16 Integration: from single-chart to many-chart

16a Curvilinear iterated integral

16b Many-chart integration

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Single-chart pieces of a manifold are combined via partitions of unity. Curvilinear iterated integrals, Stokes’ and divergence theorems take their global geometric form.

16a Curvilinear iterated integral

Recall several facts.

* The iterated integral approach (Sect. 7) decomposes an integral over the plane into integrals over parallel lines. It also decomposes an integral over 3-dimensional space into integrals over parallel planes.\(^1\)

* A 3-dimensional integral decomposes into integrals over spheres, see 14b12 and 15d12.

* However, a naive attempt to decompose an integral over the plane into integrals over curves \(y = f(x) + a\) fails (see 15d9); a new factor appears, like Jacobian.

Thus, we want to understand, whether or not a 2-dimensional integral decomposes into integrals over curves \(\varphi(\cdot) = \text{const}\), and what about a new factor; and what happens in dimension 3 (and more).

First we try dimension \(0 + 1\). Let \(\varphi \in C^1(\mathbb{R})\), \(\forall x \varphi'(x) \neq 0\). A set \(M_c = \{x : \varphi(x) = c\}\), being a singleton \(\{\varphi^{-1}(c)\}\), may be treated as a 0-dimensional manifold in \(\mathbb{R}\); naturally, \(\int_{M_c} f = f(\varphi^{-1}(c))\). Thus, generally \(\int dc \int_{M_c} f \neq \int_{\mathbb{R}} f\); rather, \(\int dc \int_{M_c} f = \int f(\varphi^{-1}(c)) dc = \int f(x)|\varphi'(x)| dx = \int f|\varphi'|\); the new factor \(|\varphi'|\) appears. Roughly, it shows how many 0-manifolds \(M_c\) appear within an infinitesimal neighborhood of \(x\).

We turn to dimension \(1 + 1\). Let \(\varphi : \mathbb{R}^2 \to \mathbb{R}\) be of class \(C^1\) near \(0\), \(\varphi(0) = 0\), \((D\varphi)_0 \neq 0\). Then \(\varphi\) is a co-chart of the set \(M_0 = \{(x, y) : \varphi(x, y) = 0\}\) around \((0, 0)\), and \(\varphi(\cdot) - c\) is a co-chart of \(M_c = \{(x, y) : \varphi(x, y) = c\}\) provided that \(c\) is small enough. We restrict ourselves to small \(c\) and small \((x, y)\), then \(M_c\) are 1-manifolds. Assuming that a function \(f \in C(\mathbb{R}^2)\) has a compact

\(^1\)Or alternatively, parallel lines. In this course we restrict ourselves to dimension \(n + 1\); for dimension \(n + m\) see the "Coarea formula".
support within the small neighborhood of \((0, 0)\), we consider \(\int dc \int_{M_c} f\). It is easy to guess that

\[(16a1) \quad \int dc \int_{M_c} f = \int_{\mathbb{R}^2} f|\nabla \varphi|,
\]

since \(|\nabla \varphi(x, y)|\) shows roughly, how many curves \(M_c\) intersect an infinitesimal neighborhood of \((x, y)\). Note that both sides of \((16a1)\) are invariant under rotations of the plane (since the volume form is well-defined for an \(n\)-manifold in an \(N\)-dimensional Euclidean space).

The case of a linear function \(\varphi\) is simple and instructive. When proving \((16a1)\) for a linear \(\varphi\) we may assume (due to the rotation invariance) that \(\varphi(x, y) = ay\). Then

\[
\int dc \int_{M_c} f = \int dc \int dx f\left(x, \frac{c}{a}\right) = a \int dy \int dx f(x, y),
\]

which proves \((16a1)\) for a linear \(\varphi\). Taking \(\varphi(x, y) = ax + by\) with \(b \neq 0\) we get

\[
M_c = \left\{ \left(x, \frac{c - ax}{b}\right) : x \in \mathbb{R} \right\}, \quad |\nabla \varphi| = \sqrt{a^2 + b^2},
\]

