14 Higher order forms; divergence theorem

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Boundary and derivative are generalized to 3-chains and 2-forms, and higher. Stokes’ theorem and divergence theorem are generalized accordingly.

14a Forms of order three

Similarly to the boundary of a singular 2-box, defined in Sect. 11d as

\[ \Gamma_{AB} + \Gamma_{BC} + \Gamma_{CD} + \Gamma_{DA} = \Gamma_{AB} + \Gamma_{BC} - \Gamma_{DC} - \Gamma_{AD}, \]

we define the boundary of a singular 3-box as follows:\(^1\)

\[ \Gamma_{ADCB} + \Gamma_{EFGH} + \Gamma_{ABFE} + \Gamma_{DHGC} + \Gamma_{AEHD} + \Gamma_{BCGF} = \]

\[ = - \Gamma_{ABCD} + \Gamma_{EFGH} - \Gamma_{AEFB} + \Gamma_{DHGC} - \Gamma_{ADHE} + \Gamma_{BCGF}. \]

Similarly to (11d1),

\[ \partial(\partial \Gamma) = 0 \quad \text{for a singular 3-box } \Gamma. \]

14a3 Exercise. Similarly to Sect. 11d, find

\[ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^3} \int_{\partial \Gamma_\varepsilon} \omega \]

where \( \Gamma_\varepsilon : [0, 1]^3 \to \mathbb{R}^n \), \( \Gamma_\varepsilon(u_1, u_2, u_3) = x + \varepsilon u_1 h_1 + \varepsilon u_2 h_2 + \varepsilon u_3 h_3 \), and \( \omega \) is an arbitrary 2-form (of class \( C^1 \)) on \( \mathbb{R}^n \).

Answer: \( (D_{h_1} \omega(\cdot, h_2, h_3))_x + (D_{h_2} \omega(\cdot, h_3, h_1))_x + (D_{h_3} \omega(\cdot, h_1, h_2))_x. \)

We proceed similarly to Def. 11d2.

\(^1\)Here we rely on our geometric intuition; for a formal approach see Sect. 14c
14a4 Definition. The exterior derivative of a 2-form $\omega$ of class $C^1$ is a 3-form $d\omega$ defined by

$$(d\omega)(\cdot, h_1, h_2, h_3) = D_{h_1}\omega(\cdot, h_2, h_3) + D_{h_2}\omega(\cdot, h_3, h_1) + D_{h_3}\omega(\cdot, h_1, h_2).$$

Wedge product was defined in Sect. 11e for two 1-forms. Now we extend it.

14a5 Definition. (a) Let $L_1, L_2$ be linear forms on $\mathbb{R}^n$. Their wedge product $L_1 \wedge L_2$ is an antisymmetric bilinear form $L^{(2)}$ on $\mathbb{R}^n$ defined by

$$L^{(2)}(a, b) = L_1(a)L_2(b) - L_1(b)L_2(a) \quad \text{for all } a, b \in \mathbb{R}^n.$$

(b) Let $L^{(1)}$ be a linear form on $\mathbb{R}^n$, and $L^{(2)}$ an antisymmetric bilinear form on $\mathbb{R}^n$. Their wedge product $L^{(1)} \wedge L^{(2)} = L^{(2)} \wedge L^{(1)}$ is an antisymmetric trilinear form $L^{(3)}$ on $\mathbb{R}^n$ defined by

$$L^{(3)}(a, b, c) = L^{(1)}(a)L^{(2)}(b, c) + L^{(1)}(b)L^{(2)}(c, a) + L^{(1)}(c)L^{(2)}(a, b)$$

for all $a, b, c \in \mathbb{R}^n$.

(Think the antisymmetry.) This definition is suggested by determinants, as follows.

A trilinear form $L$ on $\mathbb{R}^n$ is generally $L(a, b, c) = \sum_{i, j, k} c_{i, j, k}a_ib_jc_k$. If $L$ is antisymmetric then

$$L = \sum_{i < j < k} c_{i, j, k}L_{i, j, k} \quad \text{where} \quad L_{i, j, k}(a, b, c) = \begin{vmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \\ a_k & b_k & c_k \end{vmatrix}$$

(think, why). Introducing also $L_i$ and $L_{i, j}$ by

$$L_i(a) = a_i, \quad L_{i, j}(a, b) = \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}$$

we observe that $L_i \wedge L_j = L_{i, j}$ and $L_i \wedge L_{j, k} = L_{i, j, k}$ (think, why). Thus, $(L_i \wedge L_j) \wedge L_k = L_i \wedge (L_j \wedge L_k)$ (since $L_{k, i, j} = L_{i, j, k}$). Associativity follows by taking linear combinations:

$$(L_1 \wedge L_2) \wedge L_3 = L_1 \wedge (L_2 \wedge L_3) \quad \text{for all linear forms } L_1, L_2, L_3 \text{ on } \mathbb{R}^n.$$

Wedge product of differential forms is defined pointwise:

$$(\omega_1 \wedge \omega_2)(x) = \omega_1(x) \wedge \omega_2(x).$$
It follows that \((f \omega_1) \wedge (g \omega_2) = (fg)(\omega_1 \wedge \omega_2)\) for \(f, g \in C^0(\mathbb{R}^n)\). Note that \(\omega_2 \wedge \omega_1 = \pm \omega_1 \wedge \omega_2\); the sign is minus for two 1-forms, but plus for a 1-form and 2-form. By associativity, \(\omega_1 \wedge \omega_2 \wedge \omega_3\) is well-defined for three 1-forms. In particular,
\[(dx_i \wedge dx_j \wedge dx_k)(x, h_1, h_2, h_3) = L_{i,j,k}(h_1, h_2, h_3)\]
is the 3 \times 3 determinant.

