10 Improper integral

10a What is the problem
The $n$-dimensional unit ball in the $l_p$ metric,
\[ E = \{ (x_1, \ldots, x_n) : \left| x_1 \right|^p + \cdots + \left| x_n \right|^p \leq 1 \}, \]
is a Jordan measurable set, and its volume is a Riemann integral,
\[ v(E) = \int_{\mathbb{R}^n}1_E, \]
of a bounded function with bounded support. In Sect. 10g we’ll calculate it:
\[ v(E) = \frac{2^n \Gamma^n \left( \frac{1}{p} \right)}{p^n \Gamma \left( \frac{n}{p} + 1 \right)} \]
where $\Gamma$ is a function defined by
\[ \Gamma(t) = \int_0^\infty x^{t-1} e^{-x} \, dx \quad \text{for } t > 0; \]
here the integrand has no bounded support; and for $t = \frac{1}{p} < 1$ it is also unbounded (near 0). Thus we need a more general, so-called improper integral, even for calculating the volume of a bounded body!

In relatively simple cases the improper integral may be treated via \textit{ad hoc} limiting procedure adapted to the given function; for example,
\[ \int_0^\infty x^{t-1} e^{-x} \, dx = \lim_{k \to \infty} \int_{1/k}^k x^{t-1} e^{-x} \, dx. \]
In more complicated cases it is better to have a theory able to integrate rather
general functions on rather general $n$-dimensional sets. Different functions
may tend to infinity on different subsets (points, lines, surfaces), and still,
we expect $\int (af + bg) = a \int f + b \int g$ (linearity) to hold, as well as change of
variables.$^{1}$

10b  Positive integrands

We consider an open set $G \subset \mathbb{R}^n$ and functions $f : G \to [0, \infty)$ continuous
almost everywhere. We do not assume that $G$ is bounded. We also do not
assume that $G$ is Jordan measurable, even if it is bounded.$^2$ “Continuous
almost everywhere” means that the set $A \subset G$ of all discontinuity points of
$f$ satisfies $m^*(A) = 0$, recall Sect. 8f; but now $A$ need not be bounded. For
our purposes it is enough to know that $m^*(A) = 0$ if and only if $m^*(A_1) = 0$
for every bounded $A_1 \subset A$ (we may take this as the definition). We can use
the function $f \cdot \mathbb{1}_G$ equal $f$ on $G$ and 0 on $\mathbb{R}^n \setminus G$, but must be careful: $\mathbb{1}_G$
and $f \cdot \mathbb{1}_G$ need not be continuous almost everywhere.

We define

\begin{equation}
(10b1) \quad \int_G f = \sup \left\{ \int_{\mathbb{R}^n} g \left| g : \mathbb{R}^n \to \mathbb{R} \text{ integrable,} \quad 0 \leq g \leq f \text{ on } G, \quad g = 0 \text{ on } \mathbb{R}^n \setminus G \right\} \in [0, \infty].
\end{equation}

The condition on $g$ may be reformulated as $0 \leq g \leq f \cdot \mathbb{1}_G$.

10b2 Exercise. (a) Without changing this supremum we may restrict ourselves to continuous $g$ with bounded support; or, alternatively, to step functions $g$;

(b) if $f$ is bounded and $G$ is bounded, then $\int_G f = \star \int_{\mathbb{R}^n} f \cdot \mathbb{1}_G$, and in particular, $\int_G 1 = v_*(G);$\textsuperscript{3}

(c) if $f$ is bounded and $G$ is Jordan measurable, then the integral defined
by (10b1) is equal to the integral defined by (6g16).

Prove it.

\textsuperscript{1}Additional literature (for especially interested):

Bul. Univ. Petrol LVIII:2, 9–16.


\textsuperscript{2}A bounded open set need not be Jordan measurable, even if it is diffeomorphic to a
disk, as was noted in Sect. 8e (p. 138).

\textsuperscript{3}According to 8e, $v_*(G) = v(G) = m(G)$. 
10b3 Exercise. Consider the case $G = \mathbb{R}^n$, and let $\| \cdot \|$ be a norm on $\mathbb{R}^n$.

(a) Prove that

$$\int_{\mathbb{R}^n} f = \lim_{k \to \infty} \int_{\|x\| < k} \min(f(x), k) \, dx.$$ 

(b) For a locally bounded $1$ function $f$ prove that

$$\int_{\mathbb{R}^n} f = \lim_{k \to \infty} \int_{\|x\| < k} f(x) \, dx.$$ 

(c) Can it happen that $f$ is locally bounded, not bounded, and $\int_{\mathbb{R}^n} f < \infty$?

10b4 Example (Poisson). Consider

$$I = \int_{\mathbb{R}^2} e^{-|x|^2} \, dx.$$ 

On one hand, by 10b3 for the Euclidean norm,

$$I = \lim_{k \to \infty} \int_{x^2 + y^2 < k^2} e^{-(x^2 + y^2)} \, dxdy = \lim_{k \to \infty} \int_{0}^{k} r \, dr \int_{0}^{2\pi} e^{-r^2} \, d\theta = \lim_{k \to \infty} \pi \int_{0}^{k^2} e^{-u} \, du = \pi.$$ 

On the other hand, by 10b3 for $\|(x, y)\| = \max(|x|, |y|)$,

$$I = \lim_{k \to \infty} \int_{|x| < k, |y| < k} e^{-(x^2 + y^2)} \, dxdy = \lim_{k \to \infty} \left( \int_{-k}^{k} e^{-x^2} \, dx \right) \left( \int_{-k}^{k} e^{-y^2} \, dy \right) = \left( \int_{-\infty}^{+\infty} e^{-x^2} \, dx \right)^2,$$

and we obtain the celebrated Poisson formula:

$$\int_{-\infty}^{+\infty} e^{-x^2} \, dx = \sqrt{\pi}.$$ 

10b5 Exercise. Consider

$$I = \iint_{x>0, y>0} x^a y^b e^{-(x^2 + y^2)} \, dxdy \in [0, \infty]$$

\[1\] That is, bounded on every bounded subset of $\mathbb{R}^n$. 

---

**Notes:**
- The content is a part of an analysis course at Tel Aviv University, focusing on integral calculus and measure theory.
- The exercises and examples involve proving limits of integrals and exploring properties of functions under various norms and conditions.
- The Poisson formula is a well-known result in probability theory and mathematical physics, which relates to the distribution of points in space.

---

**References:**
- Analysis-III,IV
- Tel Aviv University, 2014/15

---

**Important:**
- The text is a natural representation of the document as it relates to integral calculus and analysis.
- The exercises and examples are designed to test understanding of integral properties and convergence under different norms.
for given $a, b \in \mathbb{R}$. Prove that, on one hand,

$$ I = \left( \int_0^\infty r^{a+b+1} e^{-r^2} \, dr \right) \left( \int_0^{\pi/2} \cos^a \theta \sin^b \theta \, d\theta \right), $$

and on the other hand,

$$ I = \left( \int_0^\infty x^a e^{-x^2} \, dx \right) \left( \int_0^\infty x^b e^{-x^2} \, dx \right). $$

**10b6 Exercise.** Consider $f : \mathbb{R}^2 \to [0, \infty)$ of the form $f(x) = g(|x|)$ for a given $g : [0, \infty) \to [0, \infty)$.

(a) If $g$ is integrable, then $f$ is integrable and $\int_{\mathbb{R}^2} f = 2\pi \int_0^\infty g(r) \, r \, dr$.

(b) If $g$ is continuous on $(0, \infty)$, then $\int_{\mathbb{R}^2} f = 2\pi \int_0^\infty g(r) \, r \, dr \in [0, \infty]$.

