8 Change of variables

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Change of variables is the most powerful tool for calculating multidimensional integrals. Two kinds of differentiation are instrumental: of mappings (treated in Sections 2–5) and of set functions (treated here).

8a What is the problem

The area of a disk \( \{(x, y) : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2 \) may be calculated by iterated integral,

\[
\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = \int_{-1}^{1} 2\sqrt{1-x^2} \, dx = \ldots
\]

or alternatively, in polar coordinates,

\[
\int_{0}^{1} r \, dr \int_{0}^{2\pi} \, d\phi = \int_{0}^{1} 2\pi r \, dr = \pi;
\]

the latter way is much easier! Note “\( r \, dr \)” rather than “\( dr \)” (otherwise we would get \( 2\pi \) instead of \( \pi \)).

Why the factor \( r \)? In analogy to the one-dimensional theory we may expect something like \( \frac{dx \, dy}{\frac{dr}{d\phi}} \); is it \( r \)? Well, basically, it is \( r \) because an infinitesimal rectangle \([r, r+dr] \times [\phi, \phi+d\phi]\) of area \( dr \cdot d\phi \) on the \((r, \phi)\)-plane corresponds to an infinitesimal rectangle or area \( dr \cdot r \, d\phi \) on the \((x, y)\)-plane.
The factor \( r \) is nothing but \( |\det T| \) of Sect. 6n, where \( T \) is the linear approximation to the nonlinear mapping \( (r, \varphi) \mapsto (x, y) = (r \cos \varphi, r \sin \varphi) \) near a point \((r, \varphi)\).

Thus, we need a generalization of Theorem 6n1 (the linear transformation) to nonlinear transformations. Naturally, the nonlinear case needs more effort.

**8a1 Definition.** A diffeomorphism\(^1\) between open sets \( U, V \subset \mathbb{R}^n \) is an invertible mapping \( \varphi: U \to V \) such that both \( \varphi \) and \( \varphi^{-1} \) are continuously differentiable.

By the inverse function theorem 4c5, a homeomorphism \( \varphi: U \to V \) is a diffeomorphism if and only if \( \varphi \) is continuously differentiable and \((D\varphi)_x\) is an invertible operator for all \( x \in U \) (equivalently, the Jacobian \( \det(D\varphi)_x \) does not vanish on \( U \)).

And do not forget: in contrast to dimension one, the condition \( \det(D\varphi)_x \neq 0 \) does not guarantee that \( \varphi \) is one-to-one (as noted in 4b).

**8a2 Proposition.** Let \( U, V \subset \mathbb{R}^n \) be open sets, \( \varphi: U \to V \) a diffeomorphism, and \( E \subset U \). Then the following two conditions are equivalent.

(a) \( E \) is Jordan measurable and contained in a compact subset of \( U \);
(b) \( \varphi(E) \) is Jordan measurable and contained in a compact subset of \( V \).

**8a3 Definition.** A function \( f: E \to \mathbb{R} \) on a Jordan measurable set \( E \subset \mathbb{R}^n \) is integrable (on \( E \)) if the function \( x \mapsto \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{otherwise} \end{cases} \) is integrable on \( \mathbb{R}^n \). And in this case the integral of the latter function (over \( \mathbb{R}^n \)) is \( \int_E f \).

**8a4 Exercise.** (a) Let \( E_1 \subset E_2 \) be Jordan measurable, and \( f: E_2 \to \mathbb{R} \) integrable; then \( f|_{E_1} \) is integrable.
(b) Let \( E_1, E_2 \) be Jordan measurable, and \( f: E_1 \cup E_2 \to \mathbb{R} \); if \( f|_{E_1}, f|_{E_2} \) are integrable then \( f \) is integrable.
Prove it.

**8a5 Theorem.** Let \( U, V \subset \mathbb{R}^n \) be open sets, \( \varphi: U \to V \) a diffeomorphism, \( E \subset U \) a Jordan measurable set contained in a compact subset of \( U \), and \( f: \varphi(E) \to \mathbb{R} \) an integrable function. Then \( f \circ \varphi: E \to \mathbb{R} \) is integrable, and

\[
\int_{\varphi(E)} f = \int_E (f \circ \varphi)| \det D\varphi|.
\]

\(^1\)Namely, \( C^1 \) diffeomorphism.
On the other hand, it can happen that an open set is not Jordan measurable (even if bounded); worse, it can happen that $U \subset \mathbb{R}^2$ is a disk but $V = \varphi(U)$ is open, bounded but not Jordan measurable.\footnote{The Riemann mapping theorem is instrumental. See Sect. 18.8 “Change of variables” in book: D.J.H. Garling, “A course in mathematical analysis”, vol. 2 (2014).}

\textbf{8a6 Corollary.} If, in addition, $U$ and $V$ are Jordan measurable and $D\varphi$ is bounded on $U$ then integrability of $f : V \rightarrow \mathbb{R}$ implies integrability of $(f \circ \varphi)| \det D\varphi| : U \rightarrow \mathbb{R}$, and

$$\int_V f = \int_U (f \circ \varphi)| \det D\varphi| .$$

The proofs, given in Sect. 8h, are based on a transition from set functions to (ordinary) functions, inverse to integration. (Basically, we’ll prove that $| \det D\varphi|$ is the derivative of the set function $E \mapsto v(\varphi(E))$.\footnote{Hint: $1 < r < 3$; $\cos \theta > \frac{r^2 - 2}{4r}$.} This form of differentiation, introduced and examined in Sec. 8c may be partially new to you even in dimension one.