A 2-form (of class \(C^1\)) is called closed, if its derivative is zero. The 2-form \(dx_i \wedge dx_j\) is closed, since \((dx_i \wedge dx_j)(x, h, k)\) does not depend on \(x\).

The following two exercises are similar to (11e4) and (11e5).

**14a6 Exercise.** Prove that
\[d(d\omega) = 0\]
for all 1-forms \(\omega\) of class \(C^2\) on \(\mathbb{R}^n\).

Thus, all exact 2-forms of class \(C^1\) are closed. By the way, the 2-form \(dx_i \wedge dx_j\) is exact by 13b18, or just because \(d(x_i dx_j) = dx_i \wedge dx_j\) by (11e6). Moreover,
\[(14a7)\]
\[df \wedge dg\]
is exact, therefore closed, for all \(f, g \in C^1(\mathbb{R}^n)\).

**14a8 Exercise.** Prove that
\[d(f \omega) = df \wedge \omega + f d\omega\]
for all \(f \in C^1(\mathbb{R}^n)\) and all 2-forms \(\omega\) of class \(C^1\) on \(\mathbb{R}^n\).

Therefore
\[(14a9)\]
\[d(f \omega) = df \wedge \omega\]
whenever \(\omega\) is closed.

In particular, \(d(f dx_i \wedge dx_j) = df \wedge dx_i \wedge dx_j\) for all \(f \in C^1(\mathbb{R}^n)\). Similarly to 11e7 we get the following definition equivalent to [14a4].

**14a10 Definition.** The exterior derivative of a 2-form \(\omega\) of class \(C^1\) is a 3-form \(d\omega\) defined by
\[d\omega = \sum_{i<j} df_{i,j} \wedge dx_i \wedge dx_j\]
for \(\omega = \sum_{i<j} f_{i,j} dx_i \wedge dx_j\).

We turn to change of variables, treated in Sect. 11f for 2-forms (and 1-forms, and 0-forms). Let \(\varphi \in C^1(\mathbb{R}^l \to \mathbb{R}^n)\). Recall the pullback \(\varphi^*\omega\) defined by 11f1 for all \(k\)-forms \(\omega\) on \(\mathbb{R}^n\). We generalize 11f5 and 11f6 as follows.
14a11 Exercise. Prove that
\[ \varphi^*(\omega_1 \wedge \omega_2 \wedge \omega_3) = (\varphi^*\omega_1) \wedge (\varphi^*\omega_2) \wedge (\varphi^*\omega_3) \]
for all 1-forms \( \omega_1, \omega_2, \omega_3 \) on \( \mathbb{R}^n \).

14a12 Lemma. For every 2-form \( \omega \) of class \( C^1 \) on \( \mathbb{R}^n \) and \( \varphi \in C^2(\mathbb{R}^\ell \to \mathbb{R}^n) \),
\[ \varphi^*(d\omega) = d(\varphi^*\omega). \]

Proof. We have \( \omega = \sum_{i<j} f_{i,j} \, dx_i \wedge dx_j \) and \( d\omega = \sum_{i<j} df_{i,j} \wedge dx_i \wedge dx_j \). It is sufficient to prove that \( \varphi^*(df_{i,j} \wedge dx_i \wedge dx_j) = d(\varphi^*(f_{i,j} \, dx_i \wedge dx_j)) \). We denote
\[ g_{i,j} = \varphi^* f_{i,j}, \quad y_i = \varphi^* x_i, \quad y_j = \varphi^* x_j. \]

By 11f4, \( \varphi^*(dx_i) = dy_i, \varphi^*(dx_j) = dy_j \) and \( \varphi^*(df_{i,j}) = dg_{i,j} \). By 11f5, \( \varphi^*(dx_i \wedge dx_j) = dy_i \wedge dy_j \). On the other hand, \( d(\varphi^*(f_{i,j} \, dx_i \wedge dx_j)) = d(g_{i,j} \, dy_i \wedge dy_j) = dg_{i,j} \wedge dy_i \wedge dy_j \)
by 14a11, 14a9.

14a13 Theorem. (Stokes’ theorem for \( k = 3 \))
Let \( C \) be a 3-chain in \( \mathbb{R}^n \), and \( \omega \) a 2-form of class \( C^1 \) on \( \mathbb{R}^n \). Then
\[ \int_C d\omega = \int_{\partial C} \omega. \]

Proof. It is sufficient to prove the equality \( \int_B d\omega = \int_{\partial B} \omega \) for every singular 3-box \( \Gamma \). Similarly to 11g, using (11f2) we transform the needed equality into \( \int_B \Gamma^*(d\omega) = \int_{\partial B} \Gamma^*\omega \). Similarly to 11g we may assume that \( \Gamma \) is of class \( C^2 \). Thus, 14a12 applies, and the needed equality becomes
\[ \int_B d(\Gamma^*\omega) = \int_{\partial B} \Gamma^*\omega. \]

Similarly to 11g it remains to prove the equality \( \int_B d\omega = \int_{\partial B} \omega \) for every 2-form \( \omega \) of class \( C^1 \) on the cube \( B = [0,1]^3 \subset \mathbb{R}^3 \); we consider only \( \omega = f(u_1, u_2, u_3) \, du_1 \wedge du_2 \), since the other two cases are similar.

