15 From boundary to exterior derivative

15a Chains

Recall the integral $\int_{\Gamma} \omega$ defined by (11e12).

15a1 Definition. A (singular) $k$-chain (in $\mathbb{R}^n$) is a formal linear combination of singular $k$-boxes.

That is, 

$$C = c_1 \Gamma_1 + \cdots + c_p \Gamma_p,$$

where $c_1, \ldots, c_p \in \mathbb{R}$, and $\Gamma_1, \ldots, \Gamma_p$ are singular $k$-boxes. More formally, this is a real-valued function with finite support on the (huge!) set of all singular $k$-boxes;

$$c_1 = C(\Gamma_1), \ldots, c_p = C(\Gamma_p); \quad C(\Gamma) = 0 \text{ for all other } \Gamma.$$

Clearly, all $k$-chains are a (huge) vector space, with a basis indexed by all singular $k$-boxes. Less formally we say that the singular $k$-boxes are the basis, and each singular box is (a special case of) a chain: $\Gamma = 1 \cdot \Gamma$.

15a2 Definition.

$$\int_C \omega = c_1 \int_{\Gamma_1} \omega + \cdots + c_p \int_{\Gamma_p} \omega$$

for every $k$-chain $C = c_1 \Gamma_1 + \cdots + c_p \Gamma_p$ and every $k$-form $\omega$.

Note that the integral is bilinear; $\int_C \omega$ is linear in $C$ for every $\omega$ (by construction), and linear in $\omega$ for every $C$ (since $\int_\Gamma \omega$ evidently is linear in $\omega$).
**15a3 Definition.** Two \( k \)-chains \( C_1, C_2 \) are *equivalent* if
\[
\int_{C_1} \omega = \int_{C_2} \omega \quad \text{for all } k\text{-forms } \omega \text{ (of class } C^0) .
\]

Let \( B \subset \mathbb{R}^k \) be a box, \( P \) its partition, and \( \Gamma : B \to \mathbb{R}^n \) a singular box. Then
\[
\Gamma \sim \sum_{b \in P} \Gamma|_b,
\]
since \( \Gamma \mapsto \int_{\Gamma} \omega \) is an additive function of a singular box.

Recall that singular 1-boxes are \( C^1 \)-paths.

By 11c13, equivalent paths are equivalent 1-chains.

By 11c11, the 1-chain \( \gamma + \gamma_{-1} \) is equivalent to 0; here \( \gamma_{-1} \) is the inverse path.

\[
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\]

\[
\begin{array}{ccc}
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\text{(recall 11e2)}
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\]

15b Order 0 and order 1

The case \( k = 0 \) is included as follows. The space \( \mathbb{R}^0 \) consists, by definition, of a single point \( 0 \). The only 0-dimensional box is \( \{0\} \). A singular 0-box in \( \mathbb{R}^n \) is thus \( \{x\} \) for some \( x \in \mathbb{R}^n \).

A 0-form on \( \mathbb{R}^n \) is a function \( \omega : \mathbb{R}^n \to \mathbb{R} \) (of class \( C^m \)). And
\[
\int_{\{x\}} \omega = \omega(x) ,
\]

\(^1\text{Well, more formally, it is } \{(0, x)\}.\)
of course. Accordingly, \( \int_{C} \omega = c_{1} \omega(x_{1}) + \cdots + c_{p} \omega(x_{p}) \) for a 0-chain \( C = c_{1} \{x_{1}\} + \cdots + c_{p} \{x_{p}\} \).

**15b1 Exercise.** If two 0-chains are equivalent then they are equal.
Prove it.

The *boundary* of a singular 1-box \( \gamma : [t_{0}, t_{1}] \to \mathbb{R}^{n} \) is, by definition, the 0-chain
\[
\partial \gamma = \{\gamma(t_{1})\} - \{\gamma(t_{0})\},
\]
a linear combination of two singular 0-boxes (not to be confused with \( \gamma(t_{1}) - \gamma(t_{0}) \)). Thus,
\[
\int_{\partial \gamma} \omega = \omega(\gamma(t_{1})) - \omega(\gamma(t_{0})) \quad \text{for a 0-form } \omega.
\]
The boundary of a 1-chain \( C = c_{1} \gamma_{1} + \cdots + c_{p} \gamma_{p} \) is, by definition, the 0-chain \( \partial C = c_{1} \partial \gamma_{1} + \cdots + c_{p} \partial \gamma_{p} \).

For example,

- the boundary of \( A \to B \) \( \to C \to D \) is \( -\{A\} + \{C\} + \{D\} \);
- the boundary of \( \bigcirc \to \bigcirc \to \bigcirc \) is 0.

