6 Stokes' theorem

The ultimate form of fundamental theorem of integral calculus.

6a Exterior derivative: definition

First, we consider n-forms and N-forms on $\mathbb{R}^N$ for $n = N - 1$. $n = N - 1$

Recall 4e10: for every n-form ω of class C^1 on $\mathbb{R}^N$ there exists an N-form ω' of class C^0 on $\mathbb{R}^N$ such that for every smooth set $U \subset \mathbb{R}^N$,

$$\int_{\partial U} \omega = \int_U \omega' \,.$$

This is the divergence theorem 4a3 translated into the language of differential forms.

Similarly we may translate 5c19: let $f = \operatorname{div} F$ (generalized divergence); let F correspond to ω according to (4e6), and f correspond to $\omega' = f \cdot \det$ according to (4e3); then, for arbitrary smooth U, by (4e7), $\int_{\partial U} \langle F, \mathbf{n} \rangle = \int_{\partial U} \omega$, and by (4e4), $\int_U f = \int_U \omega'$; thus, the equality $\int_U f = \int_{\partial U} \langle F, \mathbf{n} \rangle$ of 5c19 becomes $\int_{\partial U} \omega = \int_U \omega'$. We also translate 5c20: (a) such ω' is unique (for given ω); (b) if $\omega \in C^1$ then $\omega' = \operatorname{tr}(DF) \cdot \det$. In general, we call such ω' (if exists) the *generalized exterior derivative* of ω and denote it $d\omega$;

$$\int_{\partial U} \omega = \int_U d\omega \quad \text{for all smooth sets } U \,. \tag{6a1}$$

In terms of the function f that corresponds to ω' according to (4e3) and the vector field F that corresponds to ω according to (4e6) we have

$$\omega' = d\omega \quad \Longleftrightarrow \quad f = \operatorname{div} F \,. \tag{6a2}$$

Here is the translated Th. 5d6: if $\varphi : U \to V$ is of class C^1, and ω is an n-form on V that has the generalized exterior derivative, then the generalized exterior derivative of $\varphi^*\omega$ exists and is equal to $\varphi^*(d\omega)$.

The case $n = N - 1 = 0$ is treated according to 5c21. That is, a smooth set is a finite union of (separated) intervals; and $\int_{\partial U} \omega = \omega(b) - \omega(a)$ when $U = (a, b)$ and ω is a 0-form (just a continuous function). Thus, $d\omega$ exists if and only if $\omega \in C^1(\mathbb{R})$, in which case $d\omega(x, h) = \omega'(x)h$.

Now we turn to $(n-1)$-forms and n-forms on $\mathbb{R}^N$ for $1 \le n \le N$. $1 \le n \le N$

6a3 Definition. Let $U \subset \mathbb{R}^N$ be an open set, $n \in \{1, \dots, N\}$, ω an $(n-1)$-form on U. We say that an n-form ω' on U is the *generalized exterior derivative* of ω, and write $\omega' = d\omega$, if $\varphi^*\omega'$ is the generalized exterior derivative of $\varphi^*\omega$ (as defined by (6a1)) whenever $\varphi : V \to U$ is a map of class C^1, and $V \subset \mathbb{R}^n$ is an open set.

Uniqueness of ω': given $x \in U$ and $h_1, \dots, h_n \in \mathbb{R}^N$, take v and φ such that $\varphi(v) = x$ and $(D\varphi)_v e_1 = h_1$, …, $(D\varphi)_v e_n = h_n$ (try linear φ)…

6a4 Exercise. A function $f \in C^1(U)$, treated as a 0-form, has the generalized exterior derivative $df : (x, h) \mapsto (D_h f)_x$.

Prove it.

6a5 Remark. In the special case $n = N$ Definition 6a3 conforms to (6a1) by the translated 5d6.

6a6 Lemma. If $d\omega$ exists, then $d(\varphi^*\omega)$ exists and is equal to $\varphi^*(d\omega)$.