\textbf{8b Examples and exercises}

In this section we take for granted Proposition 8a2, Theorem 8a5 and Corollary 8a6 (to be proved later).

\textbf{8b1 Exercise.} (spherical coordinates in $\mathbb{R}^3$)

Consider the mapping $\Psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\Psi(r, \varphi, \theta) = (r \cos \varphi \sin \theta, r \sin \varphi \sin \theta, r \cos \theta)$.

(a) Draw the images of the planes $r = \text{const}$, $\varphi = \text{const}$, $\theta = \text{const}$, and of the lines $(\varphi, \theta) = \text{const}$, $(r, \theta) = \text{const}$, $(r, \varphi) = \text{const}$.

(b) Show that $\Psi$ is surjective but not injective.

(c) Show that $| \det D\Psi| = r^2 \sin \theta$. Find the points $(r, \varphi, \theta)$, where the operator $D\Psi$ is invertible.

(d) Let $V = (0, \infty) \times (-\pi, \pi) \times (0, \pi)$. Prove that $\Psi|_V$ is injective. Find $U = \Psi(V)$.

\textbf{8b2 Exercise.} Compute the integral $\iiint_{x^2+y^2+(z-2)^2 \leq 1} \frac{dz\,dy\,dx}{x^2+y^2+z^2}$.

Answer: $\pi \left(2 - \frac{3}{2} \log 3\right).$\footnote{Hint: $-\frac{\pi}{2} < \varphi < \frac{\pi}{4}$; $0 < r < \sqrt{2} \varphi$; $1 + \cos 2\varphi = 2 \cos^2 \varphi$; $\int \frac{d\varphi}{\cos^2 \varphi} = \tan \varphi$.}

\textbf{8b3 Exercise.} Compute the integral $\iint \frac{dz\,dy}{(1+x^2+y^2)^2}$ over one loop of the lemniscate $(x^2 + y^2)^2 = x^2 - y^2$.\footnote{Hints: use polar coordinates; $-\frac{\pi}{4} < \varphi < \frac{\pi}{4}$; $0 < r < \sqrt{\cos 2\varphi}$; $1 + \cos 2\varphi = 2 \cos^2 \varphi$; $\int \frac{d\varphi}{\cos^2 \varphi} = \tan \varphi$.}
8b4 Exercise. Compute the integral over the four-dimensional unit ball: 
\[ \iiint_{x^2+y^2+u^2+v^2 \leq 1} e^{x^2+y^2-u^2-v^2} \, dx \, dy \, du \, dv. \]

8b5 Exercise. Compute the integral \( \iiint |xyz| \, dx \, dy \, dz \) over the ellipsoid \( \{x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1\} \).
Answer: \( \frac{a^2 b^2 c^2}{6} \).

8b6 Exercise. Find the volume cut off from the unit ball by the plane \( lx + my + nz = p \).

The mean (value) of an integrable function \( f \) on a Jordan measurable set \( E \subset \mathbb{R}^n \) of non-zero volume is (by definition)
\[ \frac{1}{v(E)} \int_E f. \]
The centroid\(^4\) of \( E \) is the point \( C_E \in \mathbb{R}^n \) such that for every linear (or affine) \( f : \mathbb{R}^n \to \mathbb{R} \) the mean of \( f \) on \( E \) is equal to \( f(C_E) \). That is,
\[ C_E = \frac{1}{v(E)} \left( \int_E x_1 \, dx, \ldots, \int_E x_n \, dx \right), \]
which is often abbreviated to \( C_E = \frac{1}{v(E)} \int_E x \, dx. \)

8b7 Exercise. Find the centroids of the following bodies in \( \mathbb{R}^3 \):
(a) The cone built over the unit disk, the height of the cone is \( h \).
(b) The tetrahedron bounded by the three coordinate planes and the plane \( \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \).
(c) The hemispherical shell \( \{a^2 \leq x^2 + y^2 + z^2 \leq b^2, \ z \geq 0\} \).
(d) The octant of the ellipsoid \( \{x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1, \ x, y, z \geq 0\} \).

The solid torus in \( \mathbb{R}^3 \) with minor radius \( r \) and major radius \( R \) (for \( 0 < r < R < \infty \)) is the set
\[ \tilde{\Omega} = \{(x, y, z) : (\sqrt{x^2 + y^2} - R)^2 + z^2 \leq r^2\} \subset \mathbb{R}^3 \]
generated by rotating the disk
\[ \Omega = \{(x, z) : (x - R)^2 + z^2 \leq r^2\} \subset \mathbb{R}^2 \]
\(^1\)Hint: The integral equals \( \iiint_{x^2+y^2\leq 1} e^{x^2+y^2}(\iiint_{u^2+v^2\leq 1-(x^2+y^2)} e^{-(u^2+v^2)} \, du \, dv) \, dx \, dy \).
Now use the polar coordinates.
\(^2\)Hint: 6e14 can help.
\(^3\)Hint: 6m4 can help.
\(^4\)In other words, the barycenter of (the uniform distribution on) \( E \).
on the \((x, z)\) plane (with the center \((R, 0)\) and radius \(r\)) about the \(z\) axis.

Interestingly, the volume \(2\pi^2 R r^2\) of \(\tilde{\Omega}\) is equal to the area \(\pi r^2\) of \(\Omega\) multiplied by the distance \(2\pi R\) traveled by the center of \(\Omega\). (Thus, it is also equal to the volume of the cylinder \(\{(x, y, z) : (x, z) \in \Omega, y \in [0, 2\pi R]\}\).) Moreover, this is a special case of a general property of all solids of revolution.