Note that the map \( C \mapsto \partial C \) is linear (by construction).

Given a 0-form \( \omega \) of class \( C^{1} \) on \( \mathbb{R}^{n} \), that is, a continuously differentiable function \( \omega : \mathbb{R}^{n} \to \mathbb{R} \), its derivative \( D \omega \) may be thought of as a 1-form of class \( C^{0} \) on \( \mathbb{R}^{n} \), denoted \( d \omega \);

\[
(15b2) \quad (d \omega)(x, h) = (D \omega)_{x}(h) = (D_{h} \omega)_{x}.
\]

**15b3 Proposition.** (*Stokes’ theorem for \( k = 1 \))

Let \( C \) be a 1-chain in \( \mathbb{R}^{n} \), and \( \omega \) a 0-form of class \( C^{1} \) on \( \mathbb{R}^{n} \). Then
\[
\int_{C} d \omega = \oint_{\partial C} \omega.
\]

**Proof.** By linearity in \( C \) it is sufficient to prove it for \( C = \gamma \) (a single 1-box, that is, a path \( \gamma : [t_{0}, t_{1}] \to \mathbb{R}^{n} \)). We have
\[
\int_{\gamma} d \omega = \int_{t_{0}}^{t_{1}} d \omega(\gamma(t), \gamma'(t)) \, dt = \int_{t_{0}}^{t_{1}} (D \omega)_{\gamma(t)}(\gamma'(t)) \, dt = \int_{t_{0}}^{t_{1}} \left( \frac{d}{dt} \omega(\gamma(t)) \right) \, dt = \omega(\gamma(t_{1})) - \omega(\gamma(t_{0})) = \oint_{\partial \gamma} \omega.
\]

\( \square \)
15b4 Corollary.

\[ C_1 \sim C_2 \implies \partial C_1 = \partial C_2 \]

for arbitrary 1-chains \( C_1, C_2 \) in \( \mathbb{R}^n \).

Indeed, \( \int_{\partial C_1} \omega = \int_{C_1} d\omega = \int_{C_2} d\omega = \int_{\partial C_2} \omega \) for every 0-form \( \omega \) of class \( C^1 \).

Similarly to 15b1 it follows that \( \partial C_1 = \partial C_2 \).

The case \( k = 1 \) is special; for higher \( k \) we’ll see (in 16e9) that \( C_1 \sim C_2 \) implies \( \partial C_1 \sim \partial C_2 \) but not \( \partial C_1 = \partial C_2 \). Nothing like 15b1 holds for higher \( k \).

It is easy to prove that \( C_1 \sim C_2 \implies \partial C_1 \sim \partial C_2 \) for \( k = 1 \) without 15b1.

The only problem is that \( C^1(\mathbb{R}^n) \neq C^0(\mathbb{R}^n) \). However, \( C^1(\mathbb{R}^n) \) is dense in \( C^0(\mathbb{R}^n) \) (recall 7d28).

15c Order 1 and order 2

The boundary of a singular 2-box \( \Gamma \) is, by definition, the 1-chain

\[ \Gamma|_{AB} + \Gamma|_{BC} + \Gamma|_{CD} + \Gamma|_{DA} = \Gamma|_{AB} + \Gamma|_{BC} - \Gamma|_{DC} - \Gamma|_{AD}. \]

This is not really a definition of a 1-chain, since I did not specify the four 1-dimensional boxes (which is very easy to do); but its equivalence class is well-defined, and this is all we need for the following question.

Given a 1-form \( \omega \), can we construct a 2-form, call it \( d\omega \), such that \( \int_C d\omega = \int_{\partial C} \omega \) for all 2-chains \( C \)?

We have a function \( \Gamma \mapsto \int_{\partial \Gamma} \omega \) of a singular box; this is an additive function, since the map \( \Gamma \mapsto \partial \Gamma \) is additive (up to equivalence).

We want to differentiate this additive function in the hope that its derivative exists and is a 2-form \( d\omega \).

Note that

\[(15c1) \quad \partial(\partial \Gamma) \sim 0 \quad \text{for a singular 2-box } \Gamma \]

(try it for \( \Gamma \) of 11e2 and 11e3). By 15b3 \( \int_{\partial \Gamma} d\omega = \int_{\partial(\partial \Gamma)} \omega = 0 \) for every 0-form \( \omega \) of class \( C^1 \). It should be \( \int_{\Gamma} d(d\omega) = \int_{\partial \Gamma} d\omega = 0 \) for all \( \Gamma \), that is,
\(d(d\omega) = 0\) for every 0-form \(\omega\) of class \(C^2\). A wonder: the second derivative of a 0-form is always zero, irrespective of the second derivatives of the function! Indeed, exterior derivative is very similar to the usual derivative for 0-forms, but very dissimilar for 1-forms.