That is; we assume that ω is an $(n-1)$-form on $\mathbb{R}^N$, $d\omega$ exists, and $\varphi \in C^1(\mathbb{R}^M \to \mathbb{R}^N)$; then $d(\varphi^*\omega) = \varphi^*(d\omega)$.

Proof. Let $\psi \in C^1(\mathbb{R}^n \to \mathbb{R}^M)$; we have to prove that $\psi^*(\varphi^*(d\omega)) = d\psi^*(\varphi^*\omega)$. We have $\varphi \circ \psi \in C^1(\mathbb{R}^n \to \mathbb{R}^N)$; by 6a3, $(\varphi \circ \psi)^*(d\omega) = d(\varphi \circ \psi)^*\omega$. It remains to use the equality $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$. □

$$\begin{array}{ccccc} \omega'_0 & \xleftarrow{\ \psi^*\ } & \omega'_1 & \xleftarrow{\ \varphi^*\ } & \omega'_2 \\ {\scriptstyle d}\uparrow & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} \\ \omega_0 & \xleftarrow{\ \psi^*\ } & \omega_1 & \xleftarrow{\ \varphi^*\ } & \omega_2 \end{array}$$

6a7 Corollary (of 6a6 and 6a5)**.** Let α be an $(n-1)$-form of class C^1 on $\mathbb{R}^n$, $\varphi \in C^1(\mathbb{R}^N \to \mathbb{R}^n)$, and $\omega = \varphi^*\alpha$. Then $d\omega$ exists and is equal to $\varphi^*(d\alpha)$.

6a8 Theorem. Every differential form of class C^1 has the exterior derivative.

The proof is somewhat similar to the proof of Prop. 5c25. First, by 6a7, the claim holds for all forms that are $\varphi^*\alpha$ for some α and φ (as in 6a7). Second, if this claim holds for two forms, then it holds for their sum. It remains to prove that arbitrary form is the sum of some forms that are $\varphi^*\alpha$. This will be done in Sect. 6b.

It may seem impossible to reduce a form on $\mathbb{R}^N$ to forms on $\mathbb{R}^n$, since the former involves functions of N variables, and the latter only of n variables. But recall that in the proof of 5c25 a single vector field $G : (x_1, \dots, x_N) \mapsto (x_1, 0, \dots, 0)$ was enough! True, α is not diverse enough; however, φ is.

6b Exterior derivative: calculation

By 4e17, for $N = 2$, $n = 1$ and $\omega \in C^1$, $\omega\big(\big(\begin{smallmatrix} x \\ y \end{smallmatrix}\big), \big(\begin{smallmatrix} dx \\ dy \end{smallmatrix}\big)\big) = f_1(x,y)\,dx + f_2(x,y)\,dy$ (as in (4e14)), we have

$$d\omega = (D_1 f_2 - D_2 f_1)\det\ .$$

Let us do such calculation for $N = 3$:

$$\begin{vmatrix} F_1 & h_1 & k_1 \\ F_2 & h_2 & k_2 \\ F_3 & h_3 & k_3 \end{vmatrix} = F_1 \begin{vmatrix} h_2 & k_2 \\ h_3 & k_3 \end{vmatrix} - F_2 \begin{vmatrix} h_1 & k_1 \\ h_3 & k_3 \end{vmatrix} + F_3 \begin{vmatrix} h_1 & k_1 \\ h_2 & k_2 \end{vmatrix}\ ;$$

using the traditional notation

$$(dx_i \wedge dx_j)(h,k) = \begin{vmatrix} h_i & k_i \\ h_j & k_j \end{vmatrix}\ ,$$

we get

$$\begin{aligned} \omega = F_1\,dx_2 \wedge dx_3 - F_2\,dx_1 \wedge dx_3 + F_3\,dx_1 \wedge dx_2 = \\ = f_{1,2}\,dx_1 \wedge dx_2 + f_{1,3}\,dx_1 \wedge dx_3 + f_{2,3}\,dx_2 \wedge dx_3\ , \end{aligned}$$

where $f_{1,2} = F_3$, $f_{1,3} = -F_2$, $f_{2,3} = F_1$; $\operatorname{div} F = D_1F_1 + D_2F_2 + D_3F_3 = D_1 f_{2,3} - D_2 f_{1,3} + D_3 f_{1,2}$;

$$d\omega = (D_1 f_{2,3} - D_2 f_{1,3} + D_3 f_{1,2})\,dx_1 \wedge dx_2 \wedge dx_3\ ;$$

here we use also the traditional notation $dx_1 \wedge dx_2 \wedge dx_3$ for det.