Prove it.\(^1\)

**10b7 Exercise.** Consider $f : \mathbb{R}^n \to [0, \infty)$ of the form $f(x) = g(\|x\|)$ for a given $g : [0, \infty) \to [0, \infty)$ and a given norm $\| \cdot \|$ on $\mathbb{R}^n$.

(a) If $g$ is integrable then $f$ is integrable, and $\int_{\mathbb{R}^n} f = nV \int_0^\infty g(r) \, r^{n-1} \, dr$ where $V$ is the volume of $\{x : \|x\| < 1\}$.

(b) If $g$ is continuous on $(0, \infty)$, then $\int_{\mathbb{R}^n} f = nV \int_0^\infty g(r) \, r^{n-1} \, dr \in [0, \infty]$.

c) Let $g$ be continuous on $(0, \infty)$ and satisfy

$$ g(r) \sim r^a \quad \text{for } r \to 0^+, \quad g(r) \sim r^b \quad \text{for } r \to +\infty. $$

Then $\int f < \infty$ if and only if $b < -n < a$.

Prove it.\(^2\)

**10b8 Example.** $\int_{\mathbb{R}^n} e^{-\|x\|^2} \, dx = nV \int_0^\infty r^{n-1} e^{-r^2} \, dr$; in particular, $\int_{\mathbb{R}^n} e^{-|x|^2} \, dx = nV_n \int_0^\infty r^{n-1} e^{-r^2} \, dr$ where $V_n$ is the volume of the (usual) $n$-dimensional unit ball. On the other hand, $\int_{\mathbb{R}^n} e^{-|x|^2} \, dx = (\int_{\mathbb{R}} e^{-x^2} \, dx)^n = \pi^{n/2}$. Therefore

$$ V_n = \frac{\pi^{n/2}}{n \int_0^\infty r^{n-1} e^{-r^2} \, dr}. $$

Not unexpectedly, $V_2 = \frac{\pi}{2 \int_0^\infty r e^{-r^2} \, dr} = \pi$.

Clearly, $\int_G cf = c \int_G f$ for $c \in (0, \infty)$.

**10b9 Proposition.** $\int_G (f_1 + f_2) = \int_G f_1 + \int_G f_2 \in [0, \infty]$ for all $f_1, f_2 \geq 0$ on $G$, continuous almost everywhere.

\(^1\)Hint: (a) either polar coordinates, or 9g4; (b) use (a).

\(^2\)Hint: (a), (b) similar to 9g4, using also 9c3 and 6g12; (c) use (b).
By linearity, WLOG, prove that \( f \) continuous almost everywhere. If we have to prove that \( f \) continuous almost everywhere are equivalent: 

First we prove that \( \int_G(f_1 + f_2) \geq \int_G f_1 + \int_G f_2. \)\(^1\) Given integrable \( g_1, g_2 \) such that \( 0 \leq g_1 \leq f_1 \cdot 1_G \) and \( 0 \leq g_2 \leq f_2 \cdot 1_G, \) we have \( \int g_1 + \int g_2 = \int (g_1 + g_2) \leq \int_G(f_1 + f_2), \) since \( g_1 + g_2 \) is integrable and \( 0 \leq g_1 + g_2 \leq (f_1 + f_2) \cdot 1_G. \) The supremum in \( g_1, g_2 \) gives the claim. 

It remains to prove that \( \int_G(f_1 + f_2) \leq \int_G f_1 + \int_G f_2, \) that is, \( \int g \leq \int_G f_1 + \int_G f_2 \) for every integrable \( g \) such that \( 0 \leq g \leq (f_1 + f_2) \cdot 1_G. \) We introduce \( g_1 = \min(f_1, g), \) \( g_2 = \min(f_2, g) \) (pointwise minimum on \( G; \) and \( 0 \) on \( \mathbb{R}^n \setminus G \)) and prove that they are continuous almost everywhere (on \( \mathbb{R}^n, \) not just on \( G). \) For almost every \( x \in G, \) both \( f_1 \) and \( g \) are continuous at \( x \) and therefore \( g_1 \) is continuous at \( x. \) For almost every \( x \in \partial G, \) \( g \) is continuous at \( x, \) which ensures continuity of \( g_1 \) at \( x \) (irrespective of continuity of \( f_1), \) since \( g(x) = 0 \) \( (x \notin G). \) Thus, \( g_1 \) is continuous almost everywhere; the same holds for \( g_2. \)

By Theorem 8f1, the functions \( g_1, g_2 \) are integrable. We have \( g_1 + g_2 \geq \min(f_1 + f_2, g) = g, \) since generally, \( \min(a, c) + \min(b, c) \geq \min(a + b, c) \) for all \( a, b, c \in [0, \infty) \) (think, why). Thus, \( \int g \leq \int (g_1 + g_2) = \int g_1 + \int g_2 \leq \int_G f_1 + \int_G f_2, \) since \( 0 \leq g_1 \leq f_1 \cdot 1_G, \) \( 0 \leq g_2 \leq f_2 \cdot 1_G. \)

**10b10 Proposition** (exhaustion). For open sets \( G, G_1, G_2, \ldots \subset \mathbb{R}^n, \)

\[
G_k \uparrow G \implies \int_{G_k} f \uparrow \int_G f \in [0, \infty]
\]

for all \( f : G \to [0, \infty) \) continuous almost everywhere.

**Proof.** First of all, \( \int_{G_k} f \leq \int_{G_{k+1}} f \) (since \( 0 \leq g \leq f \cdot 1_{G_k} \) implies \( 0 \leq g \leq f \cdot 1_{G_{k+1}}), \) and similarly, \( \int_{G_k} f \leq \int_G f, \) thus \( \int_{G_k} f \uparrow \) and \( \lim_k \int_{G_k} f \leq \int_G f. \) We have to prove that \( \int_G f \leq \lim_k \int_{G_k} f. \)

Let a step function \( g : \mathbb{R}^n \to \mathbb{R} \) satisfy \( 0 \leq g \leq f \cdot 1_G; \) we have to prove that \( \int g \leq \lim_k \int_{G_k} g \), but we’ll prove that moreover, \( \int g \leq \lim_k \int_{G_k} g. \)

By linearity, WLOG, \( g = 1_{C \subset G} \) for a box \( C, C^o \subset G. \) By [10b2], \( \int_{G_k} g = \int_{G \cap C^o \cap G_k} = v_*(C^o \cap G_k) \) and \( \int_G g = v(C); \) by 8e9, \( v_*(C^o \cap G_k) \uparrow v_*(C^o \cap G), \)

**10b11 Exercise.** Let \( G_1 \subset G_2 \subset \mathbb{R}^n \) be two open sets, and \( f : G_2 \to [0, \infty) \) continuous almost everywhere. If \( f = 0 \) on \( G_2 \setminus G_1, \) then \( \int_{G_2} f = \int_{G_1} f. \)

Prove it.\(^2\)

**10b12 Exercise.** The following four conditions on a function \( f : G \to [0, \infty) \) continuous almost everywhere are equivalent:

---

\(^1\)Compare it with (6d10).