Existence of \(d\omega\) is the point of Stokes’ theorem \(15c3\). For now we’ll find a necessary condition on \(d\omega\) that ensures its uniqueness and provides an explicit formula.

Given a point \(x \in \mathbb{R}^n\) and two vectors \(h, k \in \mathbb{R}^n\), we consider small singular boxes \(\Gamma_\varepsilon: [0, 1] \times [0, 1] \to \mathbb{R}^n\),

\[\Gamma_\varepsilon(u_1, u_2) = x + \varepsilon u_1 h + \varepsilon u_2 k;\]

an additive function on \(\Gamma_\varepsilon\) should be of order \(\varepsilon^2\) as \(\varepsilon \to 0^+\); we divide it by \(\varepsilon^2\) and calculate the limit:

\[
\frac{1}{\varepsilon^2} \int_{\partial \Gamma_\varepsilon} \omega = \frac{1}{\varepsilon^2} \int_0^1 \omega(x + \varepsilon u_1 h, \varepsilon h) \, du_1 + \frac{1}{\varepsilon^2} \int_0^1 \omega(x + \varepsilon h + \varepsilon u_2 k, \varepsilon k) \, du_2 - \frac{1}{\varepsilon^2} \int_0^1 \omega(x + \varepsilon u_1 h + \varepsilon k, \varepsilon h) \, du_1 - \frac{1}{\varepsilon^2} \int_0^1 \omega(x + \varepsilon u_2 k, \varepsilon k) \, du_2 =
\]

\[
= \int_0^1 \omega(x + \varepsilon u_1 h, h) - \omega(x + \varepsilon u_1 h + \varepsilon k, h) \frac{\partial}{\partial \varepsilon} \frac{d u_1}{\varepsilon} + \frac{\partial}{\partial \varepsilon} \frac{d u_2}{\varepsilon} (D_k \omega(\cdot, h)) (x) + (D_h \omega(\cdot, k)) (x),
\]

assuming \(\omega \in C^1\). Taking into account that

\[
\frac{1}{\varepsilon^2} \int_{\Gamma_\varepsilon} d\omega \to (d\omega)(x, h, k)
\]

(for arbitrary 2-form in place of \(d\omega\)) we see that the needed \(d\omega\) (if exists) is as follows.

15c2 Definition. The exterior derivative of a 1-form \(\omega\) of class \(C^1\) is the 2-form \(d\omega\) defined by

\[
(d\omega)(\cdot, h, k) = D_k \omega(\cdot, k) - D_h \omega(\cdot, h).
\]

15c3 Theorem. (Stokes’ theorem for \(k = 2\))

Let \(C\) be a 2-chain in \(\mathbb{R}^n\), and \(\omega\) a 1-form of class \(C^1\) on \(\mathbb{R}^n\). Then

\[
\int_C d\omega = \int_{\partial C} \omega.
\]

\(^1\text{This fact will be proved for all forms of all orders, see 16e4(h).}\)
This is a special case of Theorem 15f3 to be proved much later.

15c4 Exercise. For a 1-form \( \omega = f(x,y) \, dx + g(x,y) \, dy \) on \( \mathbb{R}^2 \) (or an open subset of \( \mathbb{R}^2 \)) prove that \( (d\omega)(h,k) = (D_1g - D_2f) \det(h,k) \), that is, \( d\omega = (D_1g - D_2f) \mu_2 \), where \( \mu_2 \) is the volume form on \( \mathbb{R}^2 \).

15c5 Exercise. For the form \( \omega = -\frac{y \, dx + x \, dy}{x^2 + y^2} \) (treated in Sect. 11d) on \( \mathbb{R}^2 \) prove that \( d\omega = 0 \), but \( \int_\gamma \omega \neq 0 \) for some \( \gamma \); does it contradict 15c3?

15c6 Exercise. For the form \( \omega = -\frac{y \, dx + x \, dy}{x^2 + y^2} \) (mentioned in Sect. 11d) on \( \mathbb{R}^2 \) prove that \( d\omega = \mu_2 \). Reconsider 11d2 in the light of 15c3.