We have \( d\omega = df \wedge du_1 \wedge du_2 = \left( \frac{\partial f}{\partial u_1} \, du_1 + \frac{\partial f}{\partial u_2} \, du_2 + \frac{\partial f}{\partial u_3} \, du_3 \right) \wedge du_1 \wedge du_2 = \frac{\partial f}{\partial u_3} \, du_1 \wedge du_2 \wedge du_3 \), thus
\[ \int_B d\omega = \int_{[0,1]^3} \frac{\partial f}{\partial u_3} \, du_1 \, du_2 \, du_3 = \int_{[0,1]^2} du_1 du_2 \int_0^1 du_3 \frac{\partial f}{\partial u_3} = \int du_1 du_2 \left( f(u_1, u_2, 1) - f(u_1, u_2, 0) \right), \]
which is equal to \( \int_{\partial B} \omega \) (see (14a1)). □

\(^1\)Hint: similar to 11f5; use the \( 3 \times 3 \) determinant \( L_{i,j,k} \).
14a14 Corollary.

\[ C_1 \sim C_2 \implies \partial C_1 \sim \partial C_2 \]

for arbitrary 3-chains \( C_1, C_2 \) in \( \mathbb{R}^n \). (Similar to 11h1.)

14a15 Exercise. 1 Check that

\[ (y\, dx + x\, dy) \wedge (x\, dx \wedge dz + y\, dy \wedge dz) = (y^2 - x^2)\, dx \wedge dy \wedge dz . \]

14a16 Exercise. 2 Check that

\[ d(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy) = 3\, dx \wedge dy \wedge dz . \]

14a17 Exercise. 3 Prove that

\[ d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 - \omega_1 \wedge d\omega_2 \]

for arbitrary 1-forms \( \omega_1, \omega_2 \) on \( \mathbb{R}^n \).

Thus, if \( \omega_1 \) and \( \omega_2 \) are closed 1-forms then \( \omega_1 \wedge \omega_2 \) is a closed 2-form. (Compare it with 13b18.)

14a18 Exercise. 4 Prove a generalization of the formula for integration by parts,

\[ \int_C f\, d\omega = \int_{\partial C} f\, \omega - \int_C df \wedge \omega \]

for arbitrary 2-form \( \omega \) (of class \( C^1 \)) on \( \mathbb{R}^n \), function \( f \in C^1(\mathbb{R}^n) \), and 3-chain \( C \) in \( \mathbb{R}^n \).

14b Divergence theorem in three dimensions

A 2-form \( \omega \) on \( \mathbb{R}^3 \) corresponds to a vector field \( H \) (recall Sect. 12a), namely,

\[ \omega(x, h_1, h_2) = \det(H(x), h_1, h_2) , \]

\[ H(x) = \left( f_{2,3}(x), f_{3,1}(x), f_{1,2}(x) \right) \]

for \( \omega = \underbrace{f_{1,2}}_{H_3} \, dx_1 \wedge dx_2 + \underbrace{f_{2,3}}_{H_1} \, dx_2 \wedge dx_3 + \underbrace{f_{3,1}}_{H_2} \, dx_3 \wedge dx_1 . \)

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1Sjamaar, p. 19.
2Shurman, p. 423.
3Shurman, Th. 9.8.2 shows that in general the sign depends on the order of \( \omega_1 \).
4Shurman, Ex. 9.14.3.
14b1 Exercise. Let a vector field $E$ correspond to a 1-form $\omega_1$, and a vector field $H$ correspond to a 2-form $\omega_2$. Prove that

$$\omega_1 \wedge \omega_2 = \langle E, H \rangle dx_1 \wedge dx_2 \wedge dx_3.$$ 

For every singular 2-box $\Gamma : B \to \mathbb{R}^3$,

$$\int_\Gamma \omega = \int_B \det(H(\Gamma(u)), (D_1 \Gamma)_u, (D_2 \Gamma)_u) \, du = \int_\Gamma H$$

(recall (12a7)) is the flux of $H$ through $\Gamma$. This relation extends by linearity to 2-chains; in particular, $\int_{\partial\Gamma} \omega = \int_{\partial\Gamma} H$ is the flux of $H$ through the boundary of a singular 3-box $\Gamma$.

The derivative $d\omega$ (assuming that $\omega$ is of class $C^1$), being a 3-form on $\mathbb{R}^3$, is

$$d\omega = f \, dx_1 \wedge dx_2 \wedge dx_3$$

for some $f \in C^0(\mathbb{R}^3)$. Taking into account that $d(H_3 dx_1 \wedge dx_2) = D_3 H_3 dx_1 \wedge dx_2 \wedge dx_3$ we get

$$d\omega = (\text{div} \, H) \, dx_1 \wedge dx_2 \wedge dx_3,$$

$$\text{div} \, H = D_1 H_1 + D_2 H_2 + D_3 H_3.$$

Now we finalize the diagram (12a3) (see also (12c9)),

(14b2)

14b3 Exercise.¹ Prove that

$$\text{div}(fH) = \langle \nabla f, H \rangle + f \, \text{div} \, H$$

for all vector fields $H$ (of class $C^1$) on $\mathbb{R}^3$ and all functions $f \in C^1(\mathbb{R}^3)$.

