4 Divergence theorem and its consequences

4a Divergence and flux ........................................... 65
4b Piecewise smooth case ....................................... 67
4c Divergence of gradient: Laplacian ....................... 69
4d Laplacian at a singular point ............................... 71
4e Differential forms of order $N - 1$ ......................... 75

The divergence theorem sheds light on harmonic functions and differential forms.

4a Divergence and flux

We return to the case treated before, in the end of Sect. 3b: $G \subset \mathbb{R}^N$ is a smooth set. Recall the outward unit normal vector $n_x$ for $x \in \partial G$.

4a1 Definition. For a continuous $F : \partial G \to \mathbb{R}^N$, the (outward) flux of (the vector field) $F$ through $\partial G$ is

$$\int_{\partial G} \langle F, n \rangle .$$

(The integral is interpreted according to (2d8).)

If a vector field $F$ on $\mathbb{R}^3$ is the velocity field of a fluid, then the flux of $F$ through a surface is the amount$^1$ of fluid flowing through the surface (per unit time).$^2$ If the fluid is flowing parallel to the surface then, evidently, the flux is zero.

We continue similarly to Sect. 3b. Let $F \in C^1(G \to \mathbb{R}^N)$, with $DF$ bounded (on $G$). Recall that, by 3b6, boundedness of $DF$ on $G$ ensures that $F$ extends to $\overline{G}$ by continuity (and therefore is bounded). In such cases we always use this extension. The mapping $\tilde{F} : \mathbb{R}^N \setminus \partial G \to \mathbb{R}^N$ defined by

$$\tilde{F}(x) = \begin{cases} F(x) & \text{for } x \in G, \\ 0 & \text{for } x \notin \overline{G} \end{cases}$$

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$^1$The volume is meant, not the mass. However, these are proportional if the density (kg/m$^3$) of the matter is constant (which often holds for fluids).

$^2$See also mathinsight.
is continuous up to $\partial G$, and
\[
\tilde{F}(x - 0n_x) = F(x), \quad \tilde{F}(x + 0n_x) = 0;
\]
\[
\text{div}_{\text{sing}} \tilde{F}(x) = -\langle F(x), n_x \rangle.
\]
By Theorem 3e3 (applied to $\tilde{F}$ and $K = \partial G$),
\[
(4a2) \int_G \text{div} F = \int_{\partial G} \langle F, n \rangle,
\]
just the flux. The divergence theorem, formulated below, is thus proved.\(^1\)

**4a3 Theorem** *(Divergence theorem).* Let $G \subset \mathbb{R}^N$ be a smooth set, $F \in C^1(G \to \mathbb{R}^N)$, with $DF$ bounded on $G$. Then the integral of $\text{div} F$ over $G$ is equal to the (outward) flux of $F$ through $\partial G$.

In particular, if $\text{div} F = 0$, then $\int_{\partial G} \langle F, n \rangle = 0$.

**4a4 Exercise.** $\text{div}(fF) = f \text{div} F + \langle \nabla f, F \rangle$ whenever $f \in C^1(G)$ and $F \in C^1(G \to \mathbb{R}^N)$.
Prove it.

Thus, the divergence theorem, applied to $fF$ when $f \in C^1(G)$ with bounded $\nabla f$, and $F \in C^1(G \to \mathbb{R}^N)$ with bounded $DF$, gives a kind of integration by parts, similar to (3b12):
\[
(4a5) \int_G \langle \nabla f, F \rangle = \int_{\partial G} f \langle F, n \rangle - \int_G f \text{div} F.
\]
In particular, if $\text{div} F = 0$, then $\int_G \langle \nabla f, F \rangle = \int_{\partial G} f \langle F, n \rangle$.

Here is a useful special case. We mean by a radial function a function of the form $f : x \mapsto g(|x|)$ where $g \in C^1(0, \infty)$, and by a radial vector field $F : x \mapsto g(|x|)x$. Clearly, $f \in C^1(\mathbb{R}^N \setminus \{0\})$ and $F \in C^1(\mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N)$.

**4a6 Exercise.** (a) If $f(x) = g(|x|)$, then $\nabla f(x) = \frac{g'(|x|)}{|x|} x$;

(b) if $F(x) = g(|x|)x$, then $\text{div} F(x) = |x|g'(|x|) + Ng(|x|)$;

(c) if $F(x) = g(|x|)x$, then the (outward) flux of $F$ through the boundary of the ball $\{x : |x| < r\}$ is $cr^N g(r)$, where $c = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the area of the unit sphere.
Prove it.\(^2\)

---

\(^1\)Divergence is often explained in terms of sources and sinks (of a moving matter). But be careful: the flux of a velocity field is the amount (per unit time) as long as “amount” means “volume”. If by “amount” you mean “mass”, then you need the vector field of momentum, not velocity; multiply the velocity by the density of the matter. However, the problem disappears if the density is constant (which often holds for fluids).

\(^2\)Hint: (b) use (a) and 4a4.
Taking $G = \{ x : a < |x| < b \}$ and $F(x) = g(|x|)x$, we see that $\int_G \text{div} F = \int_a^b cr^{N-1} (rg'(r) + Ng(r)) \, dr$ by (4a6b) and (generalized) 3c8; and on the other hand, $\int_{\partial G} \langle F, n \rangle = cr^N g(r)|^b_a$ by (4a6c). Well, $\frac{d}{dr}(r^N g(r)) = r^{N-1} (rg'(r) + Ng(r))$, as it should be according to (4a2).