15d Order \( N - 1 \): forms and vector fields

Recall two types of integral over an \( n \)-manifold:

- * of an \( n \)-form \( \omega \), \( \int_{(M,O)} \omega \), defined by (12c2)-(13a4);
- * of a function \( f \), \( \int_M f \), defined by (13a7)-(13a8);

they are related by

\[
\int_M f = \int_{(M,O)} f \mu_{(M,O)},
\]

where \( \mu_{(M,O)} \) is the volume form; that is, \( \int_M f = \int_{(M,O)} \omega \) where \( \omega = f \mu_{(M,O)} \).

Interestingly, every \( n \)-form \( \omega \) on an orientable \( n \)-manifold \( M \subset \mathbb{R}^N \) is \( f \mu_{(M,O)} \) for some \( f \in C(M) \). This is a consequence of the one-dimensionality\(^1\) of the space of all antisymmetric multilinear \( n \)-forms on the tangent space \( T_xM \).

We have \( f(x) = \omega(x,e_1,\ldots,e_n) \) for some (therefore, every) orthonormal basis \((e_1,\ldots,e_n)\) of \( T_xM \) that conforms to \( O_x \). But if \( \omega \) is defined on the whole \( \mathbb{R}^N \) (not just on \( M \)), it does not lead to a function \( f \) on the whole \( \mathbb{R}^N \); indeed, in order to find \( f(x) \) we need not just \( x \) but also \( T_xM \) (and its orientation). The case \( n = N \) is simple: every \( N \)-form \( \omega \) on \( \mathbb{R}^N \) (or on an open subset of \( \mathbb{R}^N \)) is \( f \mu_N \) (for some continuous \( f \)), where \( \mu_N \) is the volume form on \( \mathbb{R}^N \); that is,

\[
\mu_N(x,h_1,\ldots,h_N) = \det(h_1,\ldots,h_N);
\omega(x,h_1,\ldots,h_N) = f(x) \det(h_1,\ldots,h_N);
\]

\[ f(x) = \omega(x,e_1,\ldots,e_N). \]

We turn to the case \( n = N - 1 \).

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\(^1\)Recall Sect. 11e and 12c.
The space of all antisymmetric multilinear \( n \)-forms \( L \) on \( \mathbb{R}^N \) is of dimension \( \binom{N}{n} = N \). Here is a useful linear one-to-one correspondence between such \( L \) and vectors \( h \in \mathbb{R}^N \):

\[
\forall h_1, \ldots, h_n \quad L(h_1, \ldots, h_n) = \det(h, h_1, \ldots, h_n) .
\]

Introducing the cross-product \( h_1 \times \cdots \times h_n \) by \(^1\)

\[
\forall h \quad \langle h, h_1 \times \cdots \times h_n \rangle = \det(h, h_1, \ldots, h_n)
\]

(it is a vector orthogonal to \( h_1, \ldots, h_n \)), we get

\[
L(h_1, \ldots, h_n) = \langle h, h_1 \times \cdots \times h_n \rangle.
\]

Doing so at every point, we get a linear one-to-one correspondence between \( n \)-forms \( \omega \) on \( \mathbb{R}^N \) and vector fields \( F \) on \( \mathbb{R}^N \):

\[
(15d1) \quad \omega(x, h_1, \ldots, h_n) = \langle F(x), h_1 \times \cdots \times h_n \rangle.
\]

Similarly, \((n - 1)\)-forms \( \omega \) on an oriented \( n \)-dimensional manifold \((M, O)\) in \( \mathbb{R}^N \) (not just \( N - n = 1 \)) are in a linear one-to-one correspondence with tangent vector fields \( F \) on \( M \), that is, \( F \in C(M \to \mathbb{R}^N) \) such that \( \forall x \in M \quad F(x) \in T_x M \).

Let \( M \subset \mathbb{R}^N \) be an orientable \( n \)-manifold (still, \( n = N - 1 \)), \( \omega \) and \( F \) as in \((15d1)\). We know that \( \omega|_M = f \mu_{(M,O)} \) for some \( f \). How is \( f \) related to \( F \)?

Given \( x \in M \), we take an orthonormal basis \((e_1, \ldots, e_n)\) of \( T_x M \), note that \( e_1 \times \cdots \times e_n = n_x \) is a unit normal vector to \( M \) at \( x \), and

\[
\langle F(x), n_x \rangle = \langle F(x), e_1 \times \cdots \times e_n \rangle = \omega(x, e_1, \ldots, e_n) = f(x) \mu_{(M,O)}(x, e_1, \ldots, e_n) = \pm f(x).
\]