\(^2\)Hint: just [10b1].
(a) \( \int_G f = 0; \)
(b) \( f(x) = 0 \) for every continuity point \( x \) of \( f; \)
(c) \( f(x) = 0 \) for almost all \( x \in G; \)
(d) the set \( \{ x \in G : f(x) = 0 \} \) is dense in \( G. \)

Prove it.\(^1\)

10c Newton potential

By the celebrated Newton’s law of universal gravitation, the gravitational force exerted by a particle of mass \( m \) at point \( \xi \) on a particle of mass \( m_0 \) at point \( x \) is \( -Gm_0mg_\xi(x) \), and \( -Gmg_\xi(\cdot) \) is the gravitational field generated by \( m \),

\[
(10c1) \quad g_\xi(x) = g_0(x - \xi) = \frac{x - \xi}{|x - \xi|^3} = -\nabla U_0(x - \xi);
\]

here the function \( U_0 : x \mapsto \frac{1}{|x|^3} \) is proportional to the gravitational potential (energy), and \( G \) is the gravitational constant.\(^2\) The reason to replace the force by the potential is simple: it is easier to work with scalar functions than with the vector ones.\(^3\)

What happens if we have a system of point masses \( \mu_1, \ldots, \mu_k \) at points \( \xi_1, \ldots, \xi_k \)? The forces are to be added, and the corresponding potential is

\[
U(x) = \sum_{j=1}^k \frac{\mu_j}{|x - \xi_j|}.
\]

A continuously distributed mass is described in physics by its density \( \rho \). Mathematically it means that the density is a point function, the mass is an additive box function, and these two functions are related according to Sect. 6a (and 8c): the mass within a box \( B \) is \( \int_B \rho \). Generally, \( \rho \) is not quite integrable but improperly integrable; and still, the mass within a box \( B \) is assumed to be \( \int_B \rho \) (improper integral) for evident physical reasons; and the total mass is \( \int_{\mathbb{R}^3} \rho \).

---

\(^1\)Hint: (b) \( \Rightarrow \) (c) \( \Rightarrow \) (d): easy; (d) \( \Rightarrow \) (a): use 10b2 a); (a) \( \Rightarrow \) (b): otherwise \( f(\cdot) \geq \varepsilon \) on some neighborhood of \( x. \)

\(^2\)\( G \approx 6.674 \cdot 10^{-11} \text{N(m/kg)}^2 \); that is, if \( m = \mu = 1 \text{ kg} \) and \( |x - \xi| = 1 \text{ m} \) then the force is \( \approx 6.674 \cdot 10^{-11} \text{ newtons}. \)

\(^3\)Knowing the force \( F \) one can write down the differential equations of motion of the particle (Newton’s second law) \( m_0\ddot{x} = F \), or \( \ddot{x} = \mathcal{G}\nabla U \) (note that \( m_0 \) does not matter). Then one hopes to integrate these equations, thus finding out where is the particle at time \( t. \)
Similarly, the potential is assumed to be $-G U_\rho$ where $U_\rho(x) = \int_{\mathbb{R}^3} \frac{\rho(\xi)}{|x - \xi|} \, d\xi$; this integral is improper (in general) and must be finite.\footnote{Mathematical rigorosity is of little interest to physicists, and still, the distinction between proper and improper integrals may be physically sound. Imagine a material ball of mass $M$ and radius $R$, consisting of a large number of uniformy distributed "particles" that are balls of mass $m$ and radius $r$. Outside the (large) ball, near its surface, the gravitational field is $GM/R^2$ in a good approximation. Inside the ball, near the surface of a "particle", the gravitational field of this single "particle" is $Gm/r^2$. Let $M = 1 \text{ kg}$, $R = 0.1 \text{ m}$, $m = 10^{-25} \text{ kg}$, $r = 10^{-14} \text{ m}$; then $M/R^2 = 100 \text{ kg/m}^2$ while $m/r^2 = 1000 \text{ kg/m}^2$. Here, a single "particle" generates a field 10 times stronger than the improper integral that will be calculated! Do not think that such parameters are physically unrealistic; these $m, r$ are the parameters of a typical atomic nucleus.}

Let us compute the potential of the homogeneous mass distribution, of density 1, within the ball of radius $R$ centered at the origin:

$$U_R(x) = \int_{|\xi| < R} \frac{d\xi}{|x - \xi|}.$$ 

Due to rotation invariance (Theorem 9c1), $U_R$ is a radial function, that is, depends only on $|x|$. Thus, it suffices to compute $U_R(x)$ at the point $x = (0, 0, a)$, $a \in [0, \infty)$. The integral is proper for $a \in (R, \infty)$ and improper for $a \in [0, R]$.

First, consider the proper integral, for $a > R$. Using the spherical coordinates $\xi = (r \cos \varphi \sin \theta, r \sin \varphi \sin \theta, r \cos \theta)$ (recall 9b3) we have

$$U_R(x) = \int_0^R dr \frac{2\pi}{2} \int_0^\pi \frac{r^2 \sin \theta \, d\theta}{\sqrt{(a - r \cos \theta)^2 + r^2 \sin^2 \theta}} = \int_0^R dr \frac{2\pi}{2} \int_0^\pi \frac{r^2 \sin \theta \, d\theta}{\sqrt{a^2 - 2ar \cos \theta + r^2}} = \int_0^R dr \frac{2\pi}{2} \int_0^\pi \frac{r^2 \sin \theta \, d\theta}{V_a(r)}.$$ 

Intuitively, the under-braced expression $V_a(r)$ is the potential of the homogeneous sphere of radius $r$; but rigorously, integration over spheres and other surfaces will be treated much later. We compute $V_a(r)$ using the variable

$$t = \sqrt{a^2 - 2ar \cos \theta + r^2}.$$ 

Then $a - r < t < a + r$, and $t \, dt = ar \sin \theta \, d\theta$. We get

$$V_a(r) = 2\pi r^2 \int_{a-r}^{a+r} \frac{t \, dt}{art} = \frac{2\pi r^2}{a} \cdot 2r = \frac{4\pi r^3}{a}.$$
Now we easily find $U_R(x)$ by integration:

\[ U_R(x) = \int_0^R V_a(r) \, dr = 4\pi \int_0^R \frac{r^2}{a} \, dr = \frac{4\pi R^3}{3a} \quad \text{for } |x| > R. \]

We turn to the case $a < R$, and treat the improper integral by exhaustion:

\[
U_R(x) = \lim_{\epsilon \to 0^+} \left( \int_{|\xi| < a-\epsilon} \frac{d\xi}{|x - \xi|} + \int_{a+\epsilon < |\xi| < R} \frac{d\xi}{|x - \xi|} \right) = \lim_{\epsilon \to 0^+} \left( \int_{a-\epsilon}^{a+\epsilon} V_a(r) \, dr + \int_{a+\epsilon}^{R} V_a(r) \, dr \right) = \int_0^R V_a(r) \, dr \in [0, \infty],
\]

the latter integral being improper, since $V_a$ need not be bounded near $a$. For $r < a$ we have $V_a(r) = 4\pi \frac{r^2}{a}$ as before. For $r > a$ we still use $t = \sqrt{a^2 - 2ar \cos \theta + r^2}$, and $t$ is still strictly increasing in $\theta \in (0, \pi)$, but now $\sqrt{a^2 - 2ar + r^2} = r - a$, thus $r - a < t < r + a$, and we get

\[ V_a(r) = 2\pi r^2 \int_{r-a}^{r+a} \frac{t \, dt}{a} = \frac{2\pi r}{a} \cdot 2a = 4\pi r. \]

A surprise: $V_a$ appears to be bounded near $a$, and extends by continuity to $(0, R)$, thus the one-dimensional integral may be treated as proper. We have

\[
U_R(x) = \int_0^R V_a(r) \, dr = \int_0^a \frac{4\pi r^2}{a} \, dr + \int_a^R 4\pi r \, dr = 4\pi \left( \frac{a^2}{3} + \frac{R^2}{2} - \frac{a^2}{2} \right) = \frac{2\pi}{3} (3R^2 - a^2) = \frac{2\pi}{3} (3R^2 - |x|^2) \quad \text{for } 0 \leq |x| < R.
\]

The case $a = R$ is easy: $U_R(x) = \int_0^R V_a(r) \, dr = \int_0^R 4\pi \frac{r^2}{a} \, dr = 4\pi \frac{R^3}{3a} = \frac{4\pi R^3}{3}$ for $|x| = R$. The function $U_R$ appears to be continuous. Finally,

\[
U_R(x) = \begin{cases} 
4\pi \frac{R^3}{3|x|} & \text{for } |x| \geq R, \\
\frac{2\pi}{3} (3R^2 - |x|^2) & \text{for } |x| \leq R.
\end{cases}
\]

Observe that $4\pi R^3/3$ is exactly the total mass of the ball. That is, together with Newton, we arrived at the conclusion that the gravitational potential, and hence the gravitational force exerted by the homogeneous ball on a particle is the same as if the whole mass of the ball were concentrated at its center, as long as the point is outside the ball. Of course, you heard about this already in the high-school.
Another important conclusion is that the potential of the homogeneous sphere does not depend on the point inside the sphere!\(^1\) Hence, the gravitational force is zero inside the sphere. The same is true for the homogeneous shell \(\{\xi : a < |\xi| < b\}\): there is no gravitational force inside the shell.