For higher N the calculation is similar, and gives for $(N-1)$-form ω

(6b1) $$\omega = \sum_{i=1}^N f_{1,\dots,i-1,i+1,\dots,N} dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_N\ ;$$

(6b2) $$d\omega = \Big(\sum_{i=1}^N (-1)^{i-1} D_i f_{1,\dots,i-1,i+1,\dots,N} \Big)\,dx_1 \wedge \dots dx_N\ ;$$

here $dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_N$ is a special case of $dx_{i_1} \wedge \dots \wedge dx_{i_n}$ defined by $1 \le n \le N$

(6b3) $$\big(dx_{i_1} \wedge \dots \wedge dx_{i_n}\big)(h_1,\dots,h_n) = \begin{vmatrix} h_{1,i_1} & \dots & h_{n,i_1} \\ \dots & \dots & \dots \\ h_{1,i_n} & \dots & h_{n,i_n} \end{vmatrix}$$

where $h_{i,j}$ is the j-th coordinate of h_i. Note the antisymmetry: $dx_2 \wedge dx_1 = -dx_1 \wedge dx_2$; and $dx_1 \wedge dx_1 = 0$.

6b4 Lemma. For every antisymmetric multilinear n-form L on $\mathbb{R}^N$,

$$L = \sum_{1 \le m_1 < \dots < m_n \le N} L(e_{m_1}, \dots, e_{m_n}) \, dx_{m_1} \wedge \dots \wedge dx_{m_n} \, .$$

Proof. Both sides of this formula are antisymmetric multilinear n-forms; we have to prove that they are equal on arbitrary $h_1, \dots, h_n \in \mathbb{R}^N$. WLOG, $h_1 = e_{p_1}, \dots, h_n = e_{p_n}$ for some $1 \le p_1 < \dots < p_n \le N$. It remains to note that

$$(dx_{m_1} \wedge \dots \wedge dx_{m_n})(e_{p_1}, \dots, e_{p_n}) = \begin{cases} 1 & \text{if } m_1 = p_1, \dots, m_n = p_n, \\ 0 & \text{otherwise.} \end{cases}$$

□

It follows that for every (differential) n-form ω on $\mathbb{R}^N$,

$$\text{(6b5)} \qquad \begin{gathered} \omega = \sum_{1 \le m_1 < \dots < m_n \le N} f_{m_1, \dots, m_n}(x) \, dx_{m_1} \wedge \dots \wedge dx_{m_n} \, , \\ f_{m_1, \dots, m_n}(x) = \omega(x, e_{m_1}, \dots, e_{m_n}) \, . \end{gathered}$$

In particular, the volume form on $\mathbb{R}^n$ is

$$\det = dx_1 \wedge \dots \wedge dx_n \, .$$

Its pullback is of special interest, and deserves a special notation:

$$\text{(6b6)} \qquad d\varphi_1 \wedge \dots \wedge d\varphi_n = \varphi^*(dx_1 \wedge \dots \wedge dx_n)$$

for $\varphi : x \mapsto \big(\varphi_1(x), \dots, \varphi_n(x)\big)$, $\varphi \in C^1(\mathbb{R}^N \to \mathbb{R}^n)$. That is,

$$\begin{aligned} (d\varphi_1 \wedge \dots \wedge d\varphi_n)(\cdot, h_1, \dots, h_n) &= \det(D_{h_1}\varphi, \dots, D_{h_n}\varphi) = \\ &= \det\big(D_{h_i}\varphi_j\big)_{i,j} = \det\langle \nabla\varphi_j, h_i \rangle_{i,j} \, . \end{aligned}$$

