$ satisfies **A1**–**A5**.*
(2) There exist a loosely closed collection $\mathcal{P}$ of closed and convex sets of probability distributions over S and a non–constant affine function $u : Y \to \mathbb{R}$, such that for every two acts[8] f and g,

$$f \succeq g \quad \Leftrightarrow \quad \max_{P \in \mathcal{P}} \min_{p \in P} \{p \cdot (u(f) - u(g))\} \geq 0.$$

To rephrase Theorem 1, under the axioms presented, there exist a utility index u and a (loosely closed) multiple–multiple priors $\mathcal{P}$ such that $f \succeq g$, if there exists at least one set of priors $P \in \mathcal{P}$ such that $p \cdot u(f) \geq p \cdot u(g)$ for every prior p in P. In other words, f is (weakly) preferred to g if there exists at least one justification, in Bewley's (2002) Knightian sense, to prefer it. Thus, preferences on L satisfying **A1**–**A5** will be referred to as *justifiable.*

Consider the example of the firm discussed in Section 3.1. Theorem 1 suggests that, subject to the assumptions described, the firms' decisions will be based on a few sets of multiple priors obtained by its agents, in the following sense. An act f is preferred to g if there exists at least one agency that prefers f to g in the Knightian sense. That is, there exists an agent for whom the expected utility of f is greater than that of g according to every prior consistent with his information.

For a numerical illustration of justifiable preferences, consider the following two agents (b and w) example. Suppose that an urn contains 30 red balls, and additional 60 balls

[8]Given vectors $x, y \in \mathbb{R}^S$, the inner–product of x and y is denoted by $x \cdot y$. The inner–product of x and a probability distribution p over S is the expectation of x with respect to p.

that are either black or white. A ball is randomly drawn from the urn and the decision maker is given the choice between lottery **R** of receiving \$100 if a red ball is drawn, and lottery **B** of receiving \$100 if a black ball is drawn. Assume that agent b acquires the information $P_b = \{p \in \Delta(r,b,w) : \ p(r) = \frac{1}{3}, p(b) \geq \frac{1}{2}\}$, and that agent w acquires the information $P_w = \{p \in \Delta(r,b,w) : \ p(r) = \frac{1}{3}, p(w) \geq \frac{1}{6}\}$. With respect to any $p \in P_b$, the expected utility of **B** is greater than that of **R**. However, there are two priors in P_w such that, with respect to one, the expected utility of **B** is greater than that of **R**, while the expected utility of **R** with respect to the other is greater than that of **B**. The model presented in Theorem 1 suggests that observing the agents, it would seem that **B** is strictly preferred to **R**. There is justification (agent b) for preferring **B** over **R**, but no justification for preferring **R** over **B**.

Next we would like to present a uniqueness result of the representation of justifiable preferences. Consider two non empty collections $\mathcal{P}', \mathcal{P}$ of probability distributions over S. We denote $\mathcal{P}' \geq \mathcal{P}$ if for every $P \in \mathcal{P}$ there exists $P' \in \mathcal{P}'$ such that $P' \subseteq P$.

Proposition 1. *Assume that $\succeq$ is a justifiable preference. Then,*
(1) The utility function u that represents $\succeq$ is unique up to a positive linear transformation.
(2) There exists a unique collection of multiple priors $\mathcal{P}^*$ that represents $\succeq$, such that if $\mathcal{P}$ represents $\succeq$ then $\mathcal{P}^* \geq \mathcal{P}$.

Remark 1. *The proof of Proposition 1 is given constructively within the proof of Theorem 1. In particular, given a multiple–multiple priors $\mathcal{P}$ that represents $\succeq$, following the dual–primal arguments in the proof of Theorem 1 would yield $\mathcal{P}^*$.*

Before turning to the proof of Theorem 1, we need to examine the geometric implications of our axioms. Let N be a finite non–empty set, and let N also denote the number of elements in N. The collection of all functions $x : N \to \mathbb{R}$ can by identified with the linear space $\mathbb{R}^N$, where multiplication of such functions by scalars and addition of functions are performed pointwise. For $x, y \in \mathbb{R}^N$, $x \geq y$ if $x(i) \geq y(i)$ for every $1 \leq i \leq N$, and $x > y$ if $x(i) > y(i)$ for every $1 \leq i \leq N$.