8b8 Proposition. (The second Pappus’s centroid theorem)\(^1\)\(^2\) Let \(\Omega \subset (0, \infty) \times \mathbb{R} \subset \mathbb{R}^2\) be a Jordan measurable set and \(\bar{\Omega} = \{(x, y, z) : (\sqrt{x^2 + y^2}, z) \in \Omega\} \subset \mathbb{R}^3\). Then \(\bar{\Omega}\) is Jordan measurable, and

\[
v_3(\bar{\Omega}) = v_2(\Omega) \cdot 2\pi x_{CE};
\]

here \(C_E = (xC_E, yC_E, zC_E)\) is the centroid of \(E\).

8b9 Exercise. Prove Prop. 8b8\(^3\)

8c Differentiating set functions

As was noted in the end of Sect. 6a, in dimension one an (ordinary) function \(\tilde{F} : \mathbb{R} \to \mathbb{R}\) leads to a set function \(F : [s, t) \mapsto \tilde{F}(t) - \tilde{F}(s)\); clearly, \(F\) is additive: \(F([r, s)) + F([s, t)) = F([r, t))\). Moreover, every additive set function \(F\) defined on one-dimensional boxes corresponds to some \(\tilde{F}\) (unique up to adding a constant); namely, \(\tilde{F}(t) = F([0, t))\).

If \(\tilde{F}\) is differentiable, \(\tilde{F}' = f\), then \(F\) and \(f\) are related by

\[
\frac{F([t - \varepsilon, t])}{\varepsilon} \to f(t), \quad \frac{F([t, t + \varepsilon])}{\varepsilon} \to f(t) \quad \text{as} \ \varepsilon \to 0 + .
\]

\(^1\)Pappus of Alexandria (\(\approx 0290–0350\)) was one of the last great Greek mathematicians of Antiquity.

\(^2\)The first Pappus’s centroid theorem, about the surface area, has to wait for Analysis 4.

\(^3\)Hint: use cylindrical coordinates: \(\Psi(r, \varphi, z) = (r \cos \varphi, r \sin \varphi, z)\).
Equivalently,
\[(8c1) \quad \frac{F([t - \varepsilon_1, t + \varepsilon_2])}{\varepsilon_1 + \varepsilon_2} \to f(t) \quad \text{as } \varepsilon_1, \varepsilon_2 \to 0^+ .\]

And if \( f \) is integrable on \([s, t]\) then\(^1\)
\[F([s, t]) = \int_{[s, t]} f .\]

In dimension 2 a similar construction exists, but is more cumbersome and less useful:
\[F([s_1, t_1] \times [s_2, t_2]) = \tilde{F}(t_1, t_2) - \tilde{F}(t_1, s_2) - \tilde{F}(s_1, t_2) + \tilde{F}(s_1, s_2) ;\]
\[\tilde{F}(s, t) = F([0, s] \times [0, t]);\]
this time \( \tilde{F} \) is unique up to adding \( \varphi(t_1) + \psi(t_2) \). In higher dimensions \( \tilde{F} \) is even less useful; we do not need it. Instead, we generalize (8c1) as follows.

First, we define an additive box function.

**8c2 Definition.** An **additive box function** \( F \) (in dimension \( n \)) is a real-valued function on the set of all boxes (in \( \mathbb{R}^n \)) such that
\[F(B) = \sum_{C \in P} F(C)\]
whenever \( P \) is a partition of a box \( B \).

Second, we define the **aspect ratio** \( \alpha(B) \) of a box \( B = [s_1, t_1] \times \cdots \times [s_n, t_n] \subset \mathbb{R}^n \) by\(^2\)
\[\alpha(B) = \frac{\max(t_1 - s_1, \ldots, t_n - s_n)}{\min(t_1 - s_1, \ldots, t_n - s_n)} .\]

Clearly, \( \alpha(B) = 1 \) if and only if \( B \) is a cube.

Third, we define the **derivative** of an additive box function \( F \) at a point \( x \) as the limit of the ratio \( \frac{F(B)}{v(B)} \) as \( B \) tends to \( x \) in the following sense:
\[(8c3) \quad B \ni x ; \quad v(B) \to 0 ; \quad \alpha(B) \to 1 .\]

\(^1\)Can you prove it (a) for continuous \( f \), (b) in general? Try 6b1 in concert with the mean value theorem. Anyway, it is the one-dimensional case of (8e4).

\(^2\)It appears that “thin” boxes (of large aspect ratio) are dangerous to the main argument of the proof (see 8h1); this is why we need to control the aspect ratio.
Symbolically,

\[ F'(x) = \lim_{{B \to x}} \frac{F(B)}{v(B)} . \]

It means: for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[ \left| \frac{F(B)}{v(B)} - F'(x) \right| \leq \varepsilon \]
for every box \( B \ni x, \text{ vol}(B) \leq \delta \) and \( \alpha(B) \leq 1 + \delta \).

If this limit exists we say that \( F \) is differentiable at \( x \) (or on \( \mathbb{R}^n \), if the limit exists for all \( x \); or on a given box, etc).

In dimension one, \( F \) is differentiable if and only if \( \tilde{F} \) is, and \( F' = \tilde{F}' \).

In general the limit need not exist, and we introduce the lower and upper derivatives,

\[ \ast F'(x) = \liminf_{{B \to x}} \frac{F(B)}{v(B)}, \quad \ast F'(x) = \limsup_{{B \to x}} \frac{F(B)}{v(B)}. \]

8d Derivative of integral

Every locally integrable\(^1\) function \( f : \mathbb{R}^n \to \mathbb{R} \) leads to an additive box function \( F : B \mapsto \int_B f \) (as was seen in Sect. 6j).