Zero gradient is trivial, but zero divergence is not. For a radial vector field, zero divergence implies that $r^N g(r)$ does not depend on $r$, that is, $g(r) = \text{const} \cdot \frac{1}{r^N}$ (and indeed, in this case $rg'(r) + Ng(r) = 0$);

$$F(x) = \frac{\text{const}}{|x|^N} x; \quad \text{div} F(x) = 0 \text{ for } x \neq 0;$$ (4a7)

$$\int_{\partial G} \langle F, n \rangle = 0 \text{ when } G \neq 0;$$

note that the latter equality fails for a ball. The flux through a sphere is

$$\int_{|x|=r} \langle F, n \rangle = \text{const} \cdot \int_{|x|=1} 1 = \text{const} \cdot \frac{2\pi^{N/2}}{\Gamma(N/2)}$$ (4a8)

where ‘const’ is as in (4a7). The same holds for arbitrary smooth set $G \ni 0$:

$$\int_{\partial G} \langle F, n \rangle = \text{const} \cdot \frac{2\pi^{N/2}}{\Gamma(N/2)}.$$ (4a9)

Proof: we take $\varepsilon > 0$ such that $\{ x : |x| \leq \varepsilon \} \subset G$; the set $G_\varepsilon = \{ x \in G : |x| > \varepsilon \}$ is smooth; by (4a7), $\int_{\partial G_\varepsilon} \langle F, n \rangle = 0$; and $\partial G_\varepsilon = \partial G \cup \{ x : |x| = \varepsilon \}$.

### 4b Piecewise smooth case

We want to apply the divergence theorem 4a3 to the open cube $G = (0,1)^N$, but for now we cannot, since the boundary $\partial G$ is not a manifold. Rather, $\partial G$ consists of $2N$ disjoint cubes of dimension $n = N - 1$ (“hyperfaces”) and a finite number$^1$ of cubes of dimensions $0, 1, \ldots, n - 1$.

For example, $\{ 1 \} \times (0,1)^n$ is a hyperface.

Each hyperface is an $n$-manifold, and has exactly two orientations. Also, the outward unit normal vector $n_x$ is well-defined at every point $x$ of a hyperface.

For example, $n_x = e_1$ for every $x \in \{ 1 \} \times (0,1)^n$.

For a function $f$ on $\partial G$ we define $\int_{\partial G} f$ as the sum of integrals over the $2N$ hyperfaces; that is,

$$\int_{\partial G} f = \sum_{i=1}^{N} \sum_{x_i=0,1} \int_{(0,1)^n} f(x_1, \ldots, x_N) \prod_{j \neq i} dx_j,$$ (4b1)

$^1$In fact, $3^N - 1 - 2N$. 
provided that these integrals are well-defined, of course.

For a vector field \( F \in C(\partial G \to \mathbb{R}^N) \) we define the flux of \( F \) through \( \partial G \) as 
\[
\int_{\partial G} \langle F, n \rangle.
\]

Note that
\[
\int_{\partial G} \langle F, n \rangle = \sum_{i=1}^{N} \sum_{x_i=0,1} (2x_i - 1) \int_{(0,1)^n} F_i(x_1, \ldots, x_N) \prod_{j \neq i} dx_j.
\]

It is surprisingly easy to prove the divergence theorem for the cube. (Just from scratch; no need to use 4a3, nor 3e3.)

**4b3 Proposition** (divergence theorem for cube). Let \( F \in C^1((0,1)^N \to \mathbb{R}^N) \), with \( DF \) bounded. Then the integral of div \( F \) over \((0,1)^N\) is equal to the (outward) flux of \( F \) through the boundary.

(As before, boundedness of \( DF \) ensures that \( F \) extends to \([0,1]^N\) by continuity; recall 3b6.)

**Proof.**
\[
\int_0^1 D_1F_1(x_1, \ldots, x_N) \, dx_1 = F_1(1, x_2, \ldots, x_N) - F_1(0, x_2, \ldots, x_N) = \sum_{x_1=0,1} (2x_1 - 1) F_1(x_1, \ldots, x_N);
\]
\[
\int_{(0,1)^N} \cdots \int D_1F_1 = \sum_{x_1=0,1} (2x_1 - 1) \int_{(0,1)^n} F_1(x_1, \ldots, x_N) \, dx_2 \ldots dx_N;
\]
similarly, for each \( i = 1, \ldots, N \),
\[
\int_{(0,1)^N} \cdots \int D_iF_i = \sum_{x_i=0,1} (2x_i - 1) \int_{(0,1)^n} F_i \prod_{j \neq i} dx_j;
\]
it remains to sum over \( i \).

The same holds for every box, of course.

A box is only one example of a bounded regular open set \( G \subset \mathbb{R}^N \) such that \( \partial G \) is not an \( n \)-manifold and still, the divergence theorem holds as 
\[
\int_G \text{div} \, F = \int_{\partial G \setminus Z} \langle F, n \rangle
\]
for some closed set \( Z \subset \partial G \) such that \( \partial G \setminus Z \) is an \( n \)-manifold of finite \( n \)-dimensional volume. For the cube (or box), \( \partial G \setminus Z \) is the union of the \( 2N \) hyperfaces, and \( Z \) is the union of cubes (or boxes) of smaller (than \( N - 1 \)) dimensions.
4b4 Definition. We say\footnote{Not a standard terminology.} that the divergence theorem holds for $G$ and $\partial G \setminus Z$, if 
$G \subset \mathbb{R}^N$ is a bounded regular open set, 
$Z \subset \partial G$ is a closed set, 
$\partial G \setminus Z$ is an $n$-manifold of finite $n$-dimensional volume, and 
$$\int_G \text{div} F = \int_{\partial G \setminus Z} \langle F, \mathbf{n} \rangle$$
for all $F \in C(\overline{G} \to \mathbb{R}^N)$ such that $F|_G \in C^1(G \to \mathbb{R}^N)$ and $DF$ is bounded on $G$.