Consider a binary relation $\succeq^*$ over a closed and convex $C \subseteq \mathbb{R}^N$. The following are some possible properties for such a relation:
(i) For every $x \in C$, $x \sim^* x$.
(ii) $x \succeq^* y$ and $z \geq x$ implies that $z \succeq^* y$.

(iii) $x > y$ implies $x \succ^* y$.
(iv) The sets $\{x \in C : \; x \succeq^* y\}$ are closed.
(v) $x \succeq^* y$ if and only if $\alpha z + (1-\alpha)x \succeq^* \alpha z + (1-\alpha)y$ for $z \in C$ and $\alpha \in [0,1]$.

Proposition 2. *Let $\succeq^*$ be a binary relation over $[0,1]^N$. Then the following are equivalent:*
(1) $\succeq^*$ satisfies (i)–(v).
(2) There exists a loosely closed collection $\mathcal{P}$ of closed and convex sets of probability distributions over N, such that

$$x \succeq^* y \quad \Leftrightarrow \quad \max_{P \in \mathcal{P}} \min_{p \in P} p \cdot (x-y) \geq 0.$$

Proof. It is easy to see that (2) implies (1). For the converse assume that $\succeq^*$ satisfies *(i)–(v)*. First we extend $\succeq^*$ to a complete binary relation over $\mathbb{R}^N$ as follows. We start with $\mathbb{R}^N_+$, the set of all nonnegative vectors is $\mathbb{R}^N$. Let $x, y \in \mathbb{R}^N_+$ and define $x \succeq^* y$ if and only if $\lambda x \succeq^* \lambda y$ for sufficiently small positive λ. To see that this is well defined, apply *(v)* where $z = 0$. We have obtained that $\succeq^*$ is (positively) homogeneous over $\mathbb{R}^N_+$, formally

$$x \succeq^* y \;\Leftrightarrow\; \lambda x \succeq^* \lambda y, \quad \text{for every } \lambda > 0.$$

Now consider $x, y \in \mathbb{R}^N$. We say that $x \succeq^* y$ if and only if there are $x', y' \in \mathbb{R}^N_+$ such that $x' \succeq^* y'$ and $x' - y' = x - y$. Since $\succeq^*$ is by now defined over all $\mathbb{R}^N_+$, it is complete over $\mathbb{R}^N$. It remains to show that $\succeq^*$ is well defined. That is, for every distinct $x, y, x', y' \in \mathbb{R}^N_+$, if $x \succeq^* y$ and $x' - y' = x - y = z$ then $x' \succeq^* y'$. Define, $w = \min(x, y)$ coordinate–wise. Then $x - w, y - w \in \mathbb{R}^N_+$, $(x-w) - (y-w) = x - y$ and $x > x - w$. $x \succeq^* y$ implies by homogeneity that $\frac{1}{2}w + \frac{1}{2}(x-w) \succeq^* \frac{1}{2}w + \frac{1}{2}(y-w)$, and again by *(v)* that $(x-w) \succeq^* (y-w)$.

Define $w' = \min(x', y')$. Note that $x' - w' = x - w = \max(z, 0)$ coordinate–wise, thus $x' - (x-w) = w'$ is in $\mathbb{R}^N_+$. The previous argument applied to x', y' and w' (that play the roles of x, y and w) shows that $x' \succeq^* y'$. We conclude that $x \succeq^* y$ if and only if $x - y \succeq^* 0$ for every $x, y \in \mathbb{R}^N$.

Let $K = \{x : x \succeq^* 0\}$. By homogeneity K is positive homogeneous, and by *(iv)* it is closed. Also, by *(ii)*, $\mathbb{R}^N_+ \subseteq K$. Moreover, K is a union of convex, closed and positively homogenous cones, each of which contains $\mathbb{R}^N_+$. Indeed, let $x \in K \setminus \mathbb{R}^N_+$, $y \in \mathbb{R}^N_+$ and $\alpha \in (0,1)$. Then, $\alpha x + (1-\alpha)y \geq \alpha x \succeq^* 0$. By *(ii)*, $\alpha x + (1-\alpha)y \in K$.

Thus, $K_x = \text{conv}\{\{\lambda \text{x} : \lambda > 0\} \cup \mathbb{R}^{\text{N}}_{+}\}$ is a closed and convex cone contained in K, and $K = \bigcup_{x \in K} K_x$.