**10c2 Exercise.** Check that all the conclusions are true when the mass distribution \(\rho\) is radial: \(\rho(\xi) = \rho(\xi') \) whenever \(|\xi| = |\xi'|\).

**10c3 Exercise.** Find the potential of the homogeneous solid ellipsoid \((x^2 + y^2)/b^2 + z^2/c^2 < 1\) at its center.

**10c4 Exercise.** Find the potential of the homogeneous solid cone of height \(h\) and radius of the base \(r\) at its vertex.

**10c5 Problem.** Show that at sufficiently large distances the potential of a solid is approximated by the potential of a point with the same total mass located at the center of mass of the solid with an error less than a constant divided by the square of the distance. The potential itself decays as the distance, so the approximation is good: its relative error is small.\(^2\)

### 10d Special functions gamma and beta

Integrating a function of two variables in one variable we get a function of the other variable. An interesting example was seen in 10e2: the function \(F(t) = \int_{0}^{\pi/2} \ln(t^2 - \sin^2 x) \, dx\) appeared to be the elementary function \(F(t) = \pi \ln \frac{t + \sqrt{t^2 - 1}}{2}\). But generally it is not elementary. Here is a much more important example. The Euler *gamma function* \(\Gamma\) is defined by

\[
(10d1) \quad \Gamma(t) = \int_{0}^{\infty} x^{t-1} e^{-x} \, dx \quad \text{for } t \in (0, \infty). 
\]

This integral is not proper for two reasons. First, the integrand is bounded near 0 for \(t \in [1, \infty)\) but unbounded for \(t \in (0, 1)\). Second, the integrand has no bounded support. In every case, using 10b10,

\[
\Gamma(t) = \lim_{k \to \infty} \int_{1/k}^{k} x^{t-1} e^{-x} \, dx < \infty,
\]

since the integrand (for a given \(t\)) is continuous on \((0, \infty)\), is \(O(x^{t-1})\) as \(x \to 0\), and (say) \(O(e^{-x/2})\) as \(x \to \infty\). Thus, \(\Gamma : (0, \infty) \to (0, \infty)\).

---

\(^1\)Since \(V_a(r)\) does not depend on \(a\) for \(a < r\).

\(^2\)This estimate is rather straightforward. A more accurate argument shows that the error is of order constant divided by the cube of the distance.

\(^3\)This is rather \(\Gamma|_{(0, \infty)}\).
Certainly, $\Gamma(1) = 1$. Integration by parts gives
\[
\int_{1/k}^{k} x^t e^{-x} \, dx = -x^t e^{-x}\bigg|_{x=1/k} + t \int_{1/k}^{k} x^{t-1} e^{-x} \, dx ;
\]
(10d2) $\Gamma(t + 1) = t\Gamma(t)$ for $t \in (0, \infty)$.

In particular,
(10d3) $\Gamma(n + 1) = n!$ for $n = 0, 1, 2, \ldots$

We note that
(10d4) $\int_{0}^{\infty} x^a e^{-x^2} \, dx = \frac{1}{2} \Gamma\left(\frac{a + 1}{2}\right)$ for $a \in (-1, \infty)$,

since $\int_{0}^{\infty} x^a e^{-x^2} \, dx = \int_{0}^{\infty} u^{a/2} e^{-u} \, \frac{du}{2\sqrt{u}}$. For $a = 0$ the Poisson formula (recall 10b4) gives
(10d5) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Thus,
(10d6) $\Gamma\left(\frac{2n + 1}{2}\right) = \frac{1}{2} \cdot \frac{3}{2} \cdot \cdots \cdot \frac{2n - 1}{2} \cdot \frac{2n}{2} \cdot \sqrt{\pi}$.

The volume $V_n$ of the $n$-dimensional unit ball (recall 10b8) is thus calculated:
(10d7)
\[
V_n = \frac{\pi^{n/2}}{\frac{n}{2} \Gamma\left(\frac{n}{2}\right)} .
\]

Not unexpectedly, $V_3 = \frac{\pi^{3/2}}{\frac{3}{2} \Gamma\left(\frac{3}{2}\right)} = \frac{\pi^{3/2}}{\frac{3}{2} \cdot \sqrt{\pi}} = \frac{4}{3} \pi$.

By 10b5, $\frac{1}{2} \Gamma\left(\frac{a + b + 2}{2}\right) \frac{\pi^{a/2}}{\Gamma\left(\frac{a}{2}\right)} = \frac{1}{2} \Gamma\left(\frac{a + 1}{2}\right) \cdot \frac{1}{2} \Gamma\left(\frac{b + 1}{2}\right)$ for $a, b \in (-1, \infty)$; that is,
(10d8) $\int_{0}^{\pi/2} \cos^{a-1} \theta \sin^{b-1} \theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a + b}{2}\right)}$ for $\alpha, \beta \in (0, \infty)$.

In particular,
(10d9) $\int_{0}^{\pi/2} \sin^{a-1} \theta \, d\theta = \int_{0}^{\pi/2} \cos^{a-1} \theta \, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{a + 1}{2}\right)}$.
The trigonometric functions can be eliminated: \[ \int_0^{\pi/2} \cos^{\alpha-1} \theta \sin^{\beta-1} \theta \, d\theta = \frac{1}{2} \int_0^{\pi/2} \cos^{\alpha-2} \theta \sin^{\beta-2} \theta \cdot 2 \sin \theta \cos \theta \, d\theta = \frac{1}{2} \int_0^{\pi/2} (1-u)^{\alpha-2} u^{\beta-2} \, du; \] thus, \[ (10d10) \quad \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \, dx = \mathcal{B}(\alpha, \beta) \quad \text{for} \quad \alpha, \beta \in (0, \infty), \] where \[ (10d11) \quad \mathcal{B}(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad \text{for} \quad \alpha, \beta \in (0, \infty) \] is another special function, the beta function.

10d12 Exercise. Check that \( \mathcal{B}(x, x) = 2^{1-2x} \mathcal{B}(x, \frac{1}{2}) \).

Hint: \( \int_0^{\pi/2} (2 \sin \theta \cos \theta)^{2x-1} \, d\theta \).

10d13 Exercise. Check the duplication formula:
\[ \Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right). \]

Hint: use 10d12.

10d14 Exercise. Calculate \( \int_0^1 x^4 \sqrt{1-x^2} \, dx \).
Answer: \( \frac{\pi}{32} \).

10d15 Exercise. Calculate \( \int_0^\infty x^m e^{-x^n} \, dx \).
Answer: \( \frac{1}{n} \Gamma\left(\frac{m+1}{n}\right) \).

10d16 Exercise. Calculate \( \int_0^1 x^m (\ln x)^n \, dx \).
Answer: \( \frac{(-1)^n n!}{(m+1)^{n+1}} \).

10d17 Exercise. Calculate \( \int_0^{\pi/2} \frac{dx}{\sqrt{\cos x}} \).
Answer: \( \frac{\Gamma^2(1/4)}{2\sqrt{\pi}} \).