\[
\int_{M_c} f = \int_{\mathbb{R}} f\left(x, \frac{c - ax}{b}\right) \sqrt{1 + \left(-\frac{a}{b}\right)^2} \, dx;
\]

\[
\int dc \int_{M_c} f = \frac{\sqrt{a^2 + b^2}}{|b|} \iint f\left(x, \frac{c - ax}{b}\right) \, dx \, dc;
\]

\[
\int_{\mathbb{R}^2} f|\nabla \varphi| = \sqrt{a^2 + b^2} \iiint f(x, y) \, dx \, dy;
\]

\[(16a1)\] becomes

\[
\frac{1}{|b|} \iiint f\left(x, \frac{c - ax}{b}\right) \, dx \, dc = \iiint f(x, y) \, dx \, dy,
\]

which follows also from the fact that the Jacobian \(\frac{\partial(x, c)}{\partial(x, y)} = |\begin{smallmatrix} 1 & 0 \\ a & b \end{smallmatrix}|\) of the mapping \((x, y) \mapsto (x, ax + by)\) is equal to \(b\).

The former argument (the rotation) fails for nonlinear \(\varphi\) (think, why), but the latter argument (the change of variables) still works, and generalizes to dimension \(n + 1\), as we’ll see soon.

Recall the implicit function theorem 5c1 (for \(c = 1\), and some notations changed): if \(x_0 \in \mathbb{R}^n\), \(y_0 \in \mathbb{R}\), \(\varphi : \mathbb{R}^{n+1} \to \mathbb{R}\) is continuously differentiable near \((x_0, y_0)\), \(\varphi(x_0, y_0) = 0\), and \(\frac{\partial \varphi}{\partial y}(x_0, y_0) \neq 0\), then there exist open neighborhoods \(U\) of \(x_0\) and \(V\) of \(y_0\) such that
(a) for every \( x \in U \) there exists one and only one \( y \in V \) satisfying \( \varphi(x, y) = 0 \);

(b) a function \( g : U \to V \) defined by \( \varphi(x, g(x)) = 0 \) is continuously differentiable, and \( \nabla g(x_0) = -\frac{1}{(\varphi_y(x_0, y_0))}\left(\frac{\partial \varphi}{\partial x}\right)(x_0, y_0) \).

Recall also the idea of the proof: a mapping \( h(x, y) = \left( x, \varphi(x, y) \right) \) is a diffeomorphism \( U \times V \to h(U \times V) \), and

\[
h^{-1}\left(\begin{array}{c} x \\ 0 \end{array}\right) = \left(\begin{array}{c} x \\ g(x) \end{array}\right).
\]

We need a bit more: there exists an open neighborhood \( W \) of \( 0 \) in \( \mathbb{R}^n \) such that for every \( c \in W \),

(a’) for every \( x \in U \) there exists one and only one \( y \in V \) satisfying \( \varphi(x, y) = c \);

(b’) a function \( g_c : U \to V \) defined by \( \varphi(x, g_c(x)) = c \) is continuously differentiable, and \( \nabla g_c(x) = -\frac{1}{(\varphi_y(x, y))}\left(\frac{\partial \varphi}{\partial x}\right)(x, y) \) whenever \( x \in U, \ y = g_c(x) \).

This is easy to prove; basically, \( h^{-1}\left(\begin{array}{c} x \\ c \end{array}\right) = \left(\begin{array}{c} x \\ g_c(x) \end{array}\right) \); for (b’), differentiate in \( x \) the relation \( \varphi(x, g_c(x)) = c \).

Thus, for every \( c \in W \) the set

\[
M_c = \{ (x, y) \in U \times V : \varphi(x, y) = c \}
\]

is an \( n \)-manifold in \( \mathbb{R}^{n+1} \); the function \( \varphi(\cdot) - c \) is a co-chart of \( M_c \); and the mapping \( U \ni x \mapsto \psi_c(x) = (x, g_c(x)) \) is a chart of the whole \( M_c \); in other words, \( M_c \) is the graph of \( g_c \). The set

\[
\bigcup_{c \in W} M_c = h^{-1}(U \times W)
\]

is an open neighborhood of \( (x_0, y_0) \).