Denote by $\mathcal{K}$ the collection of all non–empty, convex and closed cones, contained in K and contain $\mathbb{R}^N_+$. Let $\mathcal{C}$ be the subset of $\mathcal{K}$ containing all maximal elements with respect to inclusion.[9] We have that $x \in K$ if and only if there exists some convex, closed and positively homogeneous cone $C \in \mathcal{C}$, such that $x \in C$. As in Bewley (2002) (see his proof of Theorem 1, p. 105), for every $C \in \mathcal{C}$ there exists a closed and convex set P_C of probability distributions over N, such that $x \in C$ if and only if $p \cdot x \geq 0$ for every $p \in P_C$.

We would like to show next that $\mathcal{P}$ is loosely closed. Assume to the contrary that there exists some $P \in \text{CL}(\mathcal{P})$, such that there is no $P' \in \mathcal{P}$ contained in P. Thus, there exists a sequence $\{P_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}$ that converges (in the Hausdorff metric) to P, and there exists no $P' \in \mathcal{P}$ contained in P. This implies that there is a sequence of cones $\{C_{P_n}\}_{n \in \mathbb{N}} \subseteq \mathcal{C}$ that converges (in the Hausdorff metric[10]) to C_P, and there exists no $C' \in \mathcal{C}$ that contains C_P. Thus, there is $x \in C_P$ such that $x \notin K$, a contradiction to the fact that K is a closed set.

To complete the proof we are left with verifying that

$$x \in K \quad \Leftrightarrow \quad \max_{P \in \mathcal{P}} \min_{p \in P} p \cdot x \geq 0,$$

that is, there exists no $x \in K$ such that $\min_{p \in P} p \cdot x < 0$ for every $P \in \mathcal{P}$, with $\sup_{P \in \mathcal{P}} \min_{p \in P} p \cdot x = 0$. However, this result follows from $\mathcal{P}$ being loosely closed. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Since **A1**, **A2**, **A4** and **A5** are satisfied, the hypotheses of the von Neumann–Morgenstern theorem hold. The theorem assures the existence, and uniqueness up to a positive linear transformation, of an affine function $u : Y \to \mathbb{R}$, which represents the preferences restricted to L_c. By **A1** the function u is non–constant. Moreover, u can be normalized so that the minimal utility is 0 and the maximal utility is 1.

[9] $\mathcal{K}$ is a partially–ordered set with respect to inclusion. By *(iv)* maximality is well defined.

[10] The Hausdorff metric in the space of positive homogeneous, closed and convex cones (in $\mathbb{R}^N$), each of which contains the non–negative orthant, is derived by the metric over the space of closed and convex sets of probability distributions (over N). The distance between two cones is that of the identified sets of distributions.

The existence of such a utility function u induces a preference relation over $[0,1]^S$. For $f, g \in L$, $u(f) \succeq^* u(g)$ if and only $f \succeq g$. $\succeq^*$ is well defined due to axiom **A2** of $\succeq$. Furthermore, $\succeq^*$ satisfies properties *(i)–(v)*.

Now, applying Proposition 2 completes the proof. □

4. Bewley's Knightian preferences and justifiability

Bewley (2002) introduces a status quo approach towards uncertainty and axiomatizes Knightian preferences. Under Knightian preferences the decision maker prefers f to g if f dominates g in the sense that according to all priors in a given multiple priors set, the expected utility induced by f is greater than that induced by g. Formally, for every two acts f and g,

$$f \succ' g \quad \Leftrightarrow \quad \forall p \in P,\ p \cdot u(f) > p \cdot u(g),$$

where P is a given convex closed set of probability distribution over S. The obvious shortcoming of the Knightian preference order is that it is incomplete.[11]

In the case where the set of priors consists of more than one prior, there are many (i.e., a continuum of) ways to extend the Knightian preferences to a complete binary relation. An obvious way, which we call a *Bayesian extension*, is to adopt one of the many possible priors, say p, and to declare that f is preferred over g if $p \cdot u(f) \geq p \cdot u(g)$. Following one particular prior, as in the Bayesian extension, has many desirable properties such as transitivity, ambiguity aversion and time consistency when temporal aspects are involved. Gilboa et al. (2008) extend the Knightian preferences to the complete maxmin preferences.