¹Zorich, (14.18).
²Hint: 14a8 and 14b1
14b4 Exercise. Prove that
\[ \text{div}(E_1 \times E_2) = \langle \text{curl } E_1, E_2 \rangle - \langle E_1, \text{curl } E_2 \rangle \]
for all vector fields \( E_1, E_2 \) (of class \( C^1 \)) on \( \mathbb{R}^3 \).

Theorem 14a13 gives the three-dimensional divergence theorem (recall (12c8)):
\[ \int_{\partial \Gamma} H = \int_{\Gamma} \text{div } H \]
for every vector field \( H \) (of class \( C^1 \)) on \( \mathbb{R}^3 \) and every singular 3-box \( \Gamma \) in \( \mathbb{R}^3 \). Here (as in 12c) by \( \int_{\Gamma} f \) we mean \( \int_{\Gamma} f \, dx_1 \wedge dx_2 \wedge dx_3 \).

If \( \Gamma : B \to \mathbb{R}^3 \) is such that \( \Gamma|_{B^o} \) is a diffeomorphism between \( B^o \) and an open set \( G = \Gamma(B^o) \subset \mathbb{R}^3 \) then
\[ \int_{\Gamma} f(x) \, dx_1 \wedge dx_2 \wedge dx_3 = \pm \int_{G} f \]
(a similar fact in two dimensions was noted in Sect. 12c, before (12c6)). Assuming that \( \det d\Gamma > 0 \) we get \( \int_{\Gamma} (\text{div } H) \, dx_1 \wedge dx_2 \wedge dx_3 = \int_{G} \text{div } H \), and so,
\[ \int_{\partial \Gamma} H = \int_{G} \text{div } H \]
similarly to (12c6), (12c8).

In particular, spherical coordinates suggest a singular 3-box \( \Gamma_R \) that represents a ball of radius \( R \),
\[ \Gamma_R : [0, R] \times [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3, \]
\[ \Gamma_R(r, \theta, \varphi) = (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta). \]

14b8 Exercise. Prove that
\[ \int_{\Gamma_R} f(x) \, dx_1 \wedge dx_2 \wedge dx_3 = \int_{B_R} f \]
for every \( f \in C^0(B_R) \).

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1Zorich, (14.19).
2Hint: 14a17 and 14b1.
3Hint: the determinant is equal to \( r^2 \sin \theta \).
Rotation invariance follows (recall 6m4):

$$\Gamma_R \sim T \circ \Gamma_R$$

for every linear isometry $T : \mathbb{R}^3 \to \mathbb{R}^3$. By [14a14] it follows that

(14b9)

$$\partial \Gamma_R \sim T \circ \partial \Gamma_R$$

since generally $T \circ \partial \Gamma = \partial (T \circ \Gamma)$ (think, why).

**14b10 Exercise.** (a) Consider a radial vector field $F$ on $\mathbb{R}^3$,

$$F(x) = f(|x|)x, \quad f \in C^0[0, \infty)$$

(like 13c3). Check that

$$\int_{\partial \Gamma_R} F = 4\pi R^3 f(R) = 4\pi R^2 \cdot f(R) R$$

(the area of the sphere times the length of the vector).

(b) More generally, consider $F(x) = f(x)x, f \in C^0(\mathbb{R}^3)$; check that

$$\int_{\partial \Gamma_R} F = R \int_0^\pi d\theta \int_0^{2\pi} d\varphi \cdot R^3 \sin \theta \cdot f(R \sin \theta \cos \varphi, R \sin \theta \sin \varphi, R \cos \theta).$$

Postponing integration on surfaces in general, for now we define the integral of a function over the sphere $\partial B_R$ (the boundary of the ball $B_R = \{x : |x| \leq R\} \subset \mathbb{R}^3$) by

(14b11)

$$\int_{\partial B_R} f = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \cdot R^2 \sin \theta \cdot f(R \sin \theta \cos \varphi, R \sin \theta \sin \varphi, R \cos \theta)$$

for arbitrary continuous function $f$ on the sphere. Note that

$$\int_{\partial B_R} 1 = 4\pi R^2; \quad \int_{\partial \Gamma_R} f(x)x = R \int_{\partial B_R} f.$$

Now we may define the mean value of $f$ on the sphere as $\frac{1}{4\pi R^2} \int_{\partial B_R} f$. This could not be done via Riemann integral (proper or improper), since the sphere is a set of volume zero.

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1 In spherical coordinates this is easy to see for rotations about the z axis, but problematic for other axes.

2 Hint: only one (out of six) face of the boundary contributes; calculate the $3 \times 3$ determinant and integrate it.
**14b12 Exercise.** Prove that

$$\int_{B_R} f = \int_0^R \int_{\partial B_r} f$$

for all \( f \in C^0(B_R) \).