**16a2 Proposition.** For every continuous, compactly supported function \( f \) on \( \bigcup_{c \in W} M_c \),

\[
\int_W dc \int_{M_c} f = \int f|\nabla \varphi|.
\]
16a3 Exercise. Find $J_{\psi}$ given $\psi(x) = (x, g(x)) \in \mathbb{R}^{n+1}$ for $x \in \mathbb{R}^n$ and $g \in C^1(\mathbb{R}^n)$.  

Answer: $\sqrt{1 + |\nabla g|^2}$.

Proof of Prop. 16a2 For every $c \in W$,

$$\int_{M_c} f = \int_U f(x, g_c(x)) \sqrt{1 + |\nabla g_c|^2} \, dx$$

due to 16a3, thus, the function $c \mapsto \int_{M_c} f$ is continuous, and

$$\int_W dc \int_{M_c} f = \int_{U \times W} f(x, g_c(x)) \sqrt{1 + |\nabla g_c(x)|^2} \, dx \, dc.$$ 

On the other hand, $Dh = \left( \begin{array}{cc} \frac{\partial \varphi}{\partial x} & 0 \\ \frac{\partial \varphi}{\partial y} & 0 \end{array} \right)$, therefore $\det(Dh) = \frac{\partial \varphi}{\partial y}$. Also,

$$1 + |\nabla g_c(x)|^2 = 1 + \left( \frac{1}{(\frac{\partial \varphi}{\partial y}(x,y))} \right)^2 \left( \frac{\partial \varphi}{\partial x}(x,y) \right)^2 = \left| \frac{\nabla \varphi(x,y)}{\det(Dh)(x,y)} \right|^2$$

whenever $y = g_c(x)$. Finally, we apply change of variables:

$$\int_W dc \int_{M_c} f = \int_{U \times W} f(x, h^{-1}(x,c)) \left| \frac{\nabla \varphi(h^{-1}(x,c))}{\det(D(h^{-1})(x,c))} \right| \, dx \, dc =$$

$$= \int_{U \times W} f(x, h^{-1}(x,c)) \left| \frac{\nabla \varphi(h^{-1}(x,c))}{\det(D(h^{-1})(x,c))} \right| \, dx \, dc =$$

$$= \int_{U \times W} f(x, h^{-1}(x,c)) \left| \frac{\nabla \varphi(h^{-1}(x,c))}{\det(D(h^{-1})(x,c))} \right| \, dx \, dy.$$ 

\hfill $\Box$

16b Many-chart integration

Recall that $\int_{(M,\mathcal{O})} \omega$ is defined by (15c2) whenever $(M,\mathcal{O})$ is an oriented $n$-manifold and $\omega$ a single-chart $n$-form on $M$. The linearity,

$$(16b1) \quad \int_{(M,\mathcal{O})} (c_1 \omega_1 + c_2 \omega_2) = c_1 \int_{(M,\mathcal{O})} \omega_1 + c_2 \int_{(M,\mathcal{O})} \omega_2,$$

\footnote{Hint: in order to avoid working hard on a determinant, use the rotation invariance.}
is ensured by (15c2) provided that both forms $\omega_1, \omega_2$ have compact supports within the same chart.

The idea of a “partition of unity” was used in Sect. 8h (when proving Th. 8a5) in a rudimentary form: partition into integrable functions. Now we need a bit more: partition into continuous functions.\footnote{Still more will be needed in the proof of Th. 16b15: partition into $C^1$ functions. (Ultimately, partitions into $C^\infty$ functions exist, but we do not need them.)}

**16b2 Lemma.** Let $M \subset \mathbb{R}^N$ be an $n$-manifold and $K \subset M$ a compact set. Then there exist single-chart continuous functions $f_1, \ldots, f_k : M \to [0, 1]$ such that $f_1 + \cdots + f_k = 1$ on $K$.