10d18 Exercise. Check that \( \Gamma(p) \Gamma(1-p) = \int_0^\infty \frac{x^{p-1}}{1+x} \, dx \).
Hint: change \( x \) to \( t \) via \( (1+x)(1-t) = 1 \).

We mention without proof another useful formula
\[ \int_0^\infty \frac{x^{p-1}}{1+x} \, dx = \frac{\pi}{\sin \pi p} \quad \text{for} \quad 0 < p < 1. \]

There is a simple proof that that uses the residues theorem from the complex analysis course. This formula yields that \( \Gamma(t) \Gamma(1-t) = \frac{\pi}{\sin \pi t} \).
Is the function $\Gamma$ continuous?

For every compact interval $[t_0, t_1] \subset (0, \infty)$ the given function of two variables $(t, x) \mapsto x^{t-1}e^{-x}$ is Lipschitz continuous on $[t_0, t_1] \times \left[ \frac{1}{k}, k \right]$, therefore the integral is Lipschitz continuous on $[t_0, t_1]$ (recall 7b). Also,

$$\int_{1/k}^{k} x^{t-1}e^{-x} \, dx \to \Gamma(t) \quad \text{uniformly on } [t_0, t_1],$$

since $\int_{0}^{1/k} x^{t-1}e^{-x} \, dx \leq \int_{0}^{1/k} x^{\frac{1}{k}} \, dx \to 0$ as $k \to \infty$ and $\int_{k}^{\infty} x^{t-1}e^{-x} \, dx \to 0$ as $k \to \infty$. It follows that $\Gamma$ is continuous on arbitrary $[t_0, t_1]$, therefore, on the whole $(0, \infty)$.

In particular, $t\Gamma(t) = \Gamma(t+1) \to \Gamma(1) = 1$ as $t \to 0+$; that is,

$$\Gamma(t) = \frac{1}{t} + o\left(\frac{1}{t}\right) \quad \text{as } t \to 0^+.$$

Is the function $\Gamma$ differentiable?

By Theorem 7e1 the function $t \mapsto \int_{1/k}^{k} x^{t-1}e^{-x} \, dx$ is continuously differentiable, and its derivative is $t \mapsto \int_{1/k}^{k} x^{t-1}e^{-x} \ln x \, dx$; this relation results from application of Prop. 7b4 (iterated integral) to the function $(t, x) \mapsto \frac{\partial}{\partial x} x^{t-1}e^{-x} = x^{t-1}e^{-x} \ln x$ on $[t_0, t_1] \times \left[ \frac{1}{k}, k \right]$. Regrettably, iterated improper integral is not an easy matter.\footnote{If $f : [0, 1] \times [0, 1] \to [0, \infty)$ is improperly integrable, then $f_x : y \to f(x, y)$ is improperly integrable on $[0, 1]$ for almost every $x$; however, the function $\varphi : x \to \int f_x$ need not be improperly integrable. Rather, $\varphi$ is equivalent to a function semicontinuous from below (possibly, unbounded on every interval), and $\mathcal{S}\int \varphi = \int f$.}

Instead, we use exhaustion, as follows. As before,

$$\int_{1/k}^{k} x^{t-1}e^{-x} \ln x \, dx \to \int_{0}^{\infty} x^{t-1}e^{-x} \ln x \, dx \quad \text{uniformly on } [t_0, t_1]$$

(check it), therefore

$$\int_{t_0}^{t_1} dt \int_{1/k}^{k} x^{t-1}e^{-x} \ln x \, dx \to \int_{t_0}^{t_1} dt \int_{0}^{\infty} x^{t-1}e^{-x} \ln x \, dx.$$

On the other hand,

$$\int_{1/k}^{k} dx \int_{t_0}^{t_1} dt \ x^{t-1}e^{-x} \ln x = \int_{1/k}^{k} \left( x^{t-1}e^{-x}\big|_{t=t_0}^{t_1} \right) \, dx =$$

$$= \int_{1/k}^{k} x^{t_1-1}e^{-x} \, dx - \int_{1/k}^{k} x^{t_0-1}e^{-x} \, dx \to \Gamma(t_1) - \Gamma(t_0).$$
Thus, $\Gamma(t_1) - \Gamma(t_0) = \int_{t_0}^{t_1} dt \int_0^\infty x^{t-1}e^{-x} \ln x \, dx$, which implies

$$\Gamma'(t) = \int_0^\infty x^{t-1}e^{-x} \ln x \, dx.$$ 

Similarly, $\Gamma'$ is differentiable; continuing this way we get

$$\Gamma^{(k)}(t) = \int_0^\infty x^{t-1}e^{-x}(\ln x)^k \, dx \quad \text{for } k = 1, 2, \ldots$$

10e Normed space of equivalence classes

In order to integrate signed functions we reuse the simple trick of (8e1). We define

$$\int_G (g - h) = \int_G g - \int_G h$$

whenever $g, h : G \to [0, \infty)$ are continuous almost everywhere and $\int_G g < \infty$, $\int_G h < \infty$; this definition is correct, that is,

$$\int_G g_1 - \int_G h_1 = \int_G g_2 - \int_G h_2 \quad \text{whenever } g_1 - h_1 = g_2 - h_2;$$

proof:

(10e1)

$g_1 - h_1 = g_2 - h_2 \implies g_1 + h_2 = g_2 + h_1 \implies \int_G (g_1 + h_2) = \int_G (g_2 + h_1) \implies \int_G g_1 + \int_G h_2 = \int_G g_2 + \int_G h_1 \implies \int_G g_1 - \int_G h_1 = \int_G g_2 - \int_G h_2.$

10e2 Lemma. The following two conditions on a function $f : G \to \mathbb{R}$ continuous almost everywhere are equivalent:

(a) there exist $g, h : G \to [0, \infty)$, continuous almost everywhere, such that $\int_G g < \infty$, $\int_G h < \infty$ and $f = g - h$;

(b) $\int_G |f| < \infty$.

Proof. (a)$\implies$(b): $\int_G |g - h| = \int_G (|g| + |h|) = \int_G |g| + \int_G |h| < \infty$.

(b)$\implies$(a): we introduce the positive part $f^+$ and the negative part $f^-$ of $f$,

$$f^+(x) = \max(0, f(x)), \quad f^-(x) = \max(0, -f(x));$$

$$f^+ = (-f)^-; \quad f = f^+ - f^-; \quad |f| = f^+ + f^-;$$

they are continuous almost everywhere (think, why); $\int_G f^+ \leq \int_G |f| < \infty$, $\int_G f^- \leq \int_G |f| < \infty$; and $f^+ - f^- = f$. 

$\square$
We summarize:

\[ \int_G f = \int_G f^+ - \int_G f^- \]

whenever \( f : G \to \mathbb{R} \) is continuous almost everywhere and such that \( \int_G |f| < \infty \). Such functions will be called \textit{improperly integrable}\(^1\) (on \( G \)).

**10e5 Exercise.** Prove linearity: \( \int_G cf = c \int_G f \) for \( c \in \mathbb{R} \), and \( \int_G (f_1 + f_2) = \int_G f_1 + \int_G f_2 \).

Similarly to Sect. 6e, a function \( f : G \to \mathbb{R} \) continuous almost everywhere will be called \textit{negligible} if \( \int_G |f| = 0 \). Functions \( f, g \) continuous almost everywhere and such that \( f - g \) is negligible will be called equivalent. The equivalence class of \( f \) will be denoted \([f]\).

Improperly integrable functions \( f : G \to \mathbb{R} \) are a vector space. On this space, the functional \( f \mapsto \int_G |f| \) is a seminorm. The corresponding equivalence classes are a normed space (therefore also a metric space). Similarly to 6e3, the integral is a continuous linear functional on this space.