In order to get “+” rather than “\( \pm \)” we need a coordination between the orientation \( O \) and the normal vector \( n_x \). Let the basis \((e_1, \ldots, e_n)\) of \( T_x M \) conform to the orientation \( O_x \) (of \( M \) at \( x \), or equivalently, of \( T_x M \), recall Sect. 12b), then \( \mu_{(M,O)}(x, e_1, \ldots, e_n) = +1 \). The two unit normal vectors being \( \pm e_1 \times \cdots \times e_n \), we say that \( n_x = e_1 \times \cdots \times e_n \) conforms to the given orientation, and get\(^2\)

\[
\langle F(x), n_x \rangle = f(x) ; \quad \omega|_M = \langle F, n \rangle \mu_{(M,O)}.
\]

---

\(^1\)For \( N = 3 \) the cross-product is a binary operation, but for \( N > 3 \) it is not. In fact, it is possible to define the corresponding associative binary operation (the so-called exterior product, or wedge product), not on vectors but on the so-called multivectors, see \"Multivector\" and \"Exterior algebra\" in Wikipedia.

\(^2\)Not unexpectedly, in order to find \( f(x) \) we need not just \( x \) but also \( n_x \).
Integrating this over $M$, we get nothing but the flux! Recall 14c1: the flux of $F$ through $M$ is $\int_M \langle F, n \rangle$, that is, $\int_{(M, \mathcal{O})} \langle F, n \rangle \mu(M, \mathcal{O}) = \int_{(M, \mathcal{O})} \omega |_M = \int_{(M, \mathcal{O})} \omega$. Well, 14c1 treats a more special case: $M = \partial G$, and $n$ is directed outwards. Let us generalize it a little.

15d2 Definition. Let $M \subset \mathbb{R}^{n+1}$ be an orientable $n$-manifold, $F : M \to \mathbb{R}^{n+1}$ a mapping continuous almost everywhere, and $\mathcal{O}$ an orientation of $M$. The flux of (the vector field) $F$ through (the oriented hypersurface) $(M, \mathcal{O})$ is

$$\int_M \langle F, n \rangle$$

where $n$ is the unit normal vector to $M$ that conforms to $\mathcal{O}$. (The integral is treated as improper, and may converge or diverge.)

Thus,

$$\int_{(M, \mathcal{O})} \omega = \int_{M} \langle F, n \rangle$$

whenever $(M, \mathcal{O})$ is an oriented hypersurface, $n$ conforms to $\mathcal{O}$, and $F$ corresponds to $\omega$ according to (15d1).

Recall 15c4–15c6.

15d4 Exercise. For a 1-form $\omega = f(x, y) \, dx + g(x, y) \, dy$ on $\mathbb{R}^2$ (or an open subset of $\mathbb{R}^2$) prove that the corresponding vector field is $F = (F_1, F_2) = (g, -f)$, and $d\omega = (\text{div} \, F) \mu_2$.

15d5 Exercise. For the form $\omega = -y \, dx + x \, dy$ on $\mathbb{R}^2 \setminus \{0\}$ find the corresponding vector field $F$ and prove that $F$ is the gradient of the radial harmonic function (14d9).

15d6 Exercise. For the form $\omega = -y \, dx + x \, dy$ on $\mathbb{R}^2$ find the corresponding vector field $F$. Is $F$ the gradient of some function? Of some harmonic function? Find the flux of $F$ through the boundary of the triangle from 11d2.

15d7 Exercise. On $\mathbb{R}^3 \setminus \{0\}$, consider the gradient $F$ of the radial harmonic function (14d9) (for $c_1 = 1$, $c_2 = 0$), and the corresponding 2-form $\omega$. Find the integral of $\omega$ over the sphere $\{x : |x| = r\}$. 


15e Boundary

We want to apply the divergence theorem to the open cube $B = (0, 1)^N$, but for now we cannot, since the boundary $\partial B$ is not a manifold. Rather, $\partial B$ consists of $2N$ disjoint cubes of dimension $n = N - 1$ (“hyperfaces”) and a finite number\(^1\) of cubes of dimensions $0, 1, \ldots, n - 1$.

For example, $\{1\} \times (0, 1)^n$ is a hyperface.

Each hyperface is an $n$-manifold, and has exactly two orientations. Also, the outward unit normal vector $n_x$ is well-defined at every point $x$ of a hyperface.

For example, $n_x = e_1$ for every $x \in \{1\} \times (0, 1)^n$.

For a function $f$ on $\partial B$ we define $\int_{\partial B} f$ as the sum of integrals over the $2N$ hyperfaces; that is,

\[
(15e1) \quad \int_{\partial B} f = \sum_{i=1}^N \sum_{x_i=0,1} \int_{(0,1)^n} f(x_1, \ldots, x_N) \prod_{j:j \neq i} dx_j,
\]

provided that these integrals are well-defined, of course.