Therefore

$$\int_{\partial B_R} f = \frac{d}{dR} \int_{B_R} f ;$$

rotation invariance follows:

$$\int_{\partial B_R} f = \int_{\partial B_R} T \circ f$$

for every linear isometry \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \). (Compare it with (14b9).)

Similarly to Sect. 12d (before (12d4),

$$\text{div } \nabla f = \Delta f ,$$

$$\Delta = D_1D_1 + D_2D_2 + D_3D_3 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

is the Laplacian. Functions \( f \in C^2(\mathbb{R}^3) \) such that \( \Delta f = 0 \) are called harmonic.

Similarly to (12d4) we’ll prove the mean value property of a harmonic function \( u \) on \( \mathbb{R}^3 \):

$$u(0) = \frac{1}{4\pi R^2} \int_{\partial B_R} u ; \quad u(x) = \frac{1}{4\pi R^2} \int_{\partial B_R} u(x + \cdot) .$$

To this end we need Green formulas (again).

Applying (14b5) to \( H = \nabla u \) we get the first Green formula (recall (12d5))

$$\int_{\partial \Gamma} \nabla u = \int_{\Gamma} \Delta u \quad \text{for all } u \in C^2(\mathbb{R}^3) .$$

Exercise 12d6 holds in all dimensions (with the same proof):

(a) \( \text{div}(fH) = f \text{ div } H + \langle \nabla f, H \rangle \) for all \( f \in C^1(\mathbb{R}^3) \) and \( H \in C^1(\mathbb{R}^3 \rightarrow \mathbb{R}^3) \);

(b) \( \text{div}(f \nabla g) = f \Delta g + \langle \nabla f, \nabla g \rangle \) for all \( f \in C^1(\mathbb{R}^3) \) and \( g \in C^2(\mathbb{R}^3) \);

(c) \( f \Delta g - g \Delta f = \text{div}(f \nabla g - g \nabla f) \) for all \( f, g \in C^2(\mathbb{R}^3) \).

\(^1\text{Hint: first, replace } B_R \text{ with } \Gamma_R.\)

\(^2\text{Again, in spherical coordinates this is easy to see for rotations about the } z \text{ axis, but problematic for other axes.}\)
Similarly to (12d7), (12d8) we get the second Green formula
\[(14b15) \quad \int_{\partial \Gamma} u \nabla v = \int_{\Gamma} (u \Delta v + \langle \nabla u, \nabla v \rangle) \quad \text{for all } u \in C^1(\mathbb{R}^3) \text{ and } v \in C^2(\mathbb{R}^3),\]
and the third Green formula
\[(14b16) \quad \int_{\partial \Gamma} (u \nabla v - v \nabla u) = \int_{\Gamma} (u \Delta v - v \Delta u) \quad \text{for all } u, v \in C^2(\mathbb{R}^3).\]

**14b17 Exercise.** Similarly to \(\Gamma_R\) of (14b7) introduce a singular 3-box \(\Gamma_{R_1, R_2}\) that represents the spherical shell \(\{x : R_1 \leq |x| \leq R_2\} \subset \mathbb{R}^3 \) (given \(0 < R_1 < R_2 < \infty\)) and check that
\[\partial \Gamma_{R_1, R_2} \sim \partial \Gamma_{R_2} - \partial \Gamma_{R_1}.\]

Here is a three-dimensional counterpart of 12d9.

**14b18 Exercise.** (a) Let \(u\) and \(v\) be harmonic functions on a spherical shell \(\{x \in \mathbb{R}^3 : a < |x| < b\}\); prove that \(\int_{\partial \Gamma_R} (u \nabla v - v \nabla u)\) does not depend on \(R \in (a, b)\).

(b) In particular, taking \(v(z) = 1/|z|\), prove that
\[\int_{\partial \Gamma_R} u \nabla v = - \frac{1}{R^2} \int_{\partial B_R} u, \quad \int_{\partial \Gamma_R} v \nabla u = \frac{1}{R} \int_{\partial \Gamma_R} \nabla u.\]

(c) Assuming in addition that \(u\) is harmonic on the ball \(\{x \in \mathbb{R}^3 : |x| < b\}\) prove that \(\frac{1}{R^2} \int_{\partial B_R} u\) does not depend on \(R \in (0, b)\) and is equal to \(4\pi u(0)\), which proves the first equality of (14b13); the second follows by shift.

**14b19 Exercise.** *(Maximum principle for harmonic functions)*

Let \(u\) be a harmonic function on a connected open set \(G \subset \mathbb{R}^3\). If \(\sup_{x \in G} u(x) = u(x_0)\) for some \(x_0 \in G\) then \(u\) is constant.

Prove it.

The mean value may be taken on the ball rather than the sphere:

\[(14b20) \quad u(0) = \frac{3}{4\pi R^3} \int_{B_R} u, \quad u(x) = \frac{3}{4\pi R^3} \int_{B_R} u(x + \cdot).\]

**Proof:** by (14b12) and (14b13),
\[\int_{B_R} u = \int_0^R dr \int_{\partial B_r} u = \int_0^R 4\pi R^2 u(0) dr = \frac{4\pi R^3}{3} u(0).\]

1\(^\text{Hint:} u\) is harmonic by 13c4.