Another way to extend the incomplete Knightian preferences is as follows. For every two incomparable acts f, g define $f \succeq' g$. In words, whenever preferences are indecisive about the comparison of f and g, the definition implies that both, $f \succeq' g$ and $g \succeq' f$. According to the extension $\succeq'$,

$$f \succeq' g \ \Leftrightarrow \ \exists p \in P \text{ such that } p \cdot u(f) \geq p \cdot u(g). \tag{1}$$

It is clear from Eq. 1 that the latter incomparable–to–indifference extension yields a particular case of justifiable preferences: Every set in the collection of multiple priors

[11]In fact, such preferences are complete if and only if P consists of a singleton, which implies expected utility preferences.

consists of a singleton, that is, every justification is of expected utility form. This particular case of justifiable preferences could also be thought of as an alternative interpretation of Knightian preferences (see Bewley (2002, p. 97) and Eliaz and Ok (2006, p. 71-72)). However, this interpretation and perspective yield a different axiomatization, obviously similar to that in Theorem 1.

In order to formulate such an axiomatization we first need to consider two axioms. The first strengthens axiom **A1** to completeness, which is clear when motivated by a complete extension of Knightian preferences.

A1' *Relation.* $\succeq$ is reflexive, complete and non–trivial.

The second is a natural convexity axiom.

A6 *Convexity.* For any act f, the set $\{g : g \succ f\}$ is convex.

Theorem 2. *$\succeq$ is a binary relation over L. Then the following are equivalent:*
*(1) $\succeq$ satisfies **A1'** and **A2**–**A6**.*
(2) There exist a non–constant affine utility function $u : Y \to \mathbb{R}$, and a non–empty, closed and convex set P of probability distributions over S, such that for every two acts f and g,

$$f \succeq g \quad \Leftrightarrow \quad \exists p \in P \text{ such that } p \cdot u(f) \geq p \cdot u(g).$$

Moreover, P is unique and u is unique up to a positive linear transformation.

Remark 2. *The completeness axiom **A1'** does pinpoint the exact structure of a each member in the collection of multiple priors. However, the convexity axiom **A6** implies that this collection consists of singletons. The proof of Theorem 2 follows the line of that of Theorem 1 (apart from the differences discussed above) and is therefore omitted.*

5. Justifiable preferences and partially–specified probabilities

5.1. **Partially–specified probabilities.** Lehrer (2007) suggests a different perception of uncertainty than that of multiple prior, non–additive prior (Schmeidler (1989)), and others appearing in the literature. This alternative is information based. The decision maker obtains a *partially–specified probability (PSP)*, in particular, either the true probability of some, but maybe not all, events, or the true expectation of a partial list of random variables. The decision maker then evaluates the alternatives, according to her attitude to uncertainty, utilizing only the PSP and completely ignoring unavailable information.[12]

5.2. **Justifiable preferences and PSP.** In this subsection we characterize justifiable preferences, in the spirit of Theorem 2, that are generated by PSP.

Definition 2. *An act f is* primitive *if for any $\alpha \in [0,1)$:*
(i) for any constant act c such that $f \succeq c$, the inequality $h \succeq \alpha f + (1-\alpha)g$ implies $h \succeq \alpha c + (1-\alpha)g$ and $\alpha c + (1-\alpha)g \succeq h$ implies $\alpha f + (1-\alpha)g \succeq h$; and
(ii) for any constant act c such that $c \succeq f$, the inequality $\alpha f + (1-\alpha)g \succeq h$ implies $\alpha c + (1-\alpha)g \succeq h$ and $h \succeq \alpha c + (1-\alpha)g$ implies $h \succeq \alpha f + (1-\alpha)g$.

Primitive acts are essentially those that keep transitivity in a broad sense. A primitive act f satisfies simple transitivity when constants are involved. That is, $h \succeq f$ and $f \succeq c$ with c being constant imply $h \succeq c$. However, a primitive act satisfies a broader sense of transitivity. Assume, for instance, that $f \succeq c$, c being constant, and that $h \succeq \alpha f + (1-\alpha)g$, then c can replace f to produce $h \succeq \alpha c + (1-\alpha)g$. This is transitivity and mixing combined

Remark 3. *Note that the definition immediately implies that whenever f is primitive and c, d are constants such that $c \succeq f \succeq d$, then $c \succeq d$.*