The notation is consistent: if $\varphi(x_1, \dots, x_N) = (x_1, \dots, x_n)$, then $d\varphi_1 \wedge \dots \wedge d\varphi_n = dx_1 \wedge \dots \wedge dx_n$, since $D\varphi = \varphi$. Similarly, if $\varphi(x_1, \dots, x_N) = (x_{i_1}, \dots, x_{i_n})$, then $d\varphi_1 \wedge \dots \wedge d\varphi_n = dx_{i_1} \wedge \dots \wedge dx_{i_n}$.

In particular, for $\varphi \in C^1(\mathbb{R}^N \to \mathbb{R}^N)$ we have

$$\text{(6b7)} \qquad d\varphi_1 \wedge \dots \wedge d\varphi_N = (\det D\varphi)(dx_1 \wedge \dots \wedge dx_N) \, .$$

On the other hand, for $\varphi \in C^1(\mathbb{R}^N \to \mathbb{R})$ we have $(d\varphi)(\cdot, h) = \langle \nabla\varphi, h \rangle = (D_1\varphi)h_1 + \dots + (D_N\varphi)h_N$, that is,

$$\text{(6b8)} \qquad d\varphi = (D_1\varphi) \, dx_1 + \dots + (D_N\varphi) \, dx_N \, .$$

More generally, for arbitrary 1-forms $\omega_1, \dots, \omega_n$ one defines

(6b9) $$(\omega_1 \wedge \dots \wedge \omega_n)(x, h_1, \dots, h_n) = \det\big(\omega_i(x, h_j)\big)_{i,j} \,.$$

What about $d(\det)$, that is, $d(dx_1 \wedge \dots \wedge dx_n)$? If it exists, it must be 0, just because 0 is the only $(n+1)$-form on $\mathbb{R}^n$; but for now we do not know that it exists.

6b10 Proposition. $d(\det) = 0$.

It means, $d\varphi^*(dx_1 \wedge \dots \wedge dx_n) = 0$ for every $\varphi \in C^1(\mathbb{R}^N \to \mathbb{R}^n)$ where $N = n + 1$. That is, $d(d\varphi_1 \wedge \dots \wedge d\varphi_n) = 0$. $n = N - 1$

6b11 Lemma. For arbitrary $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}^{n+1}$,[1]

$$\det\big(\langle a_i, b_j\rangle\big)_{i,j} = \langle a_1 \times \dots \times a_n, b_1 \times \dots \times b_n \rangle \,.$$

Proof. Both sides of this formula are antisymmetric multilinear n-forms in $a_1, \dots, a_n$ (for given $b_1, \dots, b_n$). Thus, WLOG, $a_1 = e_{p_1}, \dots, a_n = e_{p_n}$ for some $1 \le p_1 < \dots < p_n \le n+1$. Similarly, $b_1 = e_{q_1}, \dots, b_n = e_{q_n}$ for some $1 \le q_1 < \dots < q_n \le n+1$. Now, both sides equal 1 if $p_1 = q_1, \dots, p_n = q_n$, otherwise 0. □

6b12 Lemma. The n-form $d\varphi_1 \wedge \dots \wedge d\varphi_n$ on $\mathbb{R}^N$ corresponds to the vector field $\nabla\varphi_1 \times \dots \times \nabla\varphi_n$. $n = N - 1$

Proof. $\langle \nabla\varphi_1 \times \dots \times \nabla\varphi_n, h_1 \times \dots \times h_n\rangle = \det\big(\langle \nabla\varphi_i, h_j\rangle\big)_{i,j} = (d\varphi_1 \wedge \dots \wedge d\varphi_n)(\cdot, h_1, \dots, h_n)$. □

***Proof** of Prop. 6b10.* Follows immediately from 6b12, (6a2) and 5c24: $\operatorname{div}(\nabla\varphi_1 \times \dots \times \nabla\varphi_n) = 0$. □