4b5 Exercise (product). Let $G_1 \subset \mathbb{R}^{N_1}$, $Z_1 \subset \partial G_1$, and $G_2 \subset \mathbb{R}^{N_2}$, $Z_2 \subset \partial G_2$. If the divergence theorem holds for $G_1$, $\partial G_1 \setminus Z_1$ and for $G_2$, $\partial G_2 \setminus Z_2$, then it holds for $G, \partial G \setminus Z$ where $G = G_1 \times G_2 \subset \mathbb{R}^{N_1+N_2}$ and 
$\partial G \setminus Z = ((\partial G_1 \setminus Z_1) \times G_2) \sqcup (G_1 \times (\partial G_2 \setminus Z_2))$.
Prove it.\footnote{Hint: $\text{div} F = (D_1 F_1 + \cdots + D_{N_1} F_{N_1}) + (D_{N_1+1} F_{N_1+1} + \cdots + D_{N_1+N_2} F_{N_1+N_2})$.}

An $N$-box is the product of $N$ intervals, of course. Also, a cylinder \{(x, y, z) : x^2 + y^2 < r^2, 0 < z < a\} is the product of a disk and an interval.

4c Divergence of gradient: Laplacian
Some (but not all) vector fields are gradients of scalar fields.

4c1 Definition. (a) The Laplacian $\Delta f$ of a function $f \in C^2(G)$ on an open set $G \subset \mathbb{R}^n$ is
$$\Delta f = \text{div} \nabla f .$$
(b) $f$ is harmonic, if $\Delta f = 0$.

We have $\nabla f = (D_1 f, \ldots, D_n f)$, thus, $\text{div} \nabla f = D_1 (D_1 f) + \cdots + D_n (D_n f)$; in this sense,
$$\Delta = D_1^2 + \cdots + D_n^2 = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} ,$$
the so-called Laplace operator, or Laplacian.

Any $n$-dimensional Euclidean space may be used instead of $\mathbb{R}^n$. Indeed, the gradient is well-defined in such space, and the divergence is well-defined even without Euclidean metric.

The divergence theorem\footnote{Not a standard terminology.} gives, for a smooth $G$, the so-called first Green formula

$$\int_G \Delta f = \int_{\partial G} \langle \nabla f, \mathbf{n} \rangle = \int_{\partial G} D_n f ,$$
where \((D_n f)(x) = (D_{n_x} f)_x\) is the directional derivative of \(f\) at \(x\) in the normal direction \(n_x\). Here \(f \in C^2(G)\), with bounded second derivatives.

Here is another instance of integration by parts. Let \(u \in C^1(G)\), with bounded gradient, and \(v \in C^2(G)\), with bounded second derivatives.\footnote{In fact, they are \(\text{Re} (x + iy)^m\), \(\text{Im} (x + iy)^m\) and their linear combinations.} Applying (4a5) to \(f = u\) and \(F = \nabla v\) we get
\[
\int_G \langle \nabla u, \nabla v \rangle = \int_{\partial G} u D_n v - \int_G u \Delta v,
\]
that is,\footnote{Rewriting (4c4) as}
\[
(4c3) \quad \int_G (u \Delta v + \langle \nabla u, \nabla v \rangle) = \int_{\partial G} \langle u \nabla v, n \rangle = \int G u \Delta v ,
\]
the \textit{second Green formula}. It follows that
\[
(4c4) \quad \int_G (u \Delta v - v \Delta u) = \int_{\partial G} (u D_n v - v D_n u) ,
\]
the \textit{third Green formula}; here \(u, v \in C^2(G)\), with bounded second derivatives. In particular,
\[
\int_{\partial G} u D_n v = \int_{\partial G} v D_n u \quad \text{for harmonic} \ u, v.
\]

Rewriting (4c4) as
\[
(4c5) \quad \int_G u \Delta v = \int_G v \Delta u - \int_{\partial G} v D_n u + \int_{\partial G} (D_n v) u
\]
we may say that really \(\int (u \mathbb{I}_G) \Delta v = \int v \Delta (u \mathbb{I}_G)\) where \(\Delta (u \mathbb{I}_G)\) consists of the usual Laplacian \(\Delta u \mathbb{I}_G\) sitting on \(G\) and the singular Laplacian sitting on \(\partial G\), of two terms, so-called single layer \((-D_n u)\) and double layer \(uD_n\).

Why two layers? Because the Laplacian (unlike gradient and divergence) involves second derivatives.

4c6 Exercise. Consider homogeneous polynomials on \(\mathbb{R}^2\):
\[
f(x, y) = \sum_{k=0}^m c_k x^k y^{m-k}.
\]
For \(m = 1, 2\) and 3 find all harmonic functions among these polynomials.\footnote{1}  

4c7 Exercise. On \(\mathbb{R}^2\),
\[(a) \quad \text{a function of the form}
\]
\[
f(x, y) = \sum_{k=1}^m c_k e^{a_k x + b_k y} \quad (a_k, b_k, c_k \in \mathbb{R})
\]
is harmonic only if it is constant;
(b) a function of the form
\[ f(x, y) = e^{ax} \cos by \]
is harmonic if and only if \( |a| = |b| \).
Prove it.

Now, what about a radial harmonic function? We seek a radial function \( f \) such that \( \nabla f \) is of zero divergence, that is, \( \nabla f(x) = \frac{\text{const}}{r^{N-2}} \) (recall (4a7)). By (4a6(a)), \( f(x) = g(|x|) \) where \( \frac{g'(r)}{r} = \frac{\text{const}}{r^{N-2}} \); thus, \( g(r) = \frac{\text{const}_1}{r^{N-2}} + \text{const}_2 \) for \( N \neq 2 \). We choose
\begin{equation}
(4c8) \quad f(x) = \frac{1}{|x|^{N-2}}; \quad \Delta f(x) = 0 \quad \text{for } x \neq 0.
\end{equation}
(This works also for \( N = 1 \): \( f(x) = |x| \) is harmonic on \( \mathbb{R} \setminus \{0\} \).) But for \( N = 2 \) we get \( g'(r) = \frac{\text{const}_1}{r} \); \( g(r) = \text{const}_1 \cdot \log r + \text{const}_2 \); we choose
\begin{equation}
(4c9) \quad f(x) = -\log |x| \log \frac{1}{|x|}; \quad \Delta f(x) = 0 \quad \text{for } x \neq 0.
\end{equation}
The flux of \( \nabla f \) through a sphere is
\[ \int_{|x|=r} \nabla_n f = \begin{cases} -(N-2) \frac{2\pi^{N/2}}{\Gamma(N/2)} & \text{for } N \neq 2, \\ -2\pi & \text{for } N = 2; \end{cases} \]
and, similarly to (4a9), the same holds for every smooth set \( G \ni 0 \).