**Proof.** For every $x \in K$ a function $g_x : y \mapsto (\varepsilon_x - |y - x|)^+$ is single-chart if $\varepsilon_x$ is small enough, continuous, and positive in the open $\varepsilon_x$-neighborhood of $x$. These neighborhoods are an open covering of $K$; we choose a finite subcovering and get single-chart functions $g_1, \ldots, g_k : M \to [0, \infty)$ whose sum $g = g_1 + \cdots + g_k$ is (strictly) positive on $K$. We take $\varepsilon > 0$ such that $g(x) \geq \varepsilon$ on $K$ and note that functions $f_1, \ldots, f_k : M \to [0, \infty)$ defined by

$$f_i(x) = \frac{g_i(x)}{\max(g(x), \varepsilon)}$$

have the required properties. \qed

It follows that every compactly supported $n$-form on $M$ is the sum of single-chart $n$-forms,

$$\omega = \omega_1 + \cdots + \omega_k, \quad \omega_i = f_i \omega.$$ 

It is tempting to define (assuming that $\mathcal{O}$ is an orientation of $M$)

$$(16b3) \quad \int_{(M, \mathcal{O})} \omega = \int_{(M, \mathcal{O})} \omega_1 + \cdots + \int_{(M, \mathcal{O})} \omega_k;$$

however, does this sum depend on the choice of $\omega_1, \ldots, \omega_k$? If $\omega_1 + \cdots + \omega_k = \omega = \tilde{\omega}_1 + \cdots + \tilde{\omega}_k$ then $\omega_1 + \cdots + \omega_k + (-\tilde{\omega}_1) + \cdots + (-\tilde{\omega}_k) = 0$; the question is, whether the corresponding sum of integrals must vanish, or not.

**16b4 Lemma.** Let $\omega_1, \ldots, \omega_\ell$ be single-chart $n$-forms on an $n$-manifold $M$, and $\mathcal{O}$ an orientation of $M$;

$$\text{if} \quad \omega_1 + \cdots + \omega_\ell = 0 \quad \text{then} \quad \int_{(M, \mathcal{O})} \omega_1 + \cdots + \int_{(M, \mathcal{O})} \omega_\ell = 0.$$
Proof. Lemma 16b2 gives single-chart continuous functions $f_1, \ldots, f_k$ such that $f_1 + \cdots + f_k = 1$ on a compact set that supports $\omega_1, \ldots, \omega_\ell$. By (16b1), on one hand,
\[
\sum_{j=1}^{\ell} \int_{(M,\mathcal{O})} f_i \omega_j = \int_{(M,\mathcal{O})} \sum_{j=1}^{\ell} f_i \omega_j = 0 ,
\]
since $f_i$ is single-chart; and on the other hand,
\[
\sum_{i=1}^{k} \int_{(M,\mathcal{O})} f_i \omega_j = \int_{(M,\mathcal{O})} \sum_{i=1}^{k} f_i \omega_j = \int_{(M,\mathcal{O})} \omega_j ,
\]
since $\omega_j$ is single-chart. Therefore
\[
\sum_{j=1}^{\ell} \int_{(M,\mathcal{O})} \omega_j = \sum_{j=1}^{\ell} \sum_{i=1}^{k} \int_{(M,\mathcal{O})} f_i \omega_j = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \int_{(M,\mathcal{O})} f_i \omega_j = \sum_{i=1}^{k} 0 = 0 .
\]

We see that (16b3) is indeed a correct definition of $\int_{(M,\mathcal{O})} \omega$ whenever $\omega$ is a compactly supported $n$-form on $M$.

Now we can define the $n$-dimensional volume of a compact oriented $n$-manifold $(M, \mathcal{O})$ by
\[
V_n(M, \mathcal{O}) = \int_{(M,\mathcal{O})} \mu_{(M,\mathcal{O})} \in (0, \infty)
\]
where $\mu_{(M,\mathcal{O})}$ is the volume form on $(M, \mathcal{O})$. However, the Möbius strip should have an area, too!