6b13 Corollary (of 6b10, 6a6 and (6b6))**.** $1 \le n \le N$

$$d(d\varphi_1 \wedge \dots \wedge d\varphi_n) = 0 \quad \text{for all } \varphi_1, \dots, \varphi_n \in C^1(\mathbb{R}^N) \,.$$

We see that all n-forms on $\mathbb{R}^N$ that are $d\varphi_1 \wedge \dots \wedge d\varphi_n$, and their sums, have (generalized exterior) derivatives (equal zero). But this is surely not the general case, since generally the derivative is not 0. According to 6b4, in order to get everything, it is sufficient to get $\varphi_1 \, d\varphi_2 \wedge \dots \wedge d\varphi_n$.

6b14 Proposition. $d(x_1 \, dx_2 \wedge \dots \wedge dx_n) = dx_1 \wedge \dots \wedge dx_n$.

[1] This determinant could be called "cross-Gramian".

It means, $d\varphi^*(x_1\,dx_2\wedge\dots\wedge dx_n) = \varphi^*(dx_1\wedge\dots\wedge dx_n)$ for every $\varphi\in C^1(\mathbb{R}^n\to\mathbb{R}^n)$. That is,

$$d(\varphi_1\,d\varphi_2\wedge\dots\wedge d\varphi_n) = d\varphi_1\wedge\dots\wedge d\varphi_n\,. \tag{6b15}$$

6b16 Exercise. The $(n-1)$-form $\varphi_1\,d\varphi_2\wedge\dots\wedge d\varphi_n$ on $\mathbb{R}^n$ corresponds to the vector field $\varphi_1\nabla\varphi_2\times\dots\times\nabla\varphi_n$.

Prove it.[1]

Proof *of Prop. 6b14.* Follows immediately from 6b16, (6a2), (6b7) and 5c24: $\operatorname{div}(\varphi_1\nabla\varphi_2\times\dots\times\nabla\varphi_n) = \det(D\varphi)$. □

Theorem 6a8 is thus proved.

Moreover, an arbitrary $(n-1)$-form ω of class C^1 being (6b5), we get $d\omega$ from (6b15):

$$d\omega = \sum_{1\le m_1<\dots<m_{n-1}\le N} df_{m_1,\dots,m_{n-1}}\wedge dx_{m_1}\wedge\dots\wedge dx_{m_{n-1}}\,. \tag{6b17}$$

Taking into account that $df_{m_1,\dots,m_{n-1}} = \sum_{k=1}^N \big(D_k f_{m_1,\dots,m_{n-1}}\big)dx_k$ we get

$$d\omega = \sum_{1\le m_1<\dots<m_n\le N} g_{m_1,\dots,m_n}\,dx_{m_1}\wedge\dots\wedge dx_{m_n} \quad\text{where} \tag{6b18}$$

$$g_{m_1,\dots,m_n} = \sum_{i=1}^n (-1)^{i-1} D_{m_i} f_{m_1,\dots,m_{i-1},m_{i+1},\dots,m_n}$$

(think, why); for $n=N-1$ if was seen before: (6b1), (6b2).

This (6b18) is called the (classical, not generalized) exterior derivative of a form of class C^1. We see that the classical exterior derivative is the special case of the generalized exterior derivative for the forms of class C^1.

6b19 Corollary (of 6a6 and Th. 6a8)**.** (a) $d(\varphi^*\omega) = \varphi^*(d\omega)$ whenever $d\omega$ exists and $\varphi\in C^1$;

(b) if $\omega,\varphi\in C^1$, then $d\omega$ is classical, but $d(\varphi^*\omega)$ is (generally) not;

(c) if $\omega\in C^1$ and $\varphi\in C^2$, then $d\omega$ and $d(\varphi^*\omega)$ are classical.

6b20 Corollary (of 6b13 and (6b17))**.**

$$d(d\omega)=0 \quad\text{for all } n\text{-forms } \omega \text{ of class } C^1.$$

A wonder: no second (and higher) exterior derivatives, at all!

[1] Hint: similar to 6b12.