4d Laplacian at a singular point

The function \( g(x) = 1/|x|^{N-2} \) is harmonic on \( \mathbb{R}^N \setminus \{0\} \), thus, for every \( f \in C^2 \) compactly supported within \( \mathbb{R}^N \setminus \{0\} \),
\[ \int g \Delta f = \int f \Delta g = 0. \]
It appears that for \( f \in C^2(\mathbb{R}^N) \) with a compact support,
\[ \int g \Delta f = \text{const} \cdot f(0); \]
in this sense \( g \) has a kind of singular Laplacian at the origin.

\(^1\)That is, \( f(x, y) = \text{Re} (e^{ax+iy}) \).
\(^2\text{const} = -(N-2)\text{const}_1 = -(N-2) \) for \( N \neq 2 \), and \( \text{const} = \text{const}_1 = -1 \) for \( N = 2 \).
4d1 Lemma.

\[ \int_{\mathbb{R}^N} \frac{\Delta f(x)}{|x|^{N-2}} \, dx = -(N - 2) \frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) \]

for every \( N > 2 \) and \( f \in C^2(\mathbb{R}^N) \) with a compact support.

This improper integral converges, since \( 1/|x|^{N-2} \) is improperly integrable near 0. The coefficient \( \frac{2\pi^{N/2}}{\Gamma(N/2)} \) is the \((N - 1)\)-dimensional volume of the unit sphere (recall (3c9)).

**Proof.** For arbitrary \( \varepsilon > 0 \) we consider the function \( g_\varepsilon(x) = \frac{1}{\max(|x|, \varepsilon)^{N-2}} \), and \( g(x) = 1/|x|^{N-2} \). Clearly, \( \int |g_\varepsilon - g| \to 0 \) (as \( \varepsilon \to 0 \)), and \( \int |g_\varepsilon - g||\Delta f| \to 0 \), thus, \( \int g_\varepsilon \Delta f \to \int g \Delta f \). We take \( R \in (0, \infty) \) such that \( f(x) = 0 \) for \( |x| \geq R \), introduce smooth sets \( G_1 = \{ x : |x| < \varepsilon \} \), \( G_2 = \{ x : \varepsilon < |x| < R \} \), and apply (4c4), taking into account that \( \Delta g_\varepsilon = 0 \) on \( G_1 \) and \( G_2 \):

\[ \int g_\varepsilon \Delta f = \left( \int_{G_1} + \int_{G_2} \right) g_\varepsilon \Delta f = \left( \int_{\partial G_1} + \int_{\partial G_2} \right) \left( g_\varepsilon D_n f - f D_n g_\varepsilon \right) ; \]

however, these \( D_n \) must be interpreted differently under \( \int_{\partial G_1} \) and \( \int_{\partial G_2} \):

\[ \int_{\partial G_1} g_\varepsilon D_n f = \int_{|x| = \varepsilon} \frac{1}{\varepsilon^{N-2}} D_n f ; \]

\[ \int_{\partial G_2} g_\varepsilon D_n f = \int_{|x| = \varepsilon} \frac{1}{\varepsilon^{N-2}} D_n f \]

where \( n \) is the outward normal of \( G_1 \) and inward normal of \( G_2 \); these two summands cancel each other. Further, \( \int_{\partial G_1} f D_n g_\varepsilon = \int_{|x| = \varepsilon} f \cdot 0 = 0 \) since \( g_\varepsilon \) is constant on \( G_1 \); and

\[ \int_{\partial G_2} f D_n g_\varepsilon = \int_{|x| = \varepsilon} f \cdot \frac{N - 2}{\varepsilon^{N-1}} , \]

since \( g_\varepsilon(x) = 1/|x|^{N-2} \) on \( G_2 \), and \( f(x) = 0 \) when \( |x| = R \). Finally,

\[ \int g_\varepsilon \Delta f = -(N - 2) \frac{1}{\varepsilon^{N-1}} \int_{|x| = \varepsilon} f = -(N - 2) \frac{2\pi^{N/2}}{\Gamma(N/2)} f_\varepsilon , \]

where \( f_\varepsilon \) is the mean value of \( f \) on the \( \varepsilon \)-sphere. By continuity, \( f_\varepsilon \to f(0) \) as \( \varepsilon \to 0 \); and, as we know, \( \int g_\varepsilon \Delta f \to \int g \Delta f \). \( \Box \)
4d2 Remark. For \( N = 2 \) the situation is similar:

\[
\int_{\mathbb{R}^2} \Delta f(x) \log \frac{1}{|x|} \, dx = -2\pi f(0)
\]

for every compactly supported \( f \in C^2(\mathbb{R}^2) \).

When the boundary consists of a hypersurface and an isolated point, we get a combination of (4c5) and 4d1: a singular point and two layers.