We want to define
\[
(16b5) \quad \int_M f = \int_{(G,\psi)} f \mu_{(G,\psi)}
\]
for a single-chart $f \in C(M)$; here $(G, \psi)$ is a chart such that $f$ is compactly supported within $\psi(G)$, and $\mu_{(G,\psi)}$ is the volume form on the $n$-manifold $\psi(G)$ (oriented, even if $M$ is non-orientable). To this end we need a counterpart of Lemma 15c3:
\[
\int_{(G_1,\psi_1)} f \mu_{(G_1,\psi_1)} = \int_{(G_2,\psi_2)} f \mu_{(G_2,\psi_2)}
\]
whenever $(G_1,\psi_1), (G_2,\psi_2)$ are charts such that $K \subset \psi_1(G_1) \cap \psi_2(G_2)$ for some compact $K$ that supports $f$. We do it similarly to the proof of 15c3, but
this time we split the relatively open set \( \tilde{G} = \psi_1(G_1) \cap \psi_2(G_2) \) in two relatively open sets \( \tilde{G}_-, \tilde{G}_+ \) according to the sign of \( \det D\varphi \) (having \( \psi_2^{-1} = \varphi \circ \psi_1^{-1} \) on \( G \)). It remains to take into account that \( \mu(G_1, \psi_1) = \mu(G_2, \psi_2) \) on \( \tilde{G}_+ \) but \( \mu(G_1, \psi_1) = -\mu(G_2, \psi_2) \) on \( \tilde{G}_- \).

We see that (16b5) is indeed a correct definition of \( \int_M f \) for a single-chart \( f \in C(M) \). Now, similarly to (16b2), we define

\[
(16b6) \quad \int_M f = \int_M f_1 + \cdots + \int_M f_k
\]

whenever \( f = f_1 + \cdots + f_k \) with single-chart \( f_i \in C(M) \).

16b7 Exercise. (a) Prove that (16b6) is a correct definition of \( \int_M f \) for all compactly supported \( f \in C(M) \);\(^1\)

(b) formulate and prove linearity and monotonicity of the integral.

Now it is easy to define lower and upper integrals for discontinuous compactly supported functions \( M \to \mathbb{R} \) (recall 6i2), and then, Riemann integrability and Jordan measure on an \( n \)-manifold in \( \mathbb{R}^N \). For functions with no compact support, improper integral may be used. In particular, for a non-compact manifold \( M \),

\[
V_n(M) = \sup_{f \leq 1} \int_M f = \sup_{E} V_n(E)
\]

where \( f \) runs over compactly supported continuous (or just integrable) functions, and \( E \) runs over sets Jordan measurable in \( M \). Monotone convergence of volumes (similar to 9c1) holds.

16b8 Exercise. Find the area of the (non-compact) Möbius strip 15b7.

Here is a harder exercise: find the area of the compact non-orientable 2-manifold in \( \mathbb{R}^6 \) introduced in 15b9.

**Curvilinear iterated integral revisited**

16b9 Theorem. Let \( G \subset \mathbb{R}^n \) be an open set, \( n > 1 \), \( \varphi \in C^1(G) \), \( \forall x \in G \ \nabla \varphi(x) \neq 0 \), and \( f \in C(G) \) compactly supported. Then for every \( c \in \varphi(G) \) the set \( M_c = \{ x \in G : \varphi(x) = c \} \) is an \( (n-1) \)-manifold in \( \mathbb{R}^n \), a function \( c \mapsto \int_{M_c} f \) on \( \varphi(G) \) is continuous and compactly supported, and

\[
\int_{\varphi(G)} dc \int_{M_c} f = \int_{G} f |\nabla \varphi|.
\]

\(^1\)Hint: use partitions of unity.
16b10 Remark. A function $c \mapsto V_{n-1}(M_c)$ need not be continuous on $\varphi(G)$. For a counterexample try $G = \{(x, y) : y < g(x)\} \subset \mathbb{R}^2$ and $\varphi(x, y) = y$.


16b12 Exercise. For $f \in C(\mathbb{R}^n \setminus \{0\})$ prove that

$$\int_{(0, \infty)} dr \int_{\{|x| = r\}} f = \int_{\mathbb{R}^n \setminus \{0\}} f,$$

where $\int_{(0, \infty)}$ and $\int_{\mathbb{R}^n \setminus \{0\}}$ are improper; that is, each side of the equality may be a number, $-\infty$, $+\infty$ or $\infty - \infty$.