A7 *Primitives determine preferences.* For any two acts g_1, g_2 the following are equivalent:
(i) $g_1 \succeq g_2$.
(ii) If for every two primitive acts f_1, f_2, $\alpha g_2 + (1-\alpha)f_1 \succeq^S \alpha g_1 + (1-\alpha)f_2$ for every $\alpha \in (0,1)$, then $f_1 \succeq f_2$.
According to **A7** the primitive acts are those which the preference relation should be founded on. It suggests that the fact that $\alpha g_2 + (1-\alpha)f_1$ dominates $\alpha g_1 + (1-\alpha)f_2$ for every $\alpha \in (0,1)$ cannot co–exist with $f_2 \succ f_1$, attests that the domination of $\alpha g_2 + (1-\alpha)f_1$ over $\alpha g_1 + (1-\alpha)f_2$ stems from $f_1 \succeq f_2$ and not from $g_2 \succ g_1$. **A7** states that if the primitive acts do not provide strong evidence that $g_2 \succ g_1$, by default $g_1 \succeq g_2$.

[12]PSP is related to CRISP acts defined by Ghirardato, Maccheroni and Marinacci (2004). Both PSP and CRISP acts try to capture the unambiguous acts. However, in the PSP model the primitive acts span in a sense the entire preferences (as shown in the next Theorem 3), while in Ghirardato, Maccheroni and Marinacci (2004), additional information is needed beyond the CRISP acts to determine the preference order.

Theorem 3. *For a binary relation $\succeq$ over L the following are equivalent:*
*(1) $\succeq$ satisfies **A1**-**A5** and **A7**.*
(2) There exist a convex collection of acts F, a non–constant affine function $u : Y \to \mathbb{R}$ and a prior distribution p over S, such that for every two acts g_1 and g_2, $g_1 \succeq g_2$ if and only if

$$(2)\quad \min \{p \cdot (u(f_1) - u(f_2)) : \gamma[u(f_1) - u(f_2)] \geq u(g_1) - u(g_2),\ f_1, f_2 \in F,\ \gamma > 0\} \geq 0.$$

Furthermore, u is unique up to a positive linear transformation.
(3) There exist a convex collection of acts F, a non–constant affine function $u : Y \to \mathbb{R}$ and a prior distribution p over S, such that for every two acts g_1 and g_2,

$$g_1 \succeq g_2 \quad \Leftrightarrow \quad \exists q \in P \text{ such that } q \cdot u(g_1) \geq q \cdot u(g_2),$$

where $P = \{q :\ q \cdot u(f) = p \cdot u(f) \text{ for every } f \in F\}$.
Furthermore, P is unique, F is unique and u is unique up to a positive linear transformation.

Theorem 3 item (3) is indeed the PSP version of justifiable preferences. The DM is informed of the expectations, with respect to p, of all the variables $u(f)$, $f \in F$. The set of priors P that consists of all the distributions q that are consistent with p over F induces a justifiable preference relation.

In order to prove Theorem 3, we need to establish some preliminary results regarding the collection of all primitive acts, and how they relate to one another with respect to $\succeq$.

Lemma 2. ***A2** and **A5** guarantee that every constant act is primitive.*

Proof. Suppose that f, c are constants and $f \succeq c$. By **A5**, $\alpha f + (1-\alpha)g \succeq^S \alpha c + (1-\alpha)g$ for every act g and $\alpha \in [0, 1]$. Let $h \succeq \alpha f + (1 - \alpha)g$. Then, by **A2**, $h \succeq \alpha c + (1 - \alpha)g$. All other implications required in Definition 2 are shown in a similar fashion. □

Lemma 3. ***A1**, **A2**, **A4** and **A5** guarantee that for every primitive act f there is a unique*[13] *constant act c such that $c \sim f$.*

Proof. Consider the collection L_c of all constant acts endowed with the relative topology. Completeness of $\succeq$ implies that any constant is a member of at least one of the following

[13]Here we mean unique up to the equivalence class. That is, if c, c' are constants such that $c \sim f$ and $c' \sim f$, then, by Remark 3 $c \sim c'$.

closed sets of constants: $\{c \in L_c : c \succeq f\}$ and $\{c \in L_c : f \succeq c\}$. Remark 3 guarantees that the intersection of these sets contains at most one act, and the convexity of the set of constants (which implies connectedness) implies that the intersection contains exactly one act. □

For any primitive act f denote by c_f the constant act that satisfies $c_f \sim f$.