4d3 Remark. Let \( G \subset \mathbb{R}^N \) be a smooth set, \( f \in C^2(G) \) with bounded second derivatives, and \( 0 \in G \). Then

\[
\int_G \frac{\Delta f(x)}{|x|^{N-2}} \, dx = -(N-2) \frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) - \int_{\partial G} \left( x \mapsto f(x) D_n \frac{1}{|x|^{N-2}} \right) + \int_{\partial G} \left( x \mapsto (D_n f(x)) \frac{1}{|x|^{N-2}} \right).
\]

The proof is very close to that of 4d1. The case \( N = 2 \) is similar to 4d2. The case \( G = \{ x : |x| < R \} \) is especially interesting. Here \( \partial G = \{ x : |x| = R \} \); on \( \partial G \),

\[
\frac{1}{|x|^{N-2}} = \frac{1}{R^{N-2}} \quad \text{and} \quad D_n \frac{1}{|x|^{N-2}} = -\frac{N-2}{R^{N-1}};
\]

thus,

\[
\int_{|x|<R} \frac{\Delta f(x)}{|x|^{N-2}} \, dx = -(N-2) \frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) + \frac{N-2}{R^{N-1}} \int_{|x|=R} f + \frac{1}{R^{N-2}} \int_{|x|=R} D_n f.
\]

Taking into account that \( \int_{|x|=R} D_n f = \int_{|x|<R} \Delta f \) by 4c2, we get

\[
(N-2) \frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) = -\int_{|x|<R} \left( \frac{1}{|x|^{N-2}} - \frac{1}{R^{N-2}} \right) \Delta f(x) \, dx + \frac{N-2}{R^{N-1}} \int_{|x|=R} f
\]

for \( N > 2 \); and similarly,

\[
2\pi f(0) = -\int_{|x|<R} \left( \log R - \log |x| \right) \Delta f(x) \, dx + \frac{1}{R} \int_{|x|=R} f
\]

for \( N = 2 \). In particular, for a harmonic \( f \),

\[
f(0) = \frac{\Gamma(N/2)}{2\pi^{N/2}} \frac{1}{R^{N-1}} \int_{|x|=R} f = \frac{\int_{|x|=R} f}{\int_{|x|=R} 1}
\]

for \( N \geq 2 \); the following result is thus proved (and holds also for \( N = 1 \), trivially).
4d4 Proposition (Mean value property). For every harmonic function on a ball, with bounded second derivatives, its value at the center of the ball is equal to its mean value on the boundary of the ball.\footnote{In fact, the mean value property is also sufficient for harmonicity, even if differentiability is not assumed.}

4d5 Remark. Now it is easy to understand why harmonic functions occur in physics (“the stationary heat equation”). Consider a homogeneous material solid body (in three dimensions). Fix the temperature on its boundary, and let the heat flow until a stationary state is reached. Then the temperature in the interior is a harmonic function (with the given boundary conditions).

4d6 Remark. Can the mean value property be generalized to a non-spherical boundary? We leave this question to more special courses (PDE, potential theory). But here is the idea. In 4d3 we may replace
\[\int_G \frac{1}{|x|^{N-2}} \Delta f(x) \, dx\]
with
\[\int_G \left( \frac{1}{|x|^{N-2}} + g(x) \right) \Delta f(x) \, dx\]
where $g$ is a harmonic function satisfying
\[\frac{1}{|x|^{N-2}} + g(x) = 0\]
for all $x \in \partial G$ (if we are lucky to have such $g$). Then the double layer integral $\int_{\partial G} (D_n v) u$ in (4c5), and the corresponding term in 4d3, disappears, and we get
\[(N - 2) \frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) = \int_{\partial G} \left( x \mapsto f(x) D_n \left( \frac{1}{|x|^{N-2}} + g(x) \right) \right).\]

4d7 Exercise (Maximum principle for harmonic functions).

Let $u$ be a harmonic function on a connected open set $G \subset \mathbb{R}^N$. If $\sup_{x \in G} u(x) = u(x_0)$ for some $x_0 \in G$ then $u$ is constant.

Prove it.\footnote{Hint: the set $\{x_0 : u(x_0) = \sup_{x \in G} u(x)\}$ is both open and closed in $G$.}

It appears that
\[(4d8) \quad \Delta f(x) = 2N \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left( \text{mean of } f \text{ on } \{y : |y - x| = \varepsilon\} \right) - f(x)\]

4d9 Exercise. (a) Prove that, for $N > 2$,
\[\frac{1}{R^2} \int_{|x| < R} \left( \frac{1}{|x|^{N-2}} - \frac{1}{R^{N-2}} \right) \, dx\]
does not depend on $R$; and for $N = 2$, $\frac{1}{R^2} \int_{|x| < R} (\log R - \log |x|) \, dx$ does not depend on $R$. (No need to calculate these integrals.)\footnote{Hint: change of variable.}
(b) For \( f \) of class \( C^2 \) near the origin, prove that the mean value of \( f \) on \( \{ x : |x| = \varepsilon \} \) is \( f(0) + c_2 \varepsilon^2 \Delta f(0) + o(\varepsilon^2) \) as \( \varepsilon \to 0 \), for some \( c_2, c_3, \cdots \in \mathbb{R} \) (not dependent on \( f \)).

(c) Applying (b) to \( f(x) = |x|^2 \), find \( c_2, c_3, \cdots \) and prove (4d8).

4d10 Exercise. (a) For every \( f \) integrable (properly) on \( \{ x : |x| < R \} \),

\[
\frac{\int_{|z|<R} f}{\int_{|z|<R} 1} = \int_0^R \frac{\int_{|z|=r} f}{\int_{|z|=r} 1} \frac{dr}{RN}.
\]

(b) For every bounded harmonic function on a ball, its value at the center of the ball is equal to its mean value on the ball.

Prove it.\(^1\)

4d11 Proposition. (Liouville’s theorem for harmonic functions)

Every harmonic function \( \mathbb{R}^N \to [0, \infty) \) is constant.

**Proof.** For arbitrary \( x, y \in \mathbb{R}^N \) and \( R > 0 \) we have

\[
f(x) = \frac{\int_{|z-x|<R} f(z) \, dz}{\int_{|z-x|<R} 1} \leq \frac{\int_{|z-y|<R+|x-y|} f(z) \, dz}{\int_{|z-x|<R} 1} = \left( \frac{R + |x - y|}{R} \right)^N \int_{|z-y|<R+|x-y|} f(z) \, dz = \left( \frac{R + |x - y|}{R} \right)^N f(y),
\]

since the \( R \)-neighborhood of \( x \) is contained in the \( (R + |x - y|) \)-neighborhood of \( y \). In the limit \( R \to \infty \) we get \( f(x) \leq f(y) \); similarly, \( f(y) \leq f(x) \). \( \square \)

4e Differential forms of order \( N - 1 \)

It is easy to generalize the flux, defined by 4a1, as follow.