In particular,

$$\int_{\mathbb{R}^n \setminus \{0\}} f(|x|) dx = \int_{(0, \infty)} V_{n-1}(S_r) f(r) dr,$$

where $S_r = \{x : |x| = r\}$ is the sphere. It is easy to see that $V_{n-1}(S_r) = r^{n-1}V_{n-1}(S_1)$; thus,

$$\int_{\mathbb{R}^n \setminus \{0\}} f(|x|) dx = V_{n-1}(S_1) \int_{(0, \infty)} r^{n-1} f(r) dr.$$

Now we may take $f(r) = e^{-r^2}$ and get

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \left(\int_{\mathbb{R}} e^{-t^2} dt\right)^n = (\sqrt{\pi})^n = \pi^{n/2}$$

(recall 9e); thus,

$$\pi^{n/2} = V_{n-1}(S_1) \int_{0}^{\infty} r^{n-1} e^{-r^2} dr.$$

Taking into account that

$$\int_{0}^{\infty} r^{n-1} e^{-r^2} dr = \int_{0}^{\infty} t^{(n-1)/2} e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{2} \Gamma\left(\frac{n}{2}\right)$$

(recall 9j1), we get

$$(16b13) \quad V_{n-1}(S_1) = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

1Hint: use 16a2 and a partition of unity.

2Hint: start with $f \geq 0$, approximate $f$ from below, apply 16b9.

3See also Sjamaar, Exer. 9.6.
Alternatively we may use the volume of the ball $B_1 = \{x : |x| < 1\}$,

$$V_n(B_1) = \frac{2\pi^{n/2}}{n\Gamma(n/2)},$$
calculated in Sect. 9j.

**Divergence theorem revisited**

An open set $G \subset \mathbb{R}^n$ is called *regular*, if $(G)^\circ = G$; that is, the interior of the closure of $G$ is equal to $G$. (Generally it cannot be less than $G$, but can be more than $G$; a simple example: $G = \mathbb{R} \setminus \{0\}$.) Equivalently, $G$ is regular if (and only if) $\partial G = \partial(\mathbb{R}^n \setminus G)$; that is, the boundary of the exterior of $G$ is equal to the boundary of $G$.

Let $G \subset \mathbb{R}^n$ be a bounded regular open set, $M \subset \mathbb{R}^n$ a (necessarily compact) $(n-1)$-manifold, and $\partial G = M$ (the topological boundary, nothing “singular”...). We want to prove that the flux of a vector field through $M$ is equal to the integral of its divergence over $G$. In the language of differential forms (recall 14c8, 14c9) it means a “nonsingular” Stokes’ theorem for $k = n - 1$: $\int_G d\omega = \int_M \omega$ for every $(n-1)$-form $\omega$ on $\mathbb{R}^n$. However, this makes no sense without orienting $G$ and $M$.

Recall 14c: the hyperface $\{1\} \times [-1,1]^{n-1}$ is a part of the boundary of the cube $(-1,1)^n$; the tangent space to the hyperface is spanned by vectors $e_2, \ldots, e_n$; and its orientation conforms to the basis $(e_2, \ldots, e_n)$ (in this order), while the orientation of the cube conforms to $(e_1, \ldots, e_n)$, of course. And the vector $e_1$ is the outward unit normal to the hyperface, according to the sign of the inequality $x_1 < 1$ on $(-1,1)^n$.

**16b14 Definition.** (a) A non-tangent vector $h \in \mathbb{R}^n \setminus T_xM$ is directed outwards, if $x - \varepsilon h$ belongs to $G$ and $x + \varepsilon h$ does not belong to $G$ for all $\varepsilon$ small enough;

(b) an orientation $\hat{\mathcal{O}}$ of $M$ conforms at $x \in M$ to an orientation $\mathcal{O}$ of $G$ if $(h_2, \ldots, h_n)$ conforms to $\hat{\mathcal{O}}_x$ whenever $h_1$ is directed outwards and $(h_1, h_2, \ldots, h_n)$ conforms to $\mathcal{O}_x$. (Here $h_2, \ldots, h_n \in T_xM$, $h_1 \notin T_xM$.)