2\(^\text{Hint:} the set \(\{x_0 : u(x_0) = \sup_{x \in G} u(x)\}\) is both open and closed in \(G\).
14b21 Proposition. (Liouville’s theorem for harmonic functions, dimension three)

Every harmonic function \( \mathbb{R}^3 \to [0, \infty) \) is constant.

Proof. (Nelson’s short proof)

For arbitrary \( x, y \in \mathbb{R}^3 \) and \( R > 0 \) we have

\[
    u(x) = \frac{3}{4\pi R^3} \int_{B_R} u(x + \cdot) \leq \frac{3}{4\pi R^3} \int_{B_{R+|x-y|}} u(y + \cdot) = \left( \frac{R + |x-y|}{R} \right)^3 u(y),
\]

since the \( R \)-neighborhood of \( x \) is contained in the \( (R + |x-y|) \)-neighborhood of \( y \). In the limit \( R \to \infty \) we get \( u(x) \leq u(y) \); similarly, \( u(y) \leq u(x) \).

\[ \square \]

14c Order four, and higher

In dimension four (and higher) we cannot rely on our geometric intuition as much as we did in [14a1]; we need a formal approach to orientation.

We introduce three types of cubes:

* a standard \( k \)-cube is the set \([-1, 1]^k \) in \( \mathbb{R}^k \);
* a singular \( k \)-cube in \( \mathbb{R}^n \) is a \( C^1 \) mapping \([-1, 1]^k \to \mathbb{R}^n \);
* a geometric \( k \)-cube in \( \mathbb{R}^n \) is a set \( X \subset \mathbb{R}^n \) isometric to \([-1, 1]^k \).

The group \( G_k \) of all isometries of the standard \( k \)-cube (to itself) consists of \( 2^k k! \) signed permutation matrices, like \( \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \). The determinant of such matrix is \( \pm 1 \).

Accordingly, for a given geometric \( k \)-cube in \( \mathbb{R}^n \) there exist \( 2^k k! \) isometric mappings \([-1, 1]^k \to X \). If \( \Gamma_1 \) is such mapping then others are \( \Gamma_1 \circ T \) for \( T \in G_k \); that is, they are \( \Gamma_2 \) such that \( \Gamma_1^{-1} \circ \Gamma_2 \in G_k \). All such mappings are singular \( k \)-cubes in \( \mathbb{R}^n \), not all mutually equivalent; rather,

\[
    \Gamma_1 \sim \Gamma_2 \quad \text{whenever} \quad \det(\Gamma_1^{-1} \circ \Gamma_2) = 1, \\
    \Gamma_1 \sim -\Gamma_2 \quad \text{whenever} \quad \det(\Gamma_1^{-1} \circ \Gamma_2) = -1.
\]

Thus, a geometric \( k \)-cube \( X \subset \mathbb{R}^n \) leads to two equivalence classes of singular \( k \)-cubes; these two equivalence classes will be called the two orientations of \( X \). A \( k \)-form cannot be integrated over \( X \) unless an orientation is chosen; for the other orientation the integral is the opposite number.

---

1This time, \([-1, 1]\) is technically more convenient than \([0, 1]\).
2The so-called hyperoctahedral group.
3Called also automorphisms or congruences.
The simplest case is, \( k = 1 \). A geometric 1-cube in \( \mathbb{R}^n \) is a straight interval \( X = \{ x : |A - x| + |x - B| = 2 \} \) for given \( A, B \in \mathbb{R}^n, |A - B| = 2 \). An isometry \( \gamma : [-1, 1] \to X \) defined by \( \gamma(t) = \frac{1-t}{2} A + \frac{1+t}{2} B \) is a path; denote it just \( AB \). Accordingly, \( BA \) is the other isometry \( [-1, 1] \to X, t \mapsto \frac{1-t}{2} B + \frac{1+t}{2} A \). Note that \( (BA)(t) = -(AB)(-t) \). Clearly, \( BA \sim -AB \), that is, \( \int_{BA} \omega = -\int_{AB} \omega \) for all 1-forms \( \omega \) on \( \mathbb{R}^n \).

The next case is, \( k = 2 \). Let \( X \subset \mathbb{R}^n \) be a geometric 2-cube. An isometry \( \Gamma : [-1, 1]^2 \to X \) is a singular 2-cube; denote it by \( ABCD \) where \( A = \Gamma(-1, -1), B = \Gamma(1, -1), C = \Gamma(1, 1), D = \Gamma(-1, 1) \); these are the vertices of \( X \). There are 8 isometries: \( ABCD, ADCB, BCDA, BADC, CDAB, CBAD, DABC, DCBA \); they result from \( ABCD \) via elements of the group \( G_2 \). For \( ABCD, BCDA, CDAB \) and \( DABC \) the elements of the group are rotations by \( 0, \pi/2, \pi \) and \( 3\pi/2 \), of Jacobian +1; for others, the elements of the group are reflections, of Jacobian \(-1\). Thus,

\[
ABCD \sim BCDA \sim CDAB \sim DABC \quad \text{is one orientation of } X,
\]

\[
ADCB \sim BADC \sim CBAD \sim DCBA \quad \text{is the other orientation of } X.
\]

The standard \( k \)-cube has \( 2^k \) hyperfaces

\[
\{(u_1, \ldots, u_k) \in [-1, 1]^k : u_i = a \} \quad \text{for } i \in \{1, \ldots, k\} \text{ and } a \in \{-1, 1\};
\]
each hyperface is a geometric \((k-1)\)-cube. We want to define the boundary \( \partial X \) of the standard \( k \)-cube \( X \) as the sum \( \sum \tilde{Y} \) of its hyperfaces \( Y \) treated as singular \((k-1)\)-cubes \( \tilde{Y} \); to this end we have to choose orientations of these hyperfaces. We did it already for \( k = 2, 3 \).