For a vector field $F \in C(\partial B \to \mathbb{R}^N)$ we define the flux of $F$ through $\partial B$ as $\int_{\partial B} \langle F, n \rangle$. Note that

\[
(15e2) \quad \int_{\partial B} \langle F, n \rangle = \sum_{i=1}^N \sum_{x_i=0,1} (2x_i - 1) \int_{(0,1)^n} F_i(x_1, \ldots, x_N) \prod_{j:j \neq i} dx_j.
\]

It is surprisingly easy to prove the divergence theorem for the cube. (Just from scratch; no need to use 14c3, nor 13b9.)

15e3 Proposition. Let $F \in C^1((0, 1)^N \to \mathbb{R}^N)$, with $DF$ bounded. Then the integral of $\text{div} \ F$ over $(0, 1)^N$ is equal to the (outward) flux of $F$ through the boundary.

(As before, boundedness of $DF$ ensures that $F$ extends to $[0, 1]^N$ by continuity; recall 14e4.)

Proof.

\[
\int_0^1 D_1 F_1(x_1, \ldots, x_N) \, dx_1 = F_1(1, x_2, \ldots, x_N) - F_1(0, x_2, \ldots, x_N) = \sum_{x_1=0,1} (2x_1 - 1) F_1(x_1, \ldots, x_N);
\]

\(^1\)In fact, $3^N - 1 - 2N$. 

\[
\int \cdots \int D_1 F_1 = \sum_{x_1=0,1} (2x_1 - 1) \int \cdots \int F_1(x_1, \ldots, x_N) \, dx_2 \ldots dx_N;
\]
similarly, for each \(i = 1, \ldots, N\),
\[
\int \cdots \int D_i F_i = \sum_{x_i=0,1} (2x_i - 1) \int \cdots \int F_i \prod_{j \neq i} dx_j;
\]
it remains to sum over \(i\).

The same holds for every box \(B\), of course.

Let a vector field \(F\) correspond to an \(n\)-form \(\omega\) according to (15d1). We want to think of the flux \(\int_{\partial B} \langle F, n \rangle\) as \(\int_{\partial B} \omega\); for now we cannot, since \(\partial B\) is not an \(n\)-manifold, nor an \(n\)-chain. However, we may treat \(B\) as a singular \(N\)-box \(\Gamma: B \to \mathbb{R}^N\), \(\Gamma(x) = x\), and then, according to Sections 15b, 15c, \(\partial B\) may be treated as an \(n\)-chain in two cases, \(N = 1\) and \(N = 2\). Here is the corresponding construction for arbitrary \(N\).

The \(2N\) hyperfaces of \((0, 1)^N\) are

\[H_{i,a} = \{ (x_1, \ldots, x_N) \in [0, 1]^n : x_i = a \} \quad \text{for } i = 1, \ldots, N \text{ and } a = 0, 1.\]

For each hyperface \(H_{i,a}\) we choose the orientation \(O_{i,a}\) that conforms to \(n_x\) in the sense introduced above: \(n_x = h_1 \times \cdots \times h_n\) for some (therefore, every) orthonormal basis \((h_1, \ldots, h_n)\) of the tangent space (to the hyperface) that conforms to \(O_{i,a}\). Note that \(n_x = h_1 \times \cdots \times h_n\) means \(\det(n_x, h_1, \ldots, h_n) = +1\).

Denoting by \((e_1, \ldots, e_N)\) the usual basis of \(\mathbb{R}^N\), we try the basis \((e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_N)\) of the tangent space \(\{ x : x_i = 0 \}\) to \(H_{i,a}\). We observe that \(\det(e_i, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_N) = (-1)^{i-1}\), \(n_x = (2a - 1)e_i\), and conclude that the basis \((e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_N)\) conforms to \(O_{i,a}\) if and only if \((-1)^{i-1}(2a - 1) = +1\). Thus, the \(n\)-chart \(\Delta_{i,a}\) of \(H_{i,a}\) defined by

\[\Delta_{i,a}(u_1, \ldots, u_n) = (u_1, \ldots, u_{i-1}, a, u_i, \ldots, u_n) \quad \text{for } u \in (0, 1)^n\]

conforms to \(O_{i,a}\) if and only if \((-1)^{i-1}(2a - 1) = +1\). Treating each \(\Delta_{i,a}\) (extended to \([0, 1]^n\)) as a singular \(n\)-cube, we define the \(n\)-chain \(\partial B\) as follows:

\[
\partial B = \sum_{i=1}^{N} \sum_{a=0,1} (-1)^{i-1}(2a - 1) \Delta_{i,a}.
\]

Now we have

\[
\int_{\partial B} \omega = \int_{\partial B} \langle F, n \rangle
\]
whenever $\omega$ and $F$ are related via (15d1). This equality results from
\[
(−1)^{i−1}(2a − 1)\int_{\Delta_i,a} \omega = \int_{(H_{i,a},O_{i,a})} \omega = \int_{H_{i,a}} \langle F, n \rangle
\]
by summation over $i$ and $a$.