Can we restore \( f \) from \( F \)? Surely not, since \( F \) is insensitive to a change of \( f \) on a set of volume zero (by 6g1). However, the equivalence class of \( f \) can be restored, as we’ll see soon.

We say that two functions \( f, g \) are equivalent, if \( \ast \int_B |f - g| = 0 \) for every box \( B \).

If two continuous functions are equivalent then they are equal (think, why).

8d1 Proposition. If \( F : B \mapsto \int_B f \) for a locally integrable function \( f : \mathbb{R}^n \to \mathbb{R} \), then the three functions \( \ast F', f, \ast F' \) are (pairwise) equivalent.

Proof. Given a box \( B \), we use Lipschitz functions \( f_L^-, f_L^+ : B \to \mathbb{R} \) (introduced in Sect. 6i) and their limits \( f_L^-, f_L^+ : B \to \mathbb{R};^2 \)

\[ f_L^-(x) \uparrow f_\infty^-(x), \quad f_L^+(x) \downarrow f_\infty^+(x) \quad \text{as } L \to \infty. \]

Clearly, \( f_{\infty}^- \leq f \leq f_{\infty}^+ \). We know that \( \int_B f_L^- \uparrow \int_B f \) and \( \int_B f_L^+ \downarrow \int_B f \) as \( L \to \infty \). Thus,

\[ \ast \int_B |f - f_{\infty}^+| = \int_B (f_{\infty}^+ - f) \leq \lim L \ast \int_B (f_L^+ - f) = 0, \]

\(^1\)That is, integrable on every box.

\(^2\)In fact, \( f_{\infty}^-(x) = \liminf_{{x_1 \to x}} f(x_1) \) and \( f_{\infty}^+(x) = \limsup_{{x_1 \to x}} f(x_1) \), but we do not need it.
therefore \( f \) and \( f^+_\infty \) are equivalent. Similarly, \( f \) and \( f^-\infty \) are equivalent. On the other hand,
\[
\frac{F(B)}{v(B)} = \frac{1}{v(B)} \int_B f \leq \sup_B f,
\]
therefore
\[
\ast F'(x) = \lim_{B \to x} \frac{F(B)}{v(B)} \leq \limsup_{B \to x} f^+_L = f^+_\infty(x)
\]
for all \( L \), which shows that \( \ast F' \leq f^+_\infty \). Similarly, \( \ast F' \geq f^-\infty \). We see that \( f^-\infty \leq \ast F' \leq f^+_\infty \) and \( f^-\infty , f, f^+_\infty \) are equivalent, therefore all these functions are equivalent. \( \blacksquare \)

8e Integral of derivative

8e1 Proposition. (a) If an additive box function \( F \) is differentiable on a box \( B \) then
\[
v(B) \inf_{x \in B} F'(x) \leq F(B) \leq v(B) \sup_{x \in B} F'(x).
\]
(b) For every additive box function \( F \),
\[
v(B) \inf_{x \in B} \ast F'(x) \leq F(B) \leq v(B) \sup_{x \in B} \ast F'(x).
\]

8e2 Lemma. For every partition \( P \) of a box \( B \) and every additive box function \( F \),
\[
\min_{C \in P} \frac{F(C)}{v(C)} \leq \frac{F(B)}{v(B)} \leq \max_{C \in P} \frac{F(C)}{v(C)}.
\]

Proof. Denoting \( a = \min_{C \in P} \frac{F(C)}{v(C)} \) and \( b = \max_{C \in P} \frac{F(C)}{v(C)} \) we have \( av(C) \leq F(C) \leq bv(C) \) for all \( C \in P \); the sum over \( C \) gives \( av(B) \leq F(B) \leq bv(B) \). \( \blacksquare \)

8e3 Lemma. For every box \( B \) and every \( \varepsilon > 0 \) there exists a partition \( P \) of \( B \) such that \( v(C) \leq \varepsilon \) and \( a(C) \leq 1 + \varepsilon \) for all \( C \in P \).

Proof. Given \( B = [s_1, t_1] \times \cdots \times [s_n, t_n] \), for arbitrary natural number \( K \) we define natural numbers \( k_1, \ldots, k_n \) by
\[
\frac{k_1 - 1}{K} \leq t_1 - s_1 < \frac{k_1}{K}, \ldots, \frac{k_n - 1}{K} \leq t_n - s_n < \frac{k_n}{K},
\]
divide \([s_1, t_1]\) into \( k_1 \) equal intervals, \( \ldots, [s_n, t_n]\) into \( k_n \) equal intervals, and accordingly, \( B \) into \( k_1 \ldots k_n \) equal boxes, each \( C \in P \) being a shift of \([0, \frac{t_1 - s_1}{k_1}] \times \cdots \times [0, \frac{t_n - s_n}{k_n}]\). For arbitrary \( i, j \in \{1, \ldots, n\} \) we have
\[
\frac{t_i - s_i}{k_i} = \frac{(t_i - s_i) k_j}{k_i(t_j - s_j)} \leq \frac{k_i k_j}{k_i(k_j - 1)} = \frac{k_j}{k_j - 1} = 1 + \frac{1}{k_j - 1} \leq 1 + \frac{1}{K(t_j - s_j) - 1}.
\]
Thus,

\[ \alpha(C) \leq 1 + \frac{1}{K \min(t_1-s_1, \ldots, t_n-s_n) - 1} \to 0 \quad \text{as} \quad K \to \infty. \]

Also,

\[ v(C) = \frac{t_1-s_1}{k_1} \cdots \frac{t_n-s_n}{k_n} \leq \frac{1}{K^n} \to 0 \quad \text{as} \quad K \to \infty. \]

It remains to take \( K \) large enough.