If \( G \) is Jordan measurable then the space of improperly integrable functions on \( G \) is embedded into the space of improperly integrable functions on \( \mathbb{R}^n \) by \( f \mapsto f \cdot 1_{G} \).

**10e6 Lemma.** Let \( G_1 \subset G_2 \subset \mathbb{R}^n \) be two open sets, and \( f : G_2 \to \mathbb{R} \) continuous almost everywhere. If \( f = 0 \) almost everywhere on \( G_2 \setminus G_1 \), then \( f \cdot 1_{G_1} \) is continuous almost everywhere on \( G_2 \) and equivalent to \( f \).

**Proof.** The set \( A = \{ x \in G_2 \setminus G_1 : f(x) \neq 0 \} \) is of Lebesgue measure 0. If \( f \) is continuous at \( x \in G_2 \) while \( f \cdot 1_{G_1} \) is not, then clearly \( x \in G_2 \setminus G_1 \); and moreover, \( x \in A \) (since \( \lim_{t \to x} f(t) = 0 \) implies \( \lim_{t \to x} (f(t) \cdot 1_{G_1}(t)) = 0 \)). Thus, \( f \cdot 1_{G_1} \) is continuous almost everywhere on \( G_2 \). Finally, \( f \cdot 1_{G_1} = f \) on \( G_2 \setminus A \). \( \square \)

In particular, if \( G_1 \) contains almost all points of \( G_2 \) (that is, \( G_2 \setminus G_1 \) is of Lebesgue measure 0),\(^2\) then the condition \( "f = 0 \) almost everywhere on \( G_2 \setminus G_1 \)" holds vacuously; in this case the values of \( f \) on \( G_2 \setminus G_1 \) do not influence the equivalence class of \( f \).

**10e7 Corollary.** Let \( G_1 \subset G_2 \subset \mathbb{R}^n \) be two open sets, and \( f : G_2 \to \mathbb{R} \) improperly integrable. If \( f = 0 \) almost everywhere on \( G_2 \setminus G_1 \), then \( \int_{G_2} f = \int_{G_1} f \).

\( ^1 \)In one dimension they are usually called absolutely (improperly) integrable.

\( ^2 \)Warning: this condition implies \( \nu_+(G_1) = \nu_+(G_2) \) and is implied by \( \nu_+(G_1) = \nu_+(G_2) < \infty \), but is not implied by \( \nu_+(G_1) = \nu_+(G_2) = \infty \).
Proof. First, \( \int_{G_2} f = \int_{G_2} f \cdot 1_{G_1} \) since \([f] = [f \cdot 1_{G_1}]\) by 10e6. Second, \( \int_{G_2} f^+ \cdot 1_{G_1} = \int_{G_1} f^+ \) by 10b11; the same holds for \( f^- \), and therefore for \( f^+ - f^- = f \).

Once again, if \( G_1 \) contains almost all points of \( G_2 \), then we get \( \int_{G_2} f = \int_{G_1} f \) for all \( f \) improperly integrable on \( G_2 \).

We may admit a function \( f \) partially defined on \( G \), provided that for almost every \( x \in G \), \( f \) is defined near \( x \).

1 2 In other words: \( f : G \setminus A \to \mathbb{R} \), and the (relative) closure of \( A \) in \( G \) is of Lebesgue measure 0. In this case almost all points of \( G \) belong to \((G \setminus A)^{\circ} \). Such partially defined functions may be used as well as functions defined on the whole \( G \), whenever only equivalence classes matter.

Thus, we need not hesitate saying that, for instance, \( \int_{-1}^{1} \frac{dt}{|t|^\alpha} = \frac{2}{1-\alpha} \) for \( \alpha < 1 \), even though the integrand is undefined at 0.

10e8 Proposition (Exhaustion). Let open sets \( G_1 \subset G_2 \subset \cdots \subset G \subset \mathbb{R}^n \) be such that \( \bigcup_k G_k \) contains almost all points of \( G \). Then

\[
\int_{G_k} f \to \int_G f \quad \text{as} \ k \to \infty
\]

for all \( f \) improperly integrable on \( G \).

Proof. First, the open set \( \tilde{G} = \bigcup_k G_k \) contains almost all points of \( G \), therefore \( \int_G f = \int_{\tilde{G}} f \). Second, \( G_k \uparrow \tilde{G} : 10b10 \) gives \( \int_{G_k} f = \int_{G_k} f^+ - \int_{G_k} f^- \to \int_{\tilde{G}} f^+ - \int_{\tilde{G}} f^- = \int_{\tilde{G}} f \).

In particular, if \( G_k \) are also Jordan measurable and such that \( f \) is defined and bounded on each \( G_k \), then \( \int_{G_k} f \) is the proper (Riemann) integral, and we obtain the improper integral \( \int_G f \) as the limit of proper integrals.

10e9 Proposition. Let \( G \subset \mathbb{R}^n \) be an open set, and \( f \) an improperly integrable function on \( G \).\(^3\) Then there exist Jordan measurable open sets \( G_1 \subset G_2 \subset \cdots \) such that \( G_k \subset G \), \( \bigcup_k G_k \) contains almost all points of \( G \), and \( f \) is defined and bounded on every \( G_k \).

---

\(^1\)Not just “at \( x \)!”

\(^2\)In fact, for every set \( A \subset G \) of Lebesgue measure 0 (even if dense in \( G \)), every function \( f : G \setminus A \to \mathbb{R} \) continuous almost everywhere can be extended to a function \( G \to \mathbb{R} \) continuous almost everywhere (and all such extensions evidently are mutually equivalent). Hint: \( \liminf_{t \to x, t \in G \setminus A} f(t) \leq \hat{f}(x) \leq \limsup_{t \to x, t \in G \setminus A} f(t) \) for every \( x \in A \) such that \( f \) is bounded near \( x \). Such \( \hat{f} \) is continuous at every continuity point of \( f \).

\(^3\)We admit partially defined \( f \), as explained above.
The normed space of equivalence classes, introduced above, does not admit an inner product. Now we turn to improperly square integrable functions; these are functions $f : G \to \mathbb{R}$ continuous almost everywhere and such that $\int f^2 < \infty$. If $[f] = [g]$ then $\int f^2 = \int g^2$ (check it via 10b12), thus, square integrability applies to equivalence classes. We denote the set of all square integrable equivalence classes by $\tilde{L}^2(G)$, and often write $f \in \tilde{L}^2(G)$ instead of $[f] \in \tilde{L}^2(G)$. This set is a vector space (since $(f + g)^2 \leq (f + g)^2 + (f - g)^2 = 2f^2 + 2g^2$).

If $f, g \in \tilde{L}^2(G)$ then their pointwise product $fg$ is improperly integrable (since $f^2 - 2|fg| + g^2 \geq 0$), and we define the inner product

$$\langle [f], [g] \rangle = \int fg \quad \text{(10e10)}$$

and the corresponding norm

$$\|f\|_2 = \sqrt{\langle [f], [f] \rangle}, \quad \text{that is,} \quad \|f\|_2 = \sqrt{\int f^2} \quad \text{(10e11)}$$

satisfying $[f] \neq [0] \implies \|f\|_2 > 0$ (check it via 10b12). We often write $\langle f, g \rangle$ and $\|f\|_2$ instead of $\langle [f], [g] \rangle$ and $\| [f] \|_2$.

Every 2-dimensional subspace of $\tilde{L}^2(G)$ is a Euclidean plane, which ensures the triangle inequality

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2 \quad \text{(10e12)}$$

1Not “continuous near $x$”!

2Its 2-dimensional subspace of step functions is not the Euclidean plane; you may check it similarly to the paragraph before le3.