6c Stokes' theorem

Recall (5a17): $\int_\Gamma \omega = \int_{B^\circ} \Gamma^*\omega$ for arbitrary singular n-box Γ and n-form ω. Also, we define $1 \le n \le N$

(6c1) $$\int_{\partial\Gamma} \omega = \int_{\partial B \setminus Z} \Gamma^*\omega$$

for $(n-1)$-forms ω. (As before, $\partial B \setminus Z$ is the union of the $2n$ hyperfaces of B; also, $\Gamma^*\omega$ extends from B° to ∂B by continuity.)

6c2 Theorem (Stokes' theorem)**.**

$$\int_\Gamma d\omega = \int_{\partial\Gamma} \omega$$

for every $(n-1)$-form ω of class C^1 on $\mathbb{R}^N$ and singular n-box Γ in $\mathbb{R}^N$.

Proof. By 6a3, $d(\Gamma^*\omega) = \Gamma^*(d\omega)$ on B°. By (6a2), $f = \operatorname{div} F$ on B°, where F corresponds to $\Gamma^*\omega$ according to (4e6), and f corresponds to $\Gamma^*(d\omega)$ according to (4e3). Similarly to the proof of Th. 5d8 it follows that $\int_{\partial B \setminus Z} \langle F, \mathbf{n}\rangle = \int_B f$, which means (by (4e7), (4e4)) $\int_{\partial B\setminus Z} \Gamma^*\omega = \int_B \Gamma^*(d\omega)$, that is, $\int_\Gamma d\omega = \int_{\partial\Gamma}\omega$. □

6c3 Remark. The theorem still holds (with the same proof) when $d\omega$ is the generalized exterior derivative of an $(n-1)$-form ω of class C^0.

6d Order 0 and order 1

Recall the integral of a 1-form over a path.

First, Sect. 1c (between 1c12 and 1c13):

$$\int_\gamma \omega = \int_{t_0}^{t_1} \big((f_1 \circ \gamma)\, d\gamma_1 + \dots + (f_N \circ \gamma)\, d\gamma_N \big)$$

for $\omega = f_1\, dx_1 + \dots + f_N\, dx_N$ and $\gamma(t) = \big(\gamma_1(t), \dots, \gamma_N(t)\big)$, $t \in [t_0, t_1]$; here $d\gamma_k = \gamma_k'\, dt$.

Second, (5a14):

$$\int_\gamma \omega = \int_{(t_0,t_1)} \gamma^*\omega\,.$$

Indeed, the formula $\varphi^*(f\omega) = (\varphi^* f)(\varphi^*\omega) = (f \circ \varphi)(\varphi^*\omega)$ (5a12) gives $\gamma^*(f_k\, dx_k) = (f_k \circ \gamma)\, d\gamma_k = (f_k \circ \gamma)\gamma_k'\, dt$, whence

$$\gamma^*\omega = (f_1 \circ \gamma)\gamma_1'\, dt + \dots + (f_N \circ \gamma)\gamma_N'\, dt\,,$$

and $\int_{(t_0,t_1)} \gamma^*\omega = \int_\gamma \omega$, as it should be.

Now we take $\omega = d\varphi$ (see (6b8)) for a function $\varphi \in C^1(\mathbb{R}^N)$ treated as a 0-form: $(\gamma^*\omega)(t, dt) = (D_1\varphi)_{\gamma(t)}\gamma_1'(t)\,dt + \dots + (D_N\varphi)_{\gamma(t)}\gamma_N'(t)\,dt = (\varphi \circ \gamma)'(t)\,dt$, thus,

$$\int_\gamma d\varphi = \int_{t_0}^{t_1} (\varphi \circ \gamma)'(t)\,\mathrm{d}t = \varphi(\gamma(t_1)) - \varphi(\gamma(t_0)) .$$

The integral depends on the values of φ at the endpoints only!