For a singular cube $\Gamma : [0, 1]^N \rightarrow \mathbb{R}^m$ we define $\partial \Gamma$ as the $n$-chain
\[
\partial \Gamma = \sum_{i=1}^{N} \sum_{a=0,1} (−1)^{i−1}(2a − 1)\Gamma \circ \Delta_{i,a}.
\]
Note that (15e4) is the special case for $\Gamma(x) = x$.

Here is what we get for $N = 2$ and $N = 3$:

\[
\Gamma|_{AB} + \Gamma|_{BC} + \Gamma|_{CD} + \Gamma|_{DA} = \Gamma|_{AB} + \Gamma|_{BC} - \Gamma|_{DC} - \Gamma|_{AD},
\]
\[
\Gamma|_{ADCB} + \Gamma|_{EFGH} + \Gamma|_{ABFE} + \\
+ \Gamma|_{DHGC} + \Gamma|_{AEHD} + \Gamma|_{BCGF} = \\
= -\Gamma|_{ABCD} + \Gamma|_{EFGH} - \Gamma|_{AEFB} + \\
+ \Gamma|_{DHGC} - \Gamma|_{ADHE} + \Gamma|_{BCGF}.
\]

A cube is only one example of a bounded regular open set $G \subset \mathbb{R}^{n+1}$ such that $\partial G$ is not an $n$-manifold and still, the divergence theorem holds as $\int_G \text{div} F = \int_{\partial G \setminus Z} \langle F, n \rangle$ for some closed set $Z \subset \partial G$ such that $\partial G \setminus Z$ is an $n$-manifold. In such cases we’ll say that the divergence theorem holds for $G$ and $\partial G \setminus Z$. For the cube, $\partial G \setminus Z$ is the union of the $2N$ hyperfaces, and $Z$ is the union of cubes of smaller (than $N − 1$) dimensions.

15e7 Exercise (product). Let $G_1 \subset \mathbb{R}^{N_1}$, $Z_1 \subset \partial G_1$, and $G_2 \subset \mathbb{R}^{N_2}$, $Z_2 \subset \partial G_2$. If the divergence theorem holds for $G_1$, $\partial G_1 \setminus Z_1$ and for $G_2$, $\partial G_2 \setminus Z_2$, then it holds for $G$, $\partial G \setminus Z$ where $G = G_1 \times G_2 \subset \mathbb{R}^{N_1+N_2}$ and $\partial G \setminus Z = ((\partial G_1 \setminus Z_1) \times G_2) \cup (G_1 \times (\partial G_2 \setminus Z_2))$.

Prove it.\footnote{Hint: $\text{div} F = (D_1 F_1 + \cdots + D_{N_1} F_{N_1}) + (D_{N_1+1} F_{N_1+1} + \cdots + D_{N_1+N_2} F_{N_1+N_2})$.}

An $N$-box is the product of $N$ intervals, of course. Also, a cylinder $\{(x, y, z) : x^2 + y^2 < r^2, 0 < z < a\}$ is the product of a disk and an interval.

1\text{f}_{\partial B} \omega$ is the integral of the $n$-form $\omega$ over the $n$-chain $\partial B$ defined by (15e4): $\int_{\partial B} \langle F, n \rangle$ is the flux defined by (15e2).
15f Exterior derivative

In order to find the formula for the exterior derivative \( d\omega \) of a form of arbitrary order, we could generalize the approach of Sect. [15c]. However, a shorter way is available, via divergence.

Let \( \omega \) be a \( (k-1) \)-form on \( \mathbb{R}^N \). Assuming existence of a \( k \)-form \( d\omega \) on \( \mathbb{R}^N \) such that \( \int_{\Gamma} d\omega = \int_{\partial\Gamma} \omega \) for all singular \( k \)-boxes \( \Gamma \), we want to find \( d\omega(x,h_1,\ldots,h_k) \). It is sufficient to find \( d\omega(x,e_{i_1},\ldots,e_{i_k}) \) for \( 1 \leq i_1 < \cdots < i_k \leq N \); here \((e_1,\ldots,e_N)\) is the usual basis of \( \mathbb{R}^N \). Let us find \( d\omega(x,e_1,\ldots,e_k) \); other cases are similar.