In these two cases the chosen orientations are consistent in the following sense. For every hyperface \( Y \) and every \( T \in G_k \) such that \( \det T = +1 \) (that is, \( T(\tilde{X}) = \tilde{X} \)),

\[
T(\tilde{Y}) = \tilde{T(Y)}.
\]

This consistency is necessary for Stokes’ theorem to hold, since \( T(\tilde{X}) = \tilde{X} \) must imply \( T(\partial \tilde{X}) = \partial \tilde{X} \) (recall [14a14]).

Here is a special case of the consistency condition:

\[
(14c1) \quad \text{if } T(\tilde{X}) = \tilde{X} \text{ and } T(Y) = Y \text{ then } T(\tilde{Y}) = \tilde{Y}.
\]

It is worth noting that such a condition fails for edges (rather than faces) of a 3-cube; here is a counterexample.
14c2 Example. Let $X = [-1, 1]^3$, $Y = \{ -1 \} \times \{ -1 \} \times [-1, 1]$ and $T(u_1, u_2, u_3) = (u_2, u_1, -u_3)$. Then $T$ preserves $Y$ and the orientation of $X$ but does not preserve the orientation of $Y$.

Consider the hyperface $Y_0 = \{ 1 \} \times [-1, 1]^{k-1}$ of $[-1, 1]^k$. If $T \in G_k$, $T(Y_0) = Y_0$, then

$$T = \begin{pmatrix} 1 & 0 \\ 0 & T' \end{pmatrix}$$

for some $T' \in G_{k-1}$. Thus, $\det T = \det T'$, which ensures (14c1) for $Y_0$.

Now we are in position to ensure the consistency condition in general. (This is somewhat similar to the proof of 13a1.) For each hyperface $Y$ of $[-1, 1]^k$ we choose $T_Y \in G_k$ such that $\det T_Y = +1$ and $T_Y(Y_0) = Y$. We choose an orientation of $Y_0$ and define

$$\tilde{Y} = T_Y(Y_0)$$

for all $Y$. Given hyperfaces $Y_1, Y_2$ and $T \in G_k$ such that $\det T = +1$ and $T(Y_1) = Y_2$, we have $(T_{Y_2}^{-1} \circ T \circ T_{Y_1})(Y_0) = Y_0$ and $\det(T_{Y_2}^{-1} \circ T \circ T_{Y_1}) = +1$.

Applying (14c1) to $T_{Y_2}^{-1} \circ T \circ T_{Y_1}$ and $Y_0$ we get $(T_{Y_2}^{-1} \circ T \circ T_{Y_1})(\tilde{Y}_0) = \tilde{Y}_0$; thus, $T(T(Y_1)(\tilde{Y}_0)) = T_{Y_2}(\tilde{Y}_0)$, that is, $T(\tilde{Y}_1) = \tilde{Y}_2$. (Similarly to 13a1, the choice of $T_Y$ does not really matter; think, why.)

Consistent orientations $\tilde{Y}$ are thus constructed in principle; but we need an explicit formula.

In terms of singular $(k - 1)$-cubes

$$\Delta_{i,a} : [-1, 1]^{k-1} \to [-1, 1]^k \text{ for } i \in \{ 1, \ldots, k \}, a \in \{ -1, +1 \}, \Delta_{i,a}(u_1, \ldots, u_{k-1}) = (u_1, \ldots, u_{i-1}, a, u_i, \ldots, u_{k-1}),$$

we have $\tilde{Y}_{i,a} \sim \pm \Delta_{i,a}$ where $Y_{i,a} = \{ (u_1, \ldots, u_k) \in [-1, 1]^k : u_i = a \}$ are the hyperfaces. But what are the signs?

The sign for $Y_0 = Y_{1,+}$ is rather a matter of convention; let it be +1. That is, $\tilde{Y}_0 \sim \Delta_{1,+}$. The mapping $T_{i,a} : (u_1, \ldots, u_k) \mapsto (u_2, \ldots, u_i, a u_1, u_{i+1}, \ldots, u_k)$ satisfies$^2$

$$T_{i,a}(Y_0) = Y_{i,a} ; \quad T_{i,a} \circ \Delta_{i,+} = \Delta_{i,a} ; \quad \det(T_{i,a}) = (-1)^{i-1} a .$$

$^1$More formally, $\tilde{Y}_0 \in \Delta_{1,+}$.