3The widely used notation $L^2$ is reserved for the corresponding notion in the framework of Lebesgue integration.
and the Cauchy-Schwarz inequality

\[ -\|f\|_2\|g\|_2 \leq \langle f, g \rangle \leq \|f\|_2\|g\|_2. \]  

More generally, for arbitrary \( p \in [1, \infty) \) we introduce the norm

\[ \|f\|_p = \left( \int |f|^p \right)^{1/p} \]

on the vector space \( \tilde{L}^p(G) \) of \([f]\) such that \( \int |f|^p < \infty \) (two special cases \( p = 1 \) and \( p = 2 \) being already treated). The triangle inequality

\[ \|f + g\|_p \leq \|f\|_p + \|g\|_p \]

follows from convexity of the ball \( \{ f : \int |f|^p \leq 1 \} \) (recall 1e14); convexity of the ball follows from convexity of the functional \( f \mapsto \int |f|^p \) (recall 1e13); and convexity of this functional follows from convexity of the function \( t \mapsto |t|^p \) (similarly to 1e15). The triangle inequality ensures that \( \tilde{L}^p(G) \) is a vector space. The Hölder inequality

\[ \left| \int fg \right| \leq \|f\|_p\|g\|_q \quad \text{for } f \in \tilde{L}^p(G), g \in \tilde{L}^q(G), \frac{1}{p} + \frac{1}{q} = 1, \]

is obtained similarly to 6d15(b) (but harder); first, \( ab \leq \frac{a^p}{p} + \frac{b^q}{q} \) for \( a, b \in [0, \infty) \); second,

\[ \left| \int fg \right| \leq \min_{c > 0} \left( \frac{1}{p}\|cf\|_p^p + \frac{1}{q}\|cg\|_q^q \right) = \|f\|_p\|g\|_q. \]

10f Change of variables

10f1 Theorem. Let \( U, V \subset \mathbb{R}^n \) be open sets, \( \varphi : U \to V \) a diffeomorphism, and \( f : V \to \mathbb{R} \). Then

(a) \( f \) is improperly integrable on \( V \) if and only if \( (f \circ \varphi) | \det D\varphi | \) is improperly integrable on \( U \); and

(b) in this case

\[ \int_V f = \int_U (f \circ \varphi) | \det D\varphi |. \]

Proof. We reuse the arguments from the proof of Theorem 9a1. There, \( U \) and \( V \) are assumed to be Jordan measurable, but this assumption is used only in the last paragraph of the proof. Before that we constructed Jordan measurable open sets \( V_k \uparrow V \) (denoted there by \( K_i^\circ \)) such that \( V_k \subset V, \)

\[ ^1 \text{But notations } U, V \text{ are swapped there; compare 9a1 and 9a2.} \]
the sets $\varphi^{-1}(V_k) = U_k \uparrow U$ are Jordan measurable, $U_k \subset U$, and we showed that the claim of the theorem (for Riemann integral) holds for every $f$ whose support is contained in some $V_k$, therefore, for every $f$ with a compact support inside $V$.

Now, given $f : V \to \mathbb{R}$, we note that $f$ is improperly integrable on $V$ if and only if it is improperly integrable on each $V_k$, and
\[
\lim_k \int_{V_k} |f| < \infty,
\]
and in this case $\int_V f = \lim_k \int_{V_k} f$.

Similarly,
\[\left( f \circ \varphi \right) |\det D\varphi| \text{ is improperly integrable on } U \text{ if and only if } \text{it is improperly integrable on each } U_k, \text{ and} \]
\[\lim_k \int_{U_k} |f \circ \varphi| |\det D\varphi| < \infty, \]
and in this case $\int_U (f \circ \varphi) |\det D\varphi| = \lim_k \int_{U_k} (f \circ \varphi) |\det D\varphi|$.

Thus, in order to prove the theorem for arbitrary $f$ it is sufficient to prove it for $f$ whose support is contained in some $V_k$.

Theorem 9a1, applied to the diffeomorphism $\varphi|_{U_k} : U_k \to V_k$, gives the needed claim for proper integration, that is, for bounded $f$ (boundedness of $(f \circ \varphi)|\det D\varphi|$ follows, since the determinant is bounded on $U_k$). It remains to generalize this claim to unbounded $f : V_k \to \mathbb{R}$. Taking into account that $f = f^+ - f^-$ we may assume that $f : V_k \to [0, \infty)$. We note that
\[f \text{ is improperly integrable on } V_k \text{ if and only if each } f_\ell = \min(f, \ell) \text{ is integrable on } V_k, \text{ and in this case } \int_{V_k} f = \lim_\ell \int_{V_k} f_\ell\]
(since every integrable $g$ such that $0 \leq g \leq f \cdot 1_{V_k}$ satisfies $g \leq \ell$ for some $\ell$).

Similarly, taking into account that $|\det D\varphi|$ is bounded away from 0 on $U_k$, we see that
\[\left( f \circ \varphi \right)|\det D\varphi| \text{ is improperly integrable on } U_k \text{ if and only if each } (f_\ell \circ \varphi)|\det D\varphi| \text{ is improperly integrable on } U_k, \text{ and in this case } \int_{U_k} (f \circ \varphi)|\det D\varphi| = \lim_\ell \int_{U_k} (f_\ell \circ \varphi)|\det D\varphi|.
\]
The claim follows.

\[\square\]

10f2 Exercise. Prove the equality \textbf{(10d10)} once again, avoiding \textbf{(10b5)} and trigonometric functions; to this end, consider
\[
\left( \int_0^\infty u^{\alpha+\beta-1} e^{-u} \, du \right) \left( \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \, dx \right).
\]
and change the variables \( u, x \) to \( t_1, t_2 \) as follows:
\[
\begin{align*}
\begin{cases}
    t_1 = ux \\
    t_2 = u(1 - x)
\end{cases}
\quad \begin{cases}
    u = t_1 + t_2 \\
    x = \frac{t_1}{t_1 + t_2}
\end{cases}
\]

10g  Multidimensional beta integrals of Dirichlet

10g1 Proposition.
\[
\int \cdots \int_{x_1, \ldots, x_n > 0, x_1 + \cdots + x_n < 1} x_1^{p_1-1} \cdots x_n^{p_n-1} \, dx_1 \cdots dx_n = \frac{\Gamma(p_1) \cdots \Gamma(p_n)}{\Gamma(p_1 + \cdots + p_n + 1)}
\]
for all \( p_1, \ldots, p_n > 0 \).

For the proof, we denote
\[
I(p_1, \ldots, p_n) = \int \cdots \int_{x_1, \ldots, x_n > 0, x_1 + \cdots + x_n < 1} x_1^{p_1-1} \cdots x_n^{p_n-1} \, dx_1 \cdots dx_n .
\]
This integral is improper, unless \( p_1, \ldots, p_n \geq 1 \).

10g2 Lemma. \( I(p_1, \ldots, p_n) = B(p_n, p_1 + \cdots + p_{n-1} + 1) I(p_1, \ldots, p_{n-1}) \).

Proof. We introduce proper integrals
\[
I_\varepsilon(p_1, \ldots, p_n) = \int \cdots \int_{x_1, \ldots, x_n > \varepsilon, x_1 + \cdots + x_n < 1} x_1^{p_1-1} \cdots x_n^{p_n-1} \, dx_1 \cdots dx_n
\]
for \( \varepsilon > 0 \).\footnote{If \( n \varepsilon \geq 1 \) then \( I_\varepsilon(p_1, \ldots, p_n) = 0 \), of course.} Clearly, \( I_\varepsilon(p_1, \ldots, p_n) \leq I(p_1, \ldots, p_n) \), and \( I_\varepsilon(p_1, \ldots, p_n) \to I(p_1, \ldots, p_n) \) as \( \varepsilon \to 0+ \).