For a non-regular $G$ it may happen that $x - \varepsilon h$ and $x + \varepsilon h$ both belong to $G$ (for all $\varepsilon$ small enough); but for a regular $G$ either $h$ or $(-h)$ must be directed outwards (and then the other is said to be directed inwards).

**16b15 Theorem.** Let $G \subset \mathbb{R}^n$ be a bounded regular open set, $M \subset \mathbb{R}^n$ an $(n - 1)$-manifold, $\partial G = M$, and orientations $\mathcal{O}$ of $G$ and $\hat{\mathcal{O}}$ of $M$ conform (at every point of $M$). Then

$$\int_{(G,\mathcal{O})} d\omega = \int_{(M,\hat{\mathcal{O}})} \omega$$
for every \((n-1)\)-form \(\omega\) of class \(C^1\) on \(\mathbb{R}^n\).

The divergence theorem follows.

**16b16 Theorem.** Let \(G \subset \mathbb{R}^n\) be a bounded regular open set, \(M \subset \mathbb{R}^n\) an \((n-1)\)-manifold, \(\partial G = M\). Then

\[
\int_G \text{div} \, H = \int_M \langle H, \bar{n} \rangle
\]

for every vector field \(H\) of class \(C^1\) on \(\mathbb{R}^n\); here \(\bar{n} : M \to \mathbb{R}^n\), \(\bar{n}(x)\) is the outward unit normal vector at \(x \in M\).

It remains to prove 16b15. Sometimes it is easy to construct an \(n\)-chain \(C\) such that \(C \sim (G, O)\) and \(\partial C \sim (M, \tilde{O})\) in the sense that \(\int_C \omega = \int_{(G, O)} \omega\) and \(\int_{\partial C} \omega = \int_{(M, \tilde{O})} \omega\); but in general this is problematic. Instead, one turns to a single-chart \(\omega\) via a partition of unity; and locally \(M\) is just the graph of a function.

We restrict ourselves to \(n = 2\); the general case is quite similar.

We define a *good box* (for given \(G\) and \(M\)) as an open box \(B \subset \mathbb{R}^2\) such that \(M \cap B\) is either the empty set or the graph of a function, either \(y = f(x)\) or \(x = g(y)\). More exactly, “\(y = f(x)\)” means here the following:

\[
B = U \times V \text{ for some open intervals } U, V \subset \mathbb{R}; f \in C^1(U), f(U) \subset V; \text{ and } M \cap B = \{(x, f(x)) : x \in U\}.
\]

(The case “\(x = g(y)\)” is interpreted similarly.)

The closure \(G \cup M\) of \(G\) is compact, and all good boxes are its open covering. We choose a finite covering: \(G \cup M \subset B_1 \cup \cdots \cup B_k\), and construct a corresponding partition of unity of class \(C^1\):

\[
f_1, \ldots, f_k : \mathbb{R}^n \to [0, 1] \quad \text{are continuously differentiable,}\n\]

\[
f_i(\cdot) = 0 \quad \text{outside } B_i,
\]

\[
f_1 + \cdots + f_k = 1 \quad \text{on } G \cup M.
\]

To this end, similarly to the proof of 16b2, we let \(g = g_1 + \cdots + g_k\), take \(\varepsilon\) such that \(g(\cdot) \geq \varepsilon\) on \(K\), and put

\[
f_i(x) = \frac{g_i(x)}{g(x) + \frac{\varepsilon}{2}(1 - \frac{g(x)}{\varepsilon})^2};
\]

but this time we need \(g_i \in C^1\). We obtain \(g_i\) by a linear transformation (of arguments) from (say)

\[
g(x, y) = h(x)h(y),
\]

\[
h(t) = \begin{cases} 
(1 - t^2)^2 & \text{for } -1 < t < 1, \\
0 & \text{otherwise};
\end{cases}
\]
then $f_1, \ldots, f_k$ have the required properties.