4e1 Definition. Let \( M \subset \mathbb{R}^N \) be an \( n \)-manifold, \(^2\) \( F : M \to \mathbb{R}^N \) a mapping continuous almost everywhere, and \( n : M \to \mathbb{R}^N \) a continuous mapping such that \( n_x \) is a unit normal vector to \( M \) at \( x \), for each \( x \in M \). The flux of (the vector field) \( F \) through (the hypersurface) \( M \) in the direction \( n \) is

\[
\int_M \langle F, n \rangle.
\]

(The integral is treated as improper, and may converge or diverge.)

---

\(^1\)Hint: (a) recall 13c8.

\(^2\)Necessarily orientable; see 4e9.
It is not easy to calculate this integral, even if $M$ is single-chart; the formula is complicated,
\[
\int_M \langle F, n \rangle = \int_G \langle F(\psi(u)), n_\psi(u) \rangle \sqrt{\det(\langle (D_i\psi)_u, (D_j\psi)_u \rangle)_{i,j}} \, du,
\]
and still, $n_x$ should be calculated somehow. Fortunately, there is a better formula:\footnote{A wonder: the volume form of $M$ is not needed; the volume form of $\mathbb{R}^N$ (the determinant) is used instead. Why so? Since the flux is the volume of fluid flowing through the surface (per unit time), as was noted in \ref{4a}.}
\[
\int_M \langle F, n \rangle = \pm \int_G \det(F(\psi(u)), (D_1\psi)_u, \ldots, (D_n\psi)_u) \, du
\]
(and the sign $\pm$ will be clarified soon). That is, $\int_M \langle F, n \rangle = \pm \int_M \omega$, where $\omega$ is the $n$-form defined by $\omega(x, h_1, \ldots, h_n) = \det(F(x), h_1, \ldots, h_n)$. We have to understand better this relation between vector fields and differential forms.

Recall two types of integral over an $n$-manifold:
* of an $n$-form $\omega$: $\int_{(M,\mathcal{O})}\omega$, defined by (2c2) and (2d4);
* of a function $f$, $\int_M f$, defined by (2d8) and (2d9);
they are related by
\[
\int_M f = \int_{(M,\mathcal{O})} f\mu_{(M,\mathcal{O})},
\]
where $\mu_{(M,\mathcal{O})}$ is the volume form; that is, $\int_M f = \int_{(M,\mathcal{O})} \omega$ where $\omega = f\mu_{(M,\mathcal{O})}$. Interestingly, every $n$-form $\omega$ on an orientable $n$-manifold $M \subset \mathbb{R}^N$ is $f\mu_{(M,\mathcal{O})}$ for some $f \in C(M)$. This is a consequence of the one-dimensionality\footnote{Recall Sect. 1e and 2c.} of the space of all antisymmetric multilinear $n$-forms on the tangent space $T_xM$. We have $f(x) = \omega(x, e_1, \ldots, e_n)$ for some (therefore, every) orthonormal basis $(e_1, \ldots, e_n)$ of $T_xM$ that conforms to $\mathcal{O}_x$. But if $\omega$ is defined on the whole $\mathbb{R}^N$ (not just on $M$), it does not lead to a function $f$ on the whole $\mathbb{R}^N$; indeed, in order to find $f(x)$ we need not just $x$ but also $T_xM$ (and its orientation).

The case $n = N$ is simple: every $N$-form $\omega$ on $\mathbb{R}^N$ (or on an open subset of $\mathbb{R}^N$) is $f \det$ (for some continuous $f$); here “det” denotes the volume form on $\mathbb{R}^N$; that is,
\[
\omega(x, h_1, \ldots, h_N) = f(x) \det(h_1, \ldots, h_N);
\]
\[
f(x) = \omega(x, e_1, \ldots, e_N).
\]
Note that for every open $U \subset \mathbb{R}^N$,

\[(4e4) \quad \int_U f \, \det = \int_U f(x) \, dx; \quad \int_U \det = v(U).\]

We turn to the case $n = N - 1$.

The space of all antisymmetric multilinear $n$-forms $L$ on $\mathbb{R}^N$ is of dimension $\binom{N}{n} = N$. Here is a useful linear one-to-one correspondence between such $L$ and vectors $h \in \mathbb{R}^N$:

$$\forall h_1, \ldots, h_n \quad L(h_1, \ldots, h_n) = \det(h, h_1, \ldots, h_n).$$

Introducing the cross-product $h_1 \times \cdots \times h_n$ by\footnote{For $N = 3$ the cross-product is a binary operation, but for $N > 3$ it is not. In fact, it is possible to define the corresponding associative binary operation (the so-called exterior product, or wedge product), not on vectors but on the so-called multivectors, see “Multivector” and “Exterior algebra” in Wikipedia.}

\[(4e5) \quad \forall h \quad \langle h, h_1 \times \cdots \times h_n \rangle = \det(h, h_1, \ldots, h_n)
\]

(it is a vector orthogonal to $h_1, \ldots, h_n$), we get

$$L(h_1, \ldots, h_n) = \langle h, h_1 \times \cdots \times h_n \rangle.$$

Doing so at every point, we get a linear one-to-one correspondence between $n$-forms $\omega$ on $\mathbb{R}^N$ and (continuous) vector fields $F$ on $\mathbb{R}^N$:

\[(4e6) \quad \omega(x, h_1, \ldots, h_n) = \langle F(x), h_1 \times \cdots \times h_n \rangle = \det(F(x), h_1, \ldots, h_n).
\]

Similarly, $(n - 1)$-forms $\omega$ on an oriented $n$-dimensional manifold $(M, \mathcal{O})$ in $\mathbb{R}^N$ (not just $N - n = 1$) are in a linear one-to-one correspondence with tangent vector fields $F$ on $M$, that is, $F \in C(M \to \mathbb{R}^N)$ such that $\forall x \in M \quad F(x) \in T_x M$.