$^2$Indeed, $(u_1, \ldots, u_{k-1}) \mapsto \Delta_{i,+} (1, u_1, \ldots, u_{k-1})$ in $T_{i,a}$.

\[ \Delta_{i,a} (u_1, \ldots, u_{k-1}) = (u_1, \ldots, u_{i-1}, a, u_i, \ldots, u_{k-1}), \]

\[ T_{i,a} : (u_1, \ldots, u_k) \mapsto (u_2, \ldots, u_i, a u_1, u_{i+1}, \ldots, u_k), \]

\[ T_{i,a} (Y_0) = Y_{i,a}, \quad T_{i,a} \circ \Delta_{i,+} = \Delta_{i,a}, \quad \det(T_{i,a}) = (-1)^{i-1} a. \]
By the consistency condition, \( T_{i,a}(\tilde{Y}_0) \sim \det(T_{i,a})\tilde{Y}_{i,a} \), that is, \( \tilde{Y}_{i,a} \sim \det(T_{i,a})T_{i,a} \circ \Delta_{1,+} \sim (-1)^{i-1}a\Delta_{i,a} \).

**14c3 Definition.** The boundary of a singular \( k \)-cube \( \Gamma : [-1, 1]^k \to \mathbb{R}^n \) is a \( k \)-chain

\[
\partial \Gamma = \sum_{i=1}^{k} \sum_{a=\pm 1} (-1)^{i-1}a(\Gamma \circ \Delta_{i,a}) .
\]

**14c4 Exercise.** Check that the definitions used before for \( k = 1, 2, 3 \) conform to **14c3**.

**14c5 Exercise.** Prove that \( \partial(\partial \Gamma) = 0 \) for all singular \( k \)-cubes \( \Gamma \) in \( \mathbb{R}^n \).\(^1\)

**14c6 Exercise.** Similarly to **14a3**, find

\[
\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{\partial \Gamma_\varepsilon} \omega
\]

where \( \Gamma_\varepsilon : [-1, 1]^k \to \mathbb{R}^n \), \( \Gamma_\varepsilon(u_1, \ldots, u_k) = x + \varepsilon u_1 h_1 + \cdots + \varepsilon u_k h_k \), and \( \omega \) is an arbitrary \((k-1)\)-form (of class \( C^1 \)) on \( \mathbb{R}^n \).

Answer: \( \sum_{i=1}^{k} (-1)^{i-1}(D_{h_i}\omega(\cdot, h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_k))x \).

**14c7 Definition.** The exterior derivative of a \((k-1)\)-form \( \omega \) of class \( C^1 \) is a \( k \)-form \( d\omega \) defined by

\[
(d\omega)(\cdot, h_1, \ldots, h_k) = \sum_{i=1}^{k} (-1)^{i-1}D_{h_i}\omega(\cdot, h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_k) .
\]

**14c8 Theorem.** (Stokes’ theorem)

Let \( C \) be a \( k \)-chain in \( \mathbb{R}^n \), and \( \omega \) a \((k-1)\)-form of class \( C^1 \) on \( \mathbb{R}^n \). Then

\[
\int_C d\omega = \int_{\partial C} \omega.
\]

I skip the proof. The general case is somewhat more technical than the case \( k = 3 \), but no new ideas appear in the proof. The equivalent definition of exterior derivative becomes

\[
d\omega = \sum_{i_1 < \cdots < i_k} df_{i_1,\ldots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad \text{for} \quad \omega = \sum_{i_1 < \cdots < i_k} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} ;
\]

the form \( dx_{i_1} \wedge \cdots \wedge dx_{i_k} \) is a determinant similar to \( L_{i,j,k} \) of Sect. **14a**.

Still,

\[
d(d\omega) = 0 .
\]

\(^1\)Hint: you may use the idea of **14c2** if you like.
And still,
\[ \varphi^*(d\omega) = d(\varphi^*\omega). \]

Similarly to Sect. 14b, an \((n-1)\)-form \(\omega\) on \(\mathbb{R}^n\) corresponds to a vector field \(H\), namely,
\[
\omega(x, h_1, \ldots, h_{n-1}) = \det(H(x), h_1, \ldots, h_{n-1}),
\]
\[
\omega = \sum_{i=1}^{n} (-1)^{n-1} H_i \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge dx_n.
\]

For every singular \((n-1)\)-box \(\Gamma : B \to \mathbb{R}^n\),
\[
\int_{\Gamma} \omega = \int_{B} \det(H(\Gamma(u)), (D_1\Gamma)_u, \ldots, (D_{n-1}\Gamma)_u) \, du = \int_{\Gamma} H
\]
is the flux of \(H\) through \(\Gamma\); and for an \(n\)-box \(\Gamma\), \(\int_{\partial\Gamma} \omega = \int_{\partial\Gamma} H\) is the flux of \(H\) through the boundary of \(\Gamma\).

We have
\[
d\omega = \sum_{i=1}^{n} (-1)^{n-1} dH_i \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge dx_n =
\]
\[
= \sum_{i=1}^{n} dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dH_i \wedge dx_{i+1} \wedge dx_n =
\]
\[
= \sum_{i=1}^{n} \frac{\partial H_i}{\partial x_i} \, dx_1 \wedge \cdots \wedge dx_n = (\text{div } H) \, dx_1 \wedge \cdots \wedge dx_n,
\]
\[
\text{div } H = D_1 H_1 + \cdots + D_n H_n.
\]
Thus, Th. 14c8 gives the \(n\)-dimensional divergence theorem (recall (14b1)):
\[
\int_{\partial\Gamma} H = \int_{\Gamma} \text{div } H
\]
for every vector field \(H\) (of class \(C^1\)) on \(\mathbb{R}^n\) and every singular \(n\)-box \(\Gamma\) in \(\mathbb{R}^n\).
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