Proof of Prop. 8e1. Item (a) is a special case of (b); we’ll prove (b).

Lemma 8e3 (with \( \varepsilon = 1 \)) gives a partition \( P_1 \) of \( B \) such that \( v(C) \leq 1 \) and \( \alpha(C) \leq 1 + 1 \) for all \( C \in P_1 \). Lemma 8e2 gives \( C_1 \in P_1 \) such that \( F(C_1) \geq v(B) \). We repeat the process for \( C_1 \) (in place of \( B \)) and \( \varepsilon = 1/2 \) and get \( C_2 \subset C_1 \) such that \( v(C_2) \leq 1/2 \), \( \alpha(C_2) \leq 1 + 1/2 \) and \( \frac{F(C_2)}{v(C_2)} \geq \frac{F(C_1)}{v(C_1)} \geq \frac{F(B)}{v(B)} \). Continuing this way we get boxes \( B \supset C_1 \supset C_2 \supset \ldots \), \( v(C_k) \to 0 \), \( \alpha(C_k) \to 1 \), and \( \frac{F(C_k)}{v(C_k)} \geq \frac{F(B)}{v(B)} \) for all \( k \). The intersection of all \( C_k \) is \( \{x\} \) for some \( x \in B \), and \( C_k \to x \) in the sense of (8c3). Thus, \( F'(x) \geq \limsup_k \frac{F(C_k)}{v(C_k)} \geq \frac{F(B)}{v(B)} \), and therefore \( F(B) \leq v(B) \sup_{x \in B} F'(x) \). The other inequality is proved similarly (or alternatively, turn to \((-F))\).

Combining 8e1(a) and 6b1 we get

\[ (8e4) \quad F(B) = \int_B F' \]

whenever \( F' \) exists and is integrable on \( B \). Here is a more general result.

8e5 Exercise. Prove that

\[ \star \int_B \star F' \leq F(B) \leq \int_B \star F' \]

for every box \( B \) and additive box function \( F \) such that \( \star F' \) and \( \star F' \) are bounded on \( B \).

If \( \star \int_B \star F' = \int_B \star F' \) then \( \star F' \) and \( \star F' \) are integrable and moreover, every function sandwiched between them is integrable (with the same integral).\(^1\)

In this case it is convenient to interpret \( F' \) as any such function and write

\[ F(B) = \int_B F' \]

even though \( F \) may be non-differentiable at some points. (You surely know one-dimensional examples!) However, the equality \( \star \int_B \star F' = \int_B \star F' \) may fail; here is a counterexample.

\(^1\)A similar situation appeared in Sect. 7d.
8e6 Example. There exists a nonnegative box function $F$ (in one dimension) such that $\int_{[0,1]} F' < \int_{[0,1]} F$.

We choose disjoint intervals $[s_k, t_k] \subset [0,1]$, whose union is dense on $[0,1]$, such that $\sum_k (t_k - s_k) = a \in (0,1)$, define $F$ by

$F([s,t]) = \sum_k \text{length}([s_k, t_k] \cap [s,t])$,

and observe that $F([0,1]) = a$, $0 \leq \int F' \leq \int F'$ and

$F'(x) = 1$ for all $x \in \bigcup_k (s_k, t_k)$

(think, why). Thus, $\int_{[0,1]} F' = 1$ (since the integrand is 1 on a dense set).

However, $\int_{[0,1]} F' \leq F([0,1]) = a < 1$.

8f Set function induced by mapping

Consider a mapping $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ such that the inverse image $\varphi^{-1}(B)$ of every box $B$ is a bounded set. (An example: $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x, y) = x^2 + y^2$.) It leads to a pair of box functions $F_* \leq F^*$ (in dimension $n$),

$F_*(B) = v_*(\varphi^{-1}(B^*))$, $F^*(B) = v^*(\varphi^{-1}(B))$,

generally not additive but rather superadditive and subadditive: for every partition $P$ of a box $B$,

$F_*(B) \geq \sum_{C \in P} F_*(C)$, $F^*(B) \leq \sum_{C \in P} F^*(C)$,

which follows from (6f3), (6f4) and the fact that $\varphi^{-1}(C_1^*) \cap \varphi^{-1}(C_2^*) = \varphi^{-1}(C_1^* \cap C_2^*) = \emptyset$ when $C_1^* \cap C_2^* = \emptyset$.

If $F_*(B) = F^*(B)$ then $\varphi^{-1}(B)$ is Jordan measurable, and $\varphi^{-1}(\partial B)$ is of volume zero; if this happens for all $B$ then the box function $F(B) = v(\varphi^{-1}(B))$ is additive. A useful sufficient condition is given below in terms of functions $J^-, J^+$ defined by

$J^-(x) = \liminf_{B \to x} \frac{F_*(B)}{v(B)}$, $J^+(x) = \limsup_{B \to x} \frac{F^*(B)}{v(B)}$.

---

1Equivalently, $F([s, t]) = v_*(A \cap [s, t])$ where $A = \bigcup_k [s_k, t_k]$.

2In fact, $F'$ is Lebesgue integrable, and its integral is equal to $a$. 
8f3 Proposition. If $J^-, J^+$ are locally integrable and equivalent then
\[
F_* (B) = F^* (B) = \int_B J^- = \int_B J^+
\]
for every box $B$.