Let $M \subset \mathbb{R}^N$ be an orientable $n$-manifold, $\omega$ and $F$ as in (4e6). We know that $\omega|_M = f \mu_{(M, \mathcal{O})}$ for some $f$. How is $f$ related to $F$? Given $x \in M$, we take an orthonormal basis $(e_1, \ldots, e_n)$ of $T_x M$, note that $e_1 \times \cdots \times e_n = \mathbf{n}_x$ is a unit normal vector to $M$ at $x$, and

$$\langle F(x), \mathbf{n}_x \rangle = \langle F(x), e_1 \times \cdots \times e_n \rangle = \omega(x, e_1, \ldots, e_n) = f(x) \mu_{(M, \mathcal{O})}(x, e_1, \ldots, e_n) = \pm f(x).$$

In order to get “$+$” rather than “$\pm$” we need a coordination between the orientation $\mathcal{O}$ and the normal vector $\mathbf{n}_x$. Let the basis $(e_1, \ldots, e_n)$ of $T_x M$
conform to the orientation $O_x$ (of $M$ at $x$, or equivalently, of $T_x M$, recall Sect. 2b), then $\mu_{(M,O)}(x,e_1,\ldots,e_n) = +1$. The two unit normal vectors being $\pm e_1 \times \cdots \times e_n$, we say that $n_x = e_1 \times \cdots \times e_n$ conforms to the given orientation, and get

$$\langle F(x), n_x \rangle = f(x); \quad \omega|_M = \langle F, n \rangle \mu_{(M,O)}.$$  

Integrating this over $M$, we get nothing but the flux! Recall 4e1: the flux of $F$ through $M$ is $\int_M \langle F, n \rangle$, that is, $\int_{(M,O)} \langle F, n \rangle \mu_{(M,O)} = \int_{(M,O)} \omega|_M = \int_{(M,O)} \omega$. We get (4e2), and moreover,

$$(4e7) \quad \int_M \langle F, n \rangle = \int_{(M,O)} \omega$$

for $\omega$ of (4e6) and $O$ conforming to $n$. In particular, when $M$ is single-chart, we have

$$(4e8) \quad \int_M \langle F, n \rangle = \int_G \det \left( F(\psi(u)), (D_1 \psi)_u, \ldots, (D_n \psi)_u \right) du$$

provided that $\det(n, D_1 \psi, \ldots, D_n \psi) > 0$. Necessarily, $D_1 \psi \times \cdots \times D_n \psi = cn$ for some $c \neq 0$ (since both vectors are orthogonal to the tangent space); the sign of $c$ is the sign in (4e2).

We summarize the situation with the sign.

$$(n = N - 1)$$

4e9 Remark. For an $n$-dimensional manifold $M \subset \mathbb{R}^N$, the two orientations $O_x$ at a given point $x \in M$ correspond naturally$^2$ to the two unit normal vectors $n_x$ to $M$ at $x$. Namely, for some (therefore, every) orthonormal basis $e_1, \ldots, e_n$ of $T_x M$ that conforms to $O_x$,

(a) $\det(n_x, e_1, \ldots, e_n) = +1$;

or, equivalently,

(b) $e_1 \times \cdots \times e_n = n_x$.

Alternatively (and equivalently), for arbitrary (not just orthonormal) basis,

(a') $\det(n_x, e_1, \ldots, e_n) > 0$;

(b') $e_1 \times \cdots \times e_n = cn_x$ for some $c > 0$.

Given a chart $(G, \psi)$ of $M$ around $x$ that conforms to $O_x$, we may take $e_i = (D_i \psi)_{\psi^{-1}(x)}$.

Orientations $(O_x)_{x \in M}$ of $M$ correspond naturally to continuous mappings $M \ni x \mapsto n_x \in \mathbb{R}^N$ such that for every $x \in M$, $n_x$ is a unit normal vector to $M$ at $x$. Thus, such mappings exist if and only if $M$ is orientable (and in this case, there are exactly two of them, provided that $M$ is connected).

---

$^1$Not unexpectedly, in order to find $f(x)$ we need not just $x$ but also $n_x$.

$^2$Using the orientation of $\mathbb{R}^N$ given by the determinant; the other orientation of $\mathbb{R}^N$ leads to the other correspondence.
We turn to a smooth set \( U \subset \mathbb{R}^N \). Its boundary \( \partial U \) is a hypersurface; the outward normal vector leads, according to 4e9, to an orientation of \( \partial U \). In such cases we always use this orientation. Given \( F \in C^1(U \to \mathbb{R}^N) \) with \( DF \) bounded, we may rewrite the divergence theorem 4a3, 
\[
\int_U \text{div } F = \int_{\partial U} \langle F, n \rangle,
\]
as
\[
\int_U (\text{div } F) \det = \int_{\partial U} \omega,
\]
where \( \omega \) corresponds to \( F \) according to (4e6). Taking into account that every \( n \)-form of class \( C^1 \) corresponds to some vector field, we conclude.

4e10 Proposition. For every \( n \)-form \( \omega \) of class \( C^1 \) on \( \mathbb{R}^N \) there exists an \( N \)-form \( \omega' \) on \( \mathbb{R}^N \) such that for every smooth set \( U \subset \mathbb{R}^N \),
\[
\int_{\partial U} \omega = \int_U \omega'.
\]

4e11 Remark. The same holds in the piecewise smooth case: \( \int_{\partial U \setminus Z} \omega = \int_U \omega' \) provided that the divergence theorem holds for \( U \) and \( \partial U \setminus Z \).

4e12 Example. On \( \mathbb{R}^2 \) consider a vector field \( F : (x, y) \mapsto (F_1(x, y), F_2(x, y)) \) and a curve (1-manifold) covered by a single chart \( \psi : (a, b) \to \mathbb{R}^2 \), \( \psi(t) = (\psi_1(t), \psi_2(t)) \).