Vectors \( e_1,\ldots,e_k \) span the \( k \)-dimensional subspace \( \{x : x_{k+1} = \cdots = x_N = 0\} = \mathbb{R}^k \subset \mathbb{R}^N \). We need only the restriction \( \omega|_{\mathbb{R}^k} \), and re-denote this restriction by \( \omega \).

Being a \( (k-1) \)-form on \( \mathbb{R}^k \), the form \( \omega \) corresponds to a vector field \( F : \mathbb{R}^k \to \mathbb{R}^k \) according to [15d1].

For every cube \( B \subset \mathbb{R}^k \), by [15d3] and [15e3] \( \int_{\partial B} \omega = \int_{B} \langle F, n \rangle = \int_{B} \text{div} F \).

Being a \( k \)-form on \( \mathbb{R}^k \), the form \( d\omega \) is \( \int \mu_k \) for some \( f \in C(\mathbb{R}^k) \); here \( \mu_k \) is the volume form on \( \mathbb{R}^k \). Thus, \( \int_{B} d\omega = \int_{B} f \). The needed equality \( \int_{B} d\omega = \int_{\partial B} ^{\omega} \) becomes \( \int_{B} f = \int_{B} \text{div} F \) (for all \( B \)), that is, \( f = \text{div} F \). It remains to express this equality in terms of \( \omega \) and \( d\omega \).

We have

\[
F_1(x) = \langle F(x), e_1 \rangle = \langle F(x), e_2 \times \cdots \times e_k \rangle = \omega(x, e_2, \ldots, e_k); \\
F_2(x) = \langle F(x), e_2 \rangle = \langle F(x), -e_1 \times e_3 \times \cdots \times e_k \rangle = -\omega(x, e_1, e_3, \ldots, e_k); \\
\]

and so on. Hence,

\[
\text{div} F = D_1 F_1 + \cdots + D_k F_k = \\
= D_1 \omega(\cdot, e_2, \ldots, e_k) - D_2 \omega(\cdot, e_1, e_3, \ldots, e_k) + \cdots + (-1)^{k-1} D_k \omega(\cdot, e_1, \ldots, e_{k-1}).
\]

On the other hand,

\[
d\omega(x, e_1, \ldots, e_k) = f(x) \mu_k(e_1, \ldots, e_k) = f(x) = \text{div} F(x).
\]

Finally,

\[
d\omega(\cdot, e_1, \ldots, e_k) = \\
= D_1 \omega(\cdot, e_2, \ldots, e_k) - D_2 \omega(\cdot, e_1, e_3, \ldots, e_k) + \cdots + (-1)^{k-1} D_k \omega(\cdot, e_1, \ldots, e_{k-1}).
\]

The same holds for \( e_{i_1}, \ldots, e_{i_k} \), and moreover, for arbitrary \( h_1, \ldots, h_k \in \mathbb{R}^N \), since both sides of this equality are antisymmetric multilinear forms.
15f1 Definition. The exterior derivative of a $(k - 1)$-form $\omega$ of class $C^1$ is the $k$-form $d\omega$ defined by

$$(d\omega)(\cdot, h_1, \ldots, h_k) = \sum_{i=1}^{k} (-1)^{i-1} D_{h_i} \omega(\cdot, h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_k).$$

For an $n$-form $\omega$ on $\mathbb{R}^N$, $N = n + 1$, and $B = [0,1]^N$, we have $d\omega = (\text{div } F) \mu_N$, thus, $\int_B d\omega = \int_B \text{div } F$, whence

(15f2) $$\int_B d\omega = \int_{\partial B} \omega$$

for all $n$-forms $\omega$ on $\mathbb{R}^N$, which is Stokes’ theorem for nonsingular cubes.

15f3 Theorem. (Stokes’ theorem)

Let $C$ be a $k$-chain in $\mathbb{R}^N$, and $\omega$ a $(k - 1)$-form of class $C^1$ on $\mathbb{R}^N$. Then

$$\int_C d\omega = \int_{\partial C} \omega.$$

(To be proved later, in Sect. 16d.)

15f4 Exercise. The divergence theorem holds for $G \subset \mathbb{R}^{n+1}$ and $\partial G \setminus Z$ (recall 15e7 and the paragraph before it) if and only if $\int_G d\omega = \int_{\partial G \setminus Z} \omega$ for all $n$-forms $\omega$ on $\mathbb{R}^{n+1}$.

Prove it.

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