Lemma 4. ***A1**, **A2**, **A4** and **A5** guarantee that the set of primitive acts is convex.*

Proof. Let f_1, f_2 be primitive acts and suppose that $\gamma f_1 + (1-\gamma) f_2 \succeq c$. Let $h \succeq \alpha(\gamma f_1 + (1-\gamma) f_2) + (1-\alpha) g$. We need to show that $h \succeq \alpha c + (1-\alpha) g$. Since f_1, f_2 are primitive acts, $h \succeq \alpha(\gamma c_{f_1} + (1-\gamma) c_{f_2}) + (1-\alpha) g$ and $\gamma c_{f_1} + (1-\gamma) c_{f_2} \succeq c$. Thus, Assumption 2 implies $h \succeq \alpha c + (1-\alpha) g$. All other implications required in Definition 2 are shown in a similar fashion. □

Lemma 5. ***A1**, **A2**, **A4** and **A5** guarantee that $\succeq$ is transitive over the set of primitive acts.*

Proof. Assume f_1, f_2, f_3 are primitive acts that satisfy $f_1 \succeq f_2 \succeq f_3$. Then, $c_{f_1} \succeq f_1 \succeq f_2$ which implies $c_{f_1} \succeq f_2$. On the other hand $c_{f_1} \succeq f_2 \succeq c_{f_2}$ which implies $c_{f_1} \succeq c_{f_2}$. In the same manner $c_{f_2} \succeq c_{f_3}$. By Lemma 1 we have that $c_{f_1} \succeq c_{f_3}$. Now, $f_1 \succeq c_{f_1} \succeq c_{f_3}$ implies $f_1 \succeq c_{f_3}$, and $f_1 \succeq c_{f_3} \succeq f_3$ implies $f_1 \succeq f_3$. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let $F \subseteq Y$ be the set of all primitive acts. This set is convex, includes all constant acts, and $\succeq$ is transitive over F. Therefore, there exists a utility function $u : Y \to \mathbb{R}$ and a prior distribution p over S such that for every two primitives f_1 and f_2,

$$f_1 \succeq f_2 \;\Leftrightarrow\; p \cdot u(f_1) \geq p \cdot u(f_2). \tag{3}$$

Now, define a binary relation $\succeq^*$ over all acts as follows: for every two acts g_1 and g_2, $g_1 \succeq^* g_2$ if and only if Eq. (2) holds. Let us show that $\succeq^*$ and $\succeq$ coincide.

Suppose that $g_1 \succeq^* g_2$ and assume on the contrary, that $g_2 \succ g_1$. **A7** implies that there exist two primitive acts f_1, f_2 such that $\alpha g_2 + (1-\alpha) f_1 \succeq^S \alpha g_1 + (1-\alpha) f_2$ for every $\alpha \in (0,1)$, and $f_2 \succ f_1$. By Eq. (3) we obtain $p \cdot u(f_2) > p \cdot u(f_1)$. However, $\alpha g_2 + (1-\alpha) f_1 \succeq^S \alpha g_1 + (1-\alpha) f_2$ implies that $\frac{1-\alpha}{\alpha}[u(f_1) - u(f_2)] \geq u(g_1) - u(g_2)$. Thus, the minimum in Eq. (2) is less than 0, which contradicts the assumption that $g_1 \succeq^* g_2$.

Conversely, assume that $g_1 \succeq g_2$. By **A7** the minimum in Eq. (2) is at least 0, which implies that $g_1 \succeq^* g_2$.

The equivalence of (2) and (3) is a routine application of a separation theorem and is therefore omitted. □

6. Comments

6.1. **Related literature.** Aizerman and Malishevski (1981) characterize choice functions that are rationalizable (that is, can be represented as the union of transitive and complete orders), whenever the set of outcomes is finite. Kalai, Rubinstein and Spiegler (2002) study choice functions that can be rationalized by a set of justifications where each justification is applied in a disjoint set of choice problems. They study a particular scheme that minimizes the number of justifications needed to rationalize a choice function. Cherepanov, Feddersen and Sandroni (2008) characterize choice functions that can be described as follows: In every choice problem a set of justifications is applied in order to obtain a subset of justifiable alternatives. Then, among this set an alternative is chosen according to an additional ordering. In contrast with the model presented in this paper, the chosen alternative, in all of the models described above, is determined only by the set of alternatives. These models do not accommodate justifiability.

Salant and Rubinstein (2008) develop a framework for modeling choice in the presence of framing effects. In their model, an extended choice function is a pair (A, φ): a set A of alternatives, and a frame φ. They relate the new framework to the classical model of choice correspondence, and give conditions under which there exists a "rational" relation such that an alternative is chosen in some pair (A, φ), if this alternative maximizes the relation in the feasible set. They then study a particular case of framing of limited attention.