Using the complicated formula,
\[
n_{\psi(t)} = \frac{1}{\sqrt{\psi_1'(t) + \psi_2'(t)}} \begin{pmatrix} \psi_2'(t) \\ -\psi_1'(t) \end{pmatrix}; \quad J_{\psi}(t) = \sqrt{\psi_1'^2(t) + \psi_2'^2(t)};
\]
\[
\langle F(\psi(t)), n_{\psi(t)} \rangle = \frac{1}{\sqrt{\psi_1'^2(t) + \psi_2'^2(t)}} (F_1 \psi_2' - F_2 \psi_1');
\]
flux \( = \int_a^b \langle F(\psi(t)), n_{\psi(t)} \rangle J_{\psi}(t) \, dt = \int_a^b (F_1 \psi_2' - F_2 \psi_1') \, dt \).

Alternatively, using (4e8),
\[
\det(F(\psi(t)), \psi'(t)) = \begin{vmatrix} F_1 \\ F_2 \end{vmatrix} \psi_1' = F_1 \psi_2' - F_2 \psi_1'; \quad \text{flux} = \int_a^b (F_1 \psi_2' - F_2 \psi_1') \, dt.
\]

4e13 Exercise. Fill in the details in 4e12.

4e14 Example. Continuing 4e12 consider the 1-form \( \omega(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} dx \\ dy \end{pmatrix}) = f_1(x, y) \, dx + f_2(x, y) \, dy \); it corresponds to \( F \) according to (4e6) when
\[
f_1(x, y) \, dx + f_2(x, y) \, dy = \begin{vmatrix} F_1(x, y) \\ F_2(x, y) \end{vmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}, \quad \text{that is}, \quad f_1 = -F_2, \quad f_2 = F_1.
\]
In this case,
\[
\int_M \omega = \int_a^b \omega(\psi(t), \psi'(t)) \, dt = \int_a^b (f_1(\psi(t))\psi'_1(t) + f_2(\psi(t))\psi'_2(t)) \, dt = \int_a^b (-F_2\psi'_1 + F_1\psi'_2) \, dt = \text{flux}.
\]

4e15 Exercise. Fill in the details in 4e14.

4e16 Remark. Less formally, denoting \(\psi_1(t)\) and \(\psi_2(t)\) by just \(x(t)\) and \(y(t)\) we have
\[
\int_M \omega = \int_a^b (f_1(x(t), y(t))x'(t) + f_2(x(t), y(t))y'(t)) \, dt;
\]
naturally, this is called \(\int_M (f_1 \, dx + f_2 \, dy)\).

4e17 Example. Continuing 4e12 and 4e14, we calculate the divergence:
\[
\text{div } F = D_1F_1 + D_2F_2 = D_1f_2 - D_2f_1;
\]
thus,
\[
\omega' = (\text{div } F) \det = (D_1f_2 - D_2f_1) \det;
\]
\[
\int_{\partial U} \omega = \int_U (D_1f_2 - D_2f_1)
\]
for a smooth \(U \subset \mathbb{R}^2\). If \(\partial U\) is covered (except for a single point) with a chart \(\psi : (a, b) \to \mathbb{R}^2\), \(\psi(a+) = \psi(b-)\), then 4e10 gives
\[
\int_{\partial U} (f_1 \, dx + f_2 \, dy) = \int_U (D_1f_2 - D_2f_1).
\]
This is the well-known Green’s theorem; in traditional notation,
\[
\oint_{\partial U} (L \, dx + M \, dy) = \iint_U \left( \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) \, dx \, dy.
\]

4e18 Example. The 1-form \(\omega = -\frac{y}{2} \, dx + \frac{x}{2} \, dy\) on \(\mathbb{R}^2\) (mentioned in Sect. 1d) corresponds to the vector field \(F\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} \frac{x}{2} \\ \frac{y}{2} \end{smallmatrix}\right)\), that is, \(F(x) = \frac{1}{2}x\) for \(x \in \mathbb{R}^2\). Clearly, \(\text{div } F = 1\), thus, \(\omega' = \det\); by 4e10,
\[
\int_{\partial U} \omega = v(U) \quad \text{for every smooth } U \subset \mathbb{R}^2.
\]
**Example.**
The 1-form $\omega = \frac{-y \, dx + x \, dy}{x^2 + y^2}$ on $\mathbb{R}^2 \setminus \{0\}$ (treated in Sect. 1d) corresponds to the vector field $F(\frac{x}{y}) = \frac{1}{x^2 + y^2}(\frac{x}{y})$, that is, $F(x) = \frac{x}{|x|^2}$ for $x \in \mathbb{R}^2 \setminus \{0\}$. By (4a7), $\operatorname{div} F = 0$ on $\mathbb{R}^2 \setminus \{0\}$, thus $\omega' = 0$ on $\mathbb{R}^2 \setminus \{0\}$; by 4e10 $\int_{\partial U} \omega = 0$ for every smooth $U$ such that $\overline{U} \neq 0$. On the other hand, for every smooth $U \ni \emptyset$ we have $\int_{\partial U} \omega = 2\pi$ by (4a9); compare this fact with Sect. 1d.

Similarly, in $\mathbb{R}^3$ the 2-form $\omega$ that corresponds to the vector field $F(x) = \frac{x}{|x|^3}$ satisfies $\int_{\partial U} \omega = 0$ whenever $U \ni \emptyset$, and $\int_{\partial U} \omega = 4\pi$ whenever $U \ni \emptyset$.

**Index**

- cross-product, 77
- divergence theorem, 66
- divergence theorem for cube, 68
- flux, 65, 68, 75
- Green formula first, 69
- second, 70
- third, 70
- harmonic, 69
- heat, 74
- hyperface, 67
- Laplacian, 69
- layer, 70
- Liouville’s theorem, 75
- maximum principle, 74
- mean value property, 74
- normal vector conforms to orientation, 78
- tangent vector field, 77
- $\Delta$, 69
- $h_1 \times \cdots \times h_n$, 77