In this case\(^1\)
\[
(8f4) \quad v(\varphi^{-1}(B)) = \int_B J
\]
where $J$ is any function equivalent to $J^-, J^+$.

8f5 Exercise. Prove existence of $J$ and calculate it for $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by (a) $\varphi(x, y) = x^2 + y^2$; (b) $\varphi(x, y) = \sqrt{x^2 + y^2}$; (c) $\varphi(x, y) = |x| + |y|$, taking for granted that the area of a disk is $\pi r^2$.

8f6 Exercise. Prove existence of $J$ and calculate it for $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\varphi(x, y, z) = (\sqrt{x^2 + y^2}, z)$, taking for granted Prop. 8b8.

We generalize 8e2, 8e1, 8e4.

8f7 Exercise. For every partition $P$ of a box $B$,
\[
\min_{C \in P} \frac{F_*(C)}{v(C)} \leq \frac{F_*(B)}{v(B)} \leq \frac{F^*(B)}{v(B)} \leq \max_{C \in P} \frac{F^*(C)}{v(C)}.
\]
Prove it.

8f8 Exercise.
\[
v(B) \inf_{x \in B} J^- (x) \leq F_*(B) \leq F^*(B) \leq v(B) \sup_{x \in B} J^+ (x).
\]
Prove it.

8f9 Exercise.
\[
\int_B J^- \leq F_*(B) \leq F^*(B) \leq \int_B J^+.
\]
Prove it.\(^2\)

Prop. 8f3 follows immediately.

---

\(^1\)Can this happen when $m < n$? If you are intrigued, try the inverse to the mapping of 6g11.

\(^2\)Curiously, the left-hand and the right-hand sides differ thrice: $\int, \int^*$; $\liminf, \limsup$; $v_*, v^*$.
Remark. Similar statements hold for a mapping defined on a subset of \( \mathbb{R}^m \) (rather than the whole \( \mathbb{R}^m \)). If \( \varphi : A \to \mathbb{R}^n \) for a given \( A \subset \mathbb{R}^m \) then \( \varphi^{-1}(B) \subset A \) for every \( B \), but nothing changes in (8f1), (8f2) and Prop. 8f3.

Remark. If \( J^-, J^+ \) are integrable and equivalent on a given box \( B \) (and not necessarily on every box) then \( v(\varphi^{-1}(C)) = \int_C J \) for every box \( C \subset B \).

Exercise. Calculate \( J \) for the projection mapping \( \varphi(x,y) = x \) (a) from the disk \( A = \{(x,y) : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2 \) to \( \mathbb{R} \); (b) from the annulus \( A = \{(x,y) : 1 \leq x^2 + y^2 \leq 4\} \subset \mathbb{R}^2 \) to \( \mathbb{R} \). Is \( J \) (locally) integrable?

Exercise. Calculate \( J \) for the mapping \( \varphi(x) = \sin x \) from the interval \([0, 10\pi] \subset \mathbb{R} \) to \( \mathbb{R} \). Is \( J \) (locally) integrable?

Change of variable in general

Proposition. If \( \varphi : \mathbb{R}^m \to \mathbb{R}^n \) is such that \(^1 J^- , J^+ \) are locally integrable and equivalent then for every integrable \( f : \mathbb{R}^n \to \mathbb{R} \) the function \( f \circ \varphi : \mathbb{R}^m \to \mathbb{R} \) is integrable and
\[
\int_{\mathbb{R}^m} f \circ \varphi = \int_{\mathbb{R}^n} f J .
\]

Proof. First, the claim holds when \( f = 1_B \) is the indicator of a box, since
\[
\int_{\mathbb{R}^n} f J = \int_{\mathbb{R}^m} J v(\varphi^{-1}(B)) = \int_{\mathbb{R}^m} 1_{\varphi^{-1}(B)} = \int_{\mathbb{R}^m} f \circ \varphi .
\]

Second, by linearity in \( f \) the claim holds whenever \( f \) is a step function (on some box, and 0 outside).

Third, given \( f \) integrable on a box \( B \) (and 0 outside), we consider arbitrary step functions \( g, h \) on \( B \) such that \( g \leq f \leq h \). We have \( g \circ \varphi \leq f \circ \varphi \leq h \circ \varphi \) and \( \int_{\mathbb{R}^m} g \circ \varphi = \int_B g J, \int_{\mathbb{R}^m} h \circ \varphi = \int_B h J \), thus,
\[
\int_B g J \leq \int f \circ \varphi \leq \int f \circ \varphi \leq \int_B h J , \quad \int_B g J \leq \int f J \leq \int_B h J .
\]
We take \( M \) such that \( |J(\cdot)| \leq M \) on \( B \) and get
\[
\int_B h J - \int_B g J = \int_B (h - g) J \leq M \int_B (h - g) ;
\]
thus, integrability of \( f \) implies integrability of \( f \circ \varphi \) and the needed equality for the integrals.

\(^1\)We still assume that the inverse image of a box is bounded.
8g2 Corollary. If \( \varphi : \mathbb{R}^m \to \mathbb{R}^n \) is such that \( J^-, J^+ \) are locally integrable and equivalent then:

(a) for every Jordan measurable set \( E \subset \mathbb{R}^n \) the set \( \varphi^{-1}(E) \subset \mathbb{R}^m \) is Jordan measurable;
(b) for every integrable \( f : E \to \mathbb{R} \) the function \( f \circ \varphi \) is integrable on \( \varphi^{-1}(E) \), and

\[
\int_{\varphi^{-1}(E)} f \circ \varphi = \int_E f J.
\]

Proof. (a) apply 8g1 to \( f = \mathbb{1}_E \); (b) apply 8g1 to \( f \mathbb{1}_E \).