On the other hand, $\int_{\partial\gamma} \varphi = \int_{\partial[t_o,t_1]} \gamma^*\varphi = \varphi(\gamma(t_1)) - \varphi(\gamma(t_0))$ (as explained before 6a3), and we get

$$\int_{\partial\gamma} \varphi = \int_\gamma d\varphi ,$$

as it should be.

. . . . *Vector calculus* . . . .

Recall visualization of 1-forms by vector fields (Sect. 5b, “Facet 2”): F corresponds to ω when $\omega(x,h) = \langle F(x), h\rangle$; that is, a form $\omega = f_1\,dx_1 + \dots + f_N\,dx_N$ corresponds to vector field $F(x) = \big(f_1(x), \dots, f_N(x)\big)$, and

$$\int_\gamma \omega = \int_{t_0}^{t_1} \langle F(\gamma(t)), \gamma'(t)\rangle\,\mathrm{d}t ;$$

the latter is called the integral[1] of a vector field F along a path γ. In some sense it measures how much the vector field is aligned with the path.[2] (If F is orthogonal to γ then this integral vanishes.) If the path γ is closed then this integral is called *circulation* of F around γ and denoted $\oint$; it indicates how much the vector field tends to circulate around γ.

Clearly,

$$\Big|\int_\gamma \omega\Big| \le \Big(\max_t |F(\gamma(t))|\Big) \underbrace{\int_{t_0}^{t_1} |\gamma'(t)|\,\mathrm{d}t}_{\text{length}(\gamma)} ,$$

the length of the path times the upper bound on the vector field.

If $\omega = d\varphi$, then $F = \nabla\varphi$,

$$\langle \nabla\varphi(\gamma(t)), \gamma'(t)\rangle = \big(D_{\gamma'(t)}\varphi\big)_{\gamma(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\gamma(t)) ,$$

$$\int_{t_0}^{t_1} \langle \nabla\varphi(\gamma(t)), \gamma'(t)\rangle\,\mathrm{d}t = \varphi\big(\gamma(t_1)\big) - \varphi\big(\gamma(t_0)\big) .$$

[1]Also “line integral” or “flow integral”.

[2]Nice formulation from mathinsight.

Thus,

$$\left|\varphi\big(\gamma(t_1)\big) - \varphi\big(\gamma(t_0)\big)\right| \le \Big(\max_t |\nabla\varphi(\gamma(t))|\Big) \underbrace{\int_{t_0}^{t_1} |\gamma'(t)| \,\mathrm{d}t}_{\text{length}(\gamma)} ,$$

In particular, if $\nabla\varphi = 0$, then $\varphi = \text{const}$ on each connected component of its domain, of course.

6e Order 1 and order 2

For a 1-form $\omega = f_1 \, dx_1 + \dots + f_N \, dx_N$ of class C^1,

$$d\omega = df_1 \wedge dx_1 + \dots + df_N \wedge dx_N = \sum_{i<j} (D_i f_j - D_j f_i) \, dx_i \wedge dx_j \, .$$

In particular, if $\omega = d\varphi$ for some $\varphi \in C^2(\mathbb{R}^N)$, then $d\omega = 0$, since $D_i f_j - D_j f_i = D_i D_j \varphi - D_j D_i \varphi = 0$. Moreover, $d(d\varphi) = 0$ (generalized…) for all $\varphi \in C^1(\mathbb{R}^N)$ by 6b20.

.... *Dimension $N = 2$* ....

Here $\omega = f_1 \, dx_1 + f_2 \, dx_2$;

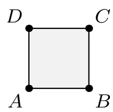
$$d\omega = (D_2 f_1) \, dx_2 \wedge dx_1 + (D_1 f_2) \, dx_1 \wedge dx_2 = (D_1 f_2 - D_2 f_1) \, dx_1 \wedge dx_2 \, ,$$

as was seen in 4e17.