Given an $(n-1)$-form $\omega$ of class $C^1$ on $\mathbb{R}^n$, we have
\[
\omega = \omega_1 + \cdots + \omega_k \quad \text{on } G \cup M,
\]
where each $\omega_i = f_i \omega$ is an $(n-1)$-form of class $C^1$, and $\omega_i = 0$ outside $B_i$.

In order to prove the equality $\int_{(G,\mathcal{O})} d\omega = \int_{(M,\mathcal{O})} \omega$ it is sufficient (due to linearity) to prove the same equality for each $\omega_i$.

The case $M \cap B_i = \emptyset$ is simple: $\int_{(M,\mathcal{O})} \omega_i = 0$ (since $\omega_i = 0$ on $M$), and $\int_{(G,\mathcal{O})} d\omega = \pm \int_{B_i} d\omega = \pm \int_{\partial B_i} \omega = 0$ (since $\omega_i = 0$ on $\partial B_i$).

It remains to consider the case “$x = g(y)$” (since the case “$y = f(x)$” is similar). That is, $B_i = V \times U$, $g : U \to V$ is continuously differentiable, and $M \cap B_i = \{(g(y), y) : y \in U\}$. We do not know which orientation of $B$ conforms to the given orientation $\mathcal{O}$ of $G$, but it does not matter, since the other orientation changes the signs of both sides of the equality.

The set $(V \times U) \setminus M$ has exactly two connected components (think, why), one of them being $G \cap (V \times U)$ (think, why). We may assume that $G \cap (V \times U) = \{(x, y) \in V \times U : x < g(y)\}$; in the other case, “$x > g(y)$”, we flip the sign of $x$.

Consider a mapping $\psi_1 : U \to \mathbb{R}^2$, $\psi_1(y) = (g(y), y)$; $(U, \psi_1)$ is a chart of the 1-manifold $M \cap (V \times U)$.

The set $G \cap (V \times U)$ may be treated as a 2-manifold (in $\mathbb{R}^2$); a mapping $\psi_2 : V \times U \to \mathbb{R}^2$,
\[
\psi_2(x, y) = \left( a + \frac{x - a}{b - a} (g(y) - a), y \right),
\]
where $(a, b) = V$, is a diffeomorphism $V \times U \to G \cap (V \times U)$; and $(V \times U, \psi_2)$ is a chart of $G \cap (V \times U)$.

These charts, $(U, \psi_1)$ and $(V \times U, \psi_2)$, lead to orientations, $\mathcal{O}_1$ on $M \cap (V \times U)$ and $\mathcal{O}_2$ on $G \cap (V \times U)$, and these two orientations conform (according to (16b14)) at every $(g(y), y) \in M \cap (V \times U)$; here is why. The vector $(g'(y), 1) \in T_{(g(y), y)} M$ conforms to $\mathcal{O}_1$; the vector $(1, 0)$ is directed

\[1\] Why prefer “$x = g(y)$” to “$y = f(x)$”? Since our preferred hyperface $\{1\} \times [-1, 1]^{n-1}$ of $[-1, 1]^n$ for $n = 2$ is “$x = 1$”, not “$y = \ldots$”.\]
outwards; and the basis \( \{(1, 0), (g'(y), 1)\} \) conforms to \( \mathcal{O}_2 \), since \( \left| \frac{1}{g'(y)} \right| > 0 \), and \( \det D\psi_2 > 0 \) as well.

We apply Stokes’ theorem to the singular box \( \Gamma : V \times U \to \mathbb{R}^2 \), \( \Gamma(x, y) = (a + \frac{x-a}{b-a}(g(y) - a), y) \), getting \( \int_{\Gamma} d\omega = \int_{\partial\Gamma} \omega \). It remains to note that

\[
\int_{\Gamma} d\omega = \int_{(G \cap (V \times U), \mathcal{O}_2)} d\omega, \quad \int_{\partial\Gamma} \omega = \int_{(M \cap (V \times U), \mathcal{O}_1)} \omega.
\]

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