The main difference between the Salant–Rubinstein model and the model presented here is that they impose the framing on the model, whereas our result derives it exogenously from the observed behavior of the decision maker. Another difference is in the framework iteself. Salant–Rubinstein assume finitely many outcomes, while we work with a more structured framework, allowing for a specific form of justifications (prior distributions over the states of nature).

Lastly, following the same arguments presented in this paper, Heller (2009) extends our Theorem 2 to choice functions by adding a uniformity axiom.

6.2. **Justifiability in alternative frameworks.** Aumann (1962) and Dubra, Maccheroni and Ok (2004) study expected utility theory without the completeness axiom. The geometric nature of their results (and the content of proofs) suggest that justifiability could be accommodated and stated in a framework of objective lotteries over outcomes.

In a game–like environment, Aumann and Dreze (2009) axiomatize an expected utility agent. It would be interesting to see whether relaxing the transitivity axiom in their framework would yield an agent that adhere to justifiability as in conjectural equilibrium.

6.3. **General justifiability.** In this paper we chose to focus on justifiability in Bewley's (2002) Knightian decision model. One can think of other models such as Gilboa and Schmeidler's (1989) maxmin expected utility model. In the justifiable version of maxmin preferences, the decision maker is characterized by a utility function u and a multiple–multiple priors $\mathcal{P}$, such that f is weakly preferred to g if and only if $\min_{p\in P} p \cdot u(f) \geq \min_{p\in P} p \cdot u(g)$, for at least one $P \in \mathcal{P}$.

For a more general notion of justifiability, denote by $\mathcal{O}^X$ the collection of all binary relations over L that agree on $\Delta(X)$, and let $\mathcal{A} \subseteq \mathcal{O}^X$ be the collection of all preferences in $\mathcal{O}^X$ that satisfy a list of axioms $\mathbb{A}$. Assume that transitivity is included in $\mathbb{A}$. We say that $\succeq$ is $\mathcal{A}$*–justifiable* if there exists a collection of elements in $\mathcal{A}$, such that, their union yields $\succeq$. In Theorem 1, for example, $\mathcal{A}$ consists of Knightian preferences. In Theorem 2, $\mathcal{A}$ consists of expected utility preferences.

Theorem 1 states that by weakening transitivity to **A2** along with the axioms of Bewley, one obtains the maximum over a collection of Knightian preferences. Theorem 2 presents a similar result for expected utility preferences. Weakening transitivity to **A2** along with the axioms of Anscombe–Aumann, yield a relation induced by taking the maximum over a collection of expected utility preferences.

We conjecture that the general notion of justifiability is hidden in the unambiguous transitivity axiom **A2**. In light of the observations above, a natural question arises as to what should the set $\mathbb{A}$ be in order to obtain $\mathcal{A}$–justifiable preferences from replacing the transitivity axiom in $\mathbb{A}$ by **A2**?

6.4. **Extensions of Knightian preferences.** A binary relation over acts is formally defined as a subset of pairs contained in the product of the set of acts with itself. The (irreflexive) Knightian preferences are the intersection of all the irreflexive Bayesian preferences induced by the priors in P. Moreover, all the irreflexive Bayesian preferences are open and convex (again, as sets of pairs) and so are the Knightian preferences. On the other hand, Eq. (1) implies that any Bayesian preference order is a subset of the justifiable preferences. Moreover, the justifiable preferences are the union of all the Bayesian extensions. The same is true for the maxmin extension. It turns out that any complete preferences contained in the justifiable preferences contain the Knightian preferences.

Two issues arise from this observation. The first can be phrased as a question: for given Knightian preferences, what are the continuous and complete extensions that are subsets of the corresponding justifiable preferences? Equivalently, what are the continuous and complete preference relations contained in given justifiable preferences?

The second issue concerns an axiomatization. Beyond the Bayesian and the maxmin preferences one can think of other preference relations, such as the maxmax, that are subsets of justifiable preferences and satisfy completeness and continuity. It would be interesting to axiomatize the continuous and complete preference relations that are subsets of a justifiable preferences (without loss of generality the set of priors could be the set of all probability distributions over S). And in the case of such preferences, what in terms of the relation itself is the unique minimal and convex set of priors that generates the corresponding justifiable preferences?

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