8g3 Remark. If \( \varphi : A \to \mathbb{R}^n \) is such that \( J^-, J^+ \) are integrable and equivalent on a given box \( B \) (and not necessarily on every box) then for every integrable \( f : B \to \mathbb{R} \) the function \( f \circ \varphi \) is integrable on \( \varphi^{-1}(B) \), and

\[
\int_{\varphi^{-1}(B)} f \circ \varphi = \int_B f J.
\]

Also, 8g2 holds for \( E \subset B \).

8g4 Exercise. (a) Prove that

\[
\int_{x^2+y^2 \leq 1} f(\sqrt{x^2+y^2}) \, dx \, dy = 2\pi \int_{[0,1]} f(r) \, r \, dr
\]

for every integrable \( f : [0,1] \to \mathbb{R} \);

(b) calculate \( \int_{x^2+y^2 \leq 1} e^{-(x^2+y^2)/2} \, dx \, dy \). (Could you do it by iterated integrals?)

8h Change of variable for a diffeomorphism

8h1 Proposition. Let \( U, V \subset \mathbb{R}^n \) be open sets and \( \varphi : V \to U \) a diffeomorphism, then\(^1\)

\[
J^-(x) = J^+(x) = |\det(D\psi)_x|
\]

for all \( x \in U \); here \( \psi = \varphi^{-1} : U \to V \).

Proof. Let \( x_0 \in U \). Denote \( T = (D\psi)_{x_0} \). By Theorem 6n1, \( v(T(E)) = |\det T| v(E) \) for every Jordan measurable \( E \subset \mathbb{R}^n \). Note that \( \varphi^{-1}(E) = \psi(E) \).

It is sufficient to prove that

\[
\frac{v_*(\psi(B))}{v(T(B))} \to 1, \quad \frac{v^*(\psi(B))}{v(T(B))} \to 1 \quad \text{as } B \to x.
\]

Similarly to Sections 3e, 4c we may assume that \( x_0 = 0, \psi(x_0) = 0 \) and \( T = \text{id} \); also, for every \( \varepsilon > 0 \) we have a neighborhood \( U_\varepsilon \) of 0 such that

\[
(1 - \varepsilon)|x_1 - x_2| \leq |y_1 - y_2| \leq (1 + \varepsilon)|x_1 - x_2|
\]

\(^1\)\( \det D\psi \) is called the Jacobian of \( \psi \) and often denoted by \( J_\psi \).
whenever \( x_1, x_2 \in U_\varepsilon \) and \( y_1 = \psi(x_1), y_2 = \psi(x_2) \). Here \(| \cdot |\) is the Euclidean norm; but we can get the same (taking a smaller neighborhood if needed) for an equivalent norm:

\[
(1 - \varepsilon) \| x_1 - x_2 \| \leq \| y_1 - y_2 \| \leq (1 + \varepsilon) \| x_1 - x_2 \|
\]

where \( \| x \| = \max(|x_1|, \ldots, |x_n|) \) for \( x = (x_1, \ldots, x_n) \).

That is, \( \{ x : \| x \| \leq r \} = [-r, r]^n \) is a cube.

We may assume that \( B \subset U_\varepsilon \) and \( \alpha(B) \leq 1 + \varepsilon \). Denoting the center of \( B \) by \( x_B \) we have

\[
\| x - x_B \| \leq r_B \implies x \in B \implies \| x - x_B \| \leq (1 + \varepsilon) r_B
\]

for some \( r_B > 0 \). It is sufficient to prove that

\[
(1 - \varepsilon)^2 (B - x_B) \subset \psi(B) - y_B \subset (1 + \varepsilon)^2 (B - x_B)
\]

(where \( y_B = \psi(x_B) \)), since this implies \((1 - \varepsilon)^2 \nu(B) \leq \nu_\ast(\psi(B)) \leq \nu_\ast(\psi(B)) \leq (1 + \varepsilon)^2 \nu(B)\).

On one hand, \( \psi(B) - y_B \subset (1 + \varepsilon)^2 (B - x_B) \) since

\[
x \in B \implies \| \psi(x) - y_B \| \leq (1 + \varepsilon) \| x - x_B \| \leq (1 + \varepsilon)^2 r_B \implies \psi(x) - y_B \in (1 + \varepsilon)^2 (B - x_B).
\]

On the other hand, \((1 - \varepsilon)^2 (B - x_B) \subset \psi(B) - y_B \) since

\[
y - y_B \in (1 - \varepsilon)^2 (B - x_B) \implies \
\| \varphi(y) - x_B \| \leq \frac{1}{1 - \varepsilon} \| y - y_B \| \leq (1 - \varepsilon)(1 + \varepsilon) r_B \leq r_B \implies \
\varphi(y) \in B \implies y - y_B \in \psi(B) - y_B.
\]

We see that \( J^- , J^+ \) are integrable and equivalent (moreover, equal and continuous) on every box \( B \subset U \). According to \( 8g_2 \) (and \( 8g_3 \)), for every Jordan measurable \( E \subset B \) and integrable \( f : E \to \mathbb{R} \),

\[
(8h_2) \quad \psi(E) \text{ is Jordan measurable},
\]

\[
(8h_3) \quad f \circ \varphi \text{ is integrable on } \psi(E), \text{ and } \int_{\psi(E)} f \circ \varphi = \int_{E} f | \det D\psi |.
\]

Given a compact subset \( K \subset U \), we generally cannot cover \( K \) by a single box \( B \subset U \), but we can cover it by a finite collection of such boxes.
Lemma. If $U \subset \mathbb{R}^n$ is open and $K \subset U$ is compact then $K \subset B_1 \cup \cdots \cup B_k \subset U$ for some boxes $B_1, \ldots, B_k$ (and some $k$).