The change of variables \( \xi = ax \) (that is, \( \xi_1 = ax_1, \ldots, \xi_n = ax_n \)) gives (by Theorem 10f1)
\[
\int \cdots \int_{\xi_1, \ldots, \xi_n > \varepsilon, x_1 + \cdots + x_n < 0} \xi_1^{p_1-1} \cdots \xi_n^{p_n-1} \, d\xi_1 \cdots d\xi_n = a^{p_1 + \cdots + p_n} I_\varepsilon(p_1, \ldots, p_n) \quad \text{for } a > 0 .
\]
We use iterated integral (proper!):

\[
I_\varepsilon(p_1, \ldots, p_n) = \int_\varepsilon^1 dx_n x_n^{p_n-1} \int \cdots \int x_1^{p_1-1} \cdots x_{n-1}^{p_{n-1}-1} dx_1 \cdots dx_{n-1} = \\
= \int_\varepsilon^1 x_n^{p_n-1}(1 - x_n)^{p_1+\cdots+p_{n-1}} I_\varepsilon/(1-x_n)(p_1, \ldots, p_{n-1}) \, dx_n.
\]

On one hand,

\[
I_\varepsilon(p_1, \ldots, p_n) \leq I(p_1, \ldots, p_{n-1}) \int_0^1 x_n^{p_n-1}(1 - x_n)^{p_1+\cdots+p_{n-1}} \, dx_n = \\
= I(p_1, \ldots, p_{n-1}) B(p_n, p_1 + \cdots + p_{n-1} + 1)
\]

for all \(\varepsilon\), therefore \(I(p_1, \ldots, p_n) \leq B(p_n, p_1 + \cdots + p_{n-1} + 1)I(p_1, \ldots, p_{n-1})\).

On the other hand, for arbitrary \(\delta > 0\),

\[
I_\varepsilon(p_1, \ldots, p_n) \geq \int_\varepsilon^{1-\delta} x_n^{p_n-1}(1 - x_n)^{p_1+\cdots+p_{n-1}} I_\varepsilon/(1-x_n)(p_1, \ldots, p_{n-1}) \, dx_n \geq \\
\geq \int_\varepsilon^{1-\delta} x_n^{p_n-1}(1 - x_n)^{p_1+\cdots+p_{n-1}} I_{\varepsilon/\delta}(p_1, \ldots, p_{n-1}) \, dx_n
\]

for all \(\varepsilon\), therefore

\[
I(p_1, \ldots, p_n) \geq \int_0^{1-\delta} x_n^{p_n-1}(1 - x_n)^{p_1+\cdots+p_{n-1}} I(p_1, \ldots, p_{n-1}) \, dx_n
\]

for all \(\delta\), and finally, \(I(p_1, \ldots, p_n) \geq B(p_n, p_1 + \cdots + p_{n-1} + 1)I(p_1, \ldots, p_{n-1})\).

\(\square\)

**Proof of Prop. 10g1**

Induction in the dimension \(n\). For \(n = 1\) the formula is obvious:

\[
\int_0^1 x_1^{p_1-1} \, dx_1 = \frac{1}{p_1} = \frac{\Gamma(p_1)}{\Gamma(p_1 + 1)}.
\]

From \(n - 1\) to \(n\): using 10g2

\[
I(p_1, \ldots, p_n) = \frac{\Gamma(p_n)\Gamma(p_1 + \cdots + p_{n-1} + 1)}{\Gamma(p_1 + \cdots + p_{n-1} + 1)} \cdot \frac{\Gamma(p_1) \cdots \Gamma(p_{n-1})}{\Gamma(p_1 + \cdots + p_{n-1} + 1)} = \\
= \frac{\Gamma(p_1) \cdots \Gamma(p_n)}{\Gamma(p_1 + \cdots + p_{n-1} + 1)}.
\]

\(\square\)
There is a seemingly more general formula,
\[
\int \cdots \int_{x_1, \ldots, x_n > 0, \atop x_1^{p_1} + \cdots + x_n^{p_n} < 1} x_1^{p_1 - 1} \cdots x_n^{p_n - 1} \, dx_1 \cdots dx_n = \frac{1}{\gamma_1 \cdots \gamma_n} \cdot \frac{\Gamma(\gamma_1) \cdots \Gamma(\gamma_n)}{\Gamma(\frac{p_1}{\gamma_1} + \cdots + \frac{p_n}{\gamma_n} + 1)},
\]
easily obtained from the previous one by the change of variables \(y_j = x_j^{\frac{1}{\gamma_j}}\).

A special case: \(p_1 = \cdots = p_n = 1, \ \gamma_1 = \cdots = \gamma_n = p;\)
\[
\int \cdots \int_{x_1, \ldots, x_n > 0, \atop x_1^{\gamma_1} + \cdots + x_n^{\gamma_n} < 1} x_1 \cdots x_n \, dx_1 \cdots dx_n = \frac{\Gamma(n\left(\frac{1}{p}\right))}{p^n \Gamma\left(n\left(\frac{1}{p}\right) + 1\right)}.
\]
We’ve found the volume of the unit ball in the metric \(l_p:\)
\[
\mathbf{v}(B_p(1)) = \frac{2^n \Gamma(n\left(\frac{1}{p}\right))}{p^n \Gamma\left(n\left(\frac{1}{p}\right) + 1\right)}.
\]
If \(p = 2,\) the formula gives us (again; see \([10d7]\)) the volume of the standard unit ball:
\[
V_n = \mathbf{v}(B_2(1)) = \frac{2^{n/2}}{n \Gamma\left(\frac{n}{2}\right)}.
\]
We also see that the volume of the unit ball in the \(l_1\)-metric equals \(\frac{2^n}{n!}.\)

Question: what does the formula give in the \(p \to \infty\) limit?

\textbf{10g3 Exercise.} Show that
\[
\int_{x_1 + \cdots + x_n < 1, \atop x_1, \ldots, x_n > 0} \varphi(x_1 + \cdots + x_n) \, dx_1 \cdots dx_n = \frac{1}{(n - 1)!} \int_0^1 \varphi(s) s^{n-1} \, ds
\]
for every “good” function \(\varphi : [0, 1] \to \mathbb{R}\) and, more generally,
\[
\int \cdots \int_{x_1 + \cdots + x_n < 1, \atop x_1, \ldots, x_n > 0} \varphi(x_1 + \cdots + x_n) x_1^{p_1 - 1} \cdots x_n^{p_n - 1} \, dx_1 \cdots dx_n =
\]
\[
= \frac{\Gamma(p_1) \cdots \Gamma(p_n)}{\Gamma(p_1 + \cdots + p_n)} \int_0^1 \varphi(u) u^{p_1 + \cdots + p_n - 1} \, du.
\]
Hint: consider
\[
\int_0^1 \, ds \, \varphi'(s) \int \cdots \int_{x_1 + \cdots + x_n < s, \atop x_1, \ldots, x_n > 0} x_1^{p_1 - 1} \cdots x_n^{p_n - 1} \, dx_1 \cdots dx_n.
\]
Index

almost all points, 174
beta function, 171
Cauchy-Schwarz inequality, 177
continuous almost everywhere, 162
equivalent, 174
exhaustion, 165 175
gamma function, 169
gravitational constant, 166
H"older inequality, 177
improper integral
\text{signed}, 173 174
\text{unsigned}, 162
improperly integrable, 174
inner product, 176
Lebesgue measure zero, 162
linearity, 174
negligible, 174
Newton’s law, 166
partially defined, 175
Poisson formula, 163
square integrable, 176
triangle inequality, 176 177
volume of ball, 170 181
B, 171
\[ f \], 174
\[ f \cdot \mathbb{1}_{G} \], 162
\[ f^+, f^- \], 173
G, 162
\hat{G}, 166
\Gamma, 169
\hat{L}^2(G), 176
\hat{L}^p(G), 177