Proof. The number $\varepsilon = \inf_{x \in K} \text{dist}(x, \mathbb{R}^n \setminus U)$ is not 0, since the function $x \mapsto \text{dist}(x, \mathbb{R}^n \setminus U)$ is continuous (moreover, Lip(1)) on $K$. For $\delta = \frac{\varepsilon}{2\sqrt{n}}$ each $\delta$-pixel (recall the end of Sect. 6k) intersecting $K$ is contained in $U$.

Corollary. $\psi(E)$ is Jordan measurable whenever $E \subset U$ is a Jordan measurable set contained in a compact subset of $U$.

Proof. $E \subset B_1 \cup \cdots \cup B_k$; sets $\psi(E \cap B_i)$ are Jordan measurable by (8h2); their union $\psi(E)$ is thus Jordan measurable.

Proposition 8a2 follows immediately. Theorem 8a5 needs a bit more effort.

Given $A = B_1 \cup \cdots \cup B_k$ and $f : A \to \mathbb{R}$, can we represent it as $f = f_1 + \cdots + f_k$ where each $f_i$ vanishes outside $B_i$? Yes, we can; such technique is called “partition of unity” and will be used repeatedly in Analysis 4. This time its use is quite trivial, and could be avoided easily, but I do not want to miss a good opportunity to get acquainted with it.

We define functions $\rho_1, \ldots, \rho_k : A \to [0, 1]$ by

$$
\rho_i(x) = \begin{cases} 
\frac{1}{1_{B_1}(x) + \cdots + 1_{B_k}(x)} & \text{if } x \in B_i, \\
0 & \text{otherwise.}
\end{cases}
$$

Clearly, $\rho_1 + \cdots + \rho_k = 1$ on $A$, each $\rho_i$ vanishes outside $B_i$ and is integrable on $B_i$ (just because it is a step function).

Given an integrable $f : A \to \mathbb{R}$, we introduce $f_1 = f \rho_1, \ldots, f_k = f \rho_k$; by (8h3), $\int_{\psi(B_i)} f_i \circ \varphi = \int_{B_i} f_i \det D\psi |$, that is, $\int_{\psi(A)} f_i \circ \varphi = \int_{A} f_i \det D\psi |$; the sum over $i = 1, \ldots, k$ gives $\int_{\psi(A)} f \circ \varphi = \int_{A} f \det D\psi |$. Applying it to $f 1_E$ for a Jordan measurable $E \subset A$ we get

$$
\int_{\psi(E)} f \circ \varphi = \int_{E} f \det D\psi |
$$

for integrable $f : E \to \mathbb{R}$.

In order to get Theorem 8a5 it remains to change notation. First, denote $g = f \circ \varphi$, then $f = g \circ \psi$, and $\int_{\psi(E)} g = \int_{E} (g \circ \psi) \det D\psi |$. Second, rename $g$ into $f$ and $\psi$ into $\varphi$.

\footnote{Do you want to propose a simpler construction of $\rho_1, \ldots, \rho_k$? Well, you can; but let me exercise the construction that will be reused in less trivial situations in Analysis 4. I intentionally work with arbitrary (not just almost disjoint) boxes.}
Proof of Corollary 8a6. Given $\delta > 0$, 6k11 gives us a compact Jordan measurable set $E_1 \subset U$ such that $v(U \setminus E_1) \leq \delta$. Similarly, compact $F_1 \subset V$, $v(V \setminus F_1) \leq \delta$. By 8a2, $\varphi(E_1)$ and $\varphi^{-1}(F_1)$ are Jordan measurable. Introducing $E = E_1 \cup \varphi^{-1}(F_1)$ and $F = F_1 \cup \varphi(E_1)$ we see that the sets $E \subset U$ and $F \subset V$ are compact, Jordan measurable, $v(U \setminus E) \leq \delta$, $v(V \setminus F) \leq \delta$ and $F = \varphi(E)$. By 8a5, $\int_F f = \int_E (f \circ \varphi) |\det D\varphi|$. By Prop. 6d15, the function $(f \circ \varphi) |\det D\varphi|$ on $U$ is approximated by integrable functions $(f \circ \varphi) |\det D\varphi| \mathbb{1}_E$. By 8a5, $\int_U (f \circ \varphi) |\det D\varphi| \leq (\sup_{V} |f|)(\sup_{U} |\det D\varphi|)\delta$. The inequality
\[
\int_{U \setminus E} (f \circ \varphi) |\det D\varphi| \leq (\sup_{V} |f|)(\sup_{U} |\det D\varphi|)\delta
\]
shows that the function $(f \circ \varphi) |\det D\varphi|$ on $U$ is approximated by integrable functions $(f \circ \varphi) |\det D\varphi| \mathbb{1}_E$. By Prop. 6d15, the function $(f \circ \varphi) |\det D\varphi|$ is integrable on $U$, and $\int_U (f \circ \varphi) |\det D\varphi|$ is approximated by $\int_E (f \circ \varphi) |\det D\varphi| = \int_F f$. Also $\int_V f$ is approximated by $\int_F f$. In the limit we get $\int_V f = \int_U (f \circ \varphi) |\det D\varphi|$.

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