Thus, Stokes' theorem $\int_\Gamma d\omega = \int_{\partial\Gamma} \omega$, that is, $\int_\Gamma (D_1 f_2 - D_2 f_1) \, dx_1 \wedge dx_2 = \int_{\partial\Gamma} (f_1 \, dx_1 + f_2 \, dx_2)$, is a "singular" counterpart of Green's theorem $\int_U (D_1 f_2 - D_2 f_1) \, dx_1 \wedge dx_2 = \int_{\partial U} (f_1 \, dx_1 + f_2 \, dx_2)$ mentioned in 4e17.

Denoting the given 2-box by $ABCD$ we have

$$\int_{\partial\Gamma} \omega = \Big(\int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} \Big) \omega$$



(think, why); in this sense $\partial(ABCD) = AB + BC + CD + DA = AB + BC - DC - AD$, a formal linear combination of 1-boxes, so-called 1-chain; it may also be treated as a (piecewise smooth) path $\gamma = \partial\Gamma$.

.... *Vector calculus for $N = 2$* ....

Dimension 2 is special: 1-forms and $(N-1)$-forms are the same when $N = 2$. In 4e14 the 1-form $\omega = f_1 \, dx_1 + f_2 \, dx_2$ was treated as an $(N-1)$-form that corresponds to vector field $H(x) = \big(f_2(x), -f_1(x)\big)$ ("Facet 3" in Sect. 5b); then, treating $\partial\Gamma$ as a path γ, we have

$$\int_{\partial\Gamma} \omega = \int_\gamma (-H_2\,dx_1 + H_1\,dx_2) = \int_{t_0}^{t_1} (H_1\gamma_2' - H_2\gamma_1')\,\mathrm{d}t = \text{flux}$$

by 4e12. Also, $\operatorname{div} H = D_1H_1 + D_2H_2 = D_1f_2 - D_2f_1$, that is, $d\omega = (\operatorname{div} H)\,dx_1 \wedge dx_2$. Thus, Stokes' theorem $\int_\Gamma d\omega = \int_{\partial\Gamma} \omega$ turns into the "singular" divergence theorem: the flux of H through $\partial\Gamma$ is equal to the integral of $\operatorname{div} H$ over Γ.

Alternatively we may visualize the 1-form $\omega = f_1\,dx_1 + f_2\,dx_2$ by the vector field $E(x) = \big(f_1(x), f_2(x)\big)$ ("Facet 2" in Sect. 5b), then $\int_{\partial\Gamma} \omega$ is the circulation of E around $\partial\Gamma$; as was noted, Stokes' theorem gives a "singular" generalization of Green's theorem:

$$\oint_{\partial\Gamma} (E_1\,dx + E_2\,dy) = \int_\Gamma (D_1E_2 - D_2E_1)\,dxdy\,.$$

Clearly,

$$\left| \oint_{\partial\Gamma} (E_1\,dx + E_2\,dy) \right| \le \big(\max |D_1E_2 - D_2E_1|\big)\,\operatorname{area}(\Gamma)\,,$$

where $\operatorname{area}(\Gamma) = \int_B J_\Gamma$, J_Γ being the (generalized) Jacobian (introduced in Sect. 2c).

Note that rotation by $+\pi/2$ turns $H(x)$ into $E(x)$.

.... *Dimension* $N = 3$ ....

Here $\omega = f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3$; $\quad d\omega =$

$$(D_2f_1\,dx_2+D_3f_1\,dx_3)\wedge dx_1+(D_3f_2\,dx_3+D_1f_2\,dx_1)\wedge dx_2+(D_1f_3\,dx_1+D_2f_3\,dx_2)\wedge dx_3$$
$$= (D_1f_2-D_2f_1)dx_1\wedge dx_2+(D_2f_3-D_3f_2)dx_2\wedge dx_3+(D_3f_1-D_1f_3)dx_3\wedge dx_1\,.$$

.... *Vector calculus for* $N = 3$ ....

Dimension 3 is special, too: the four special cases $n = 0$, $n = 1$, $n = N-1$, $n = N$ exhaust all $n = 0, \dots, N$ when $N = 3$. Thus, all n-forms are visualized easily: 0-forms and 3-forms by functions, 1-forms and 2-forms by vector fields (using "Facet 2" and "Facet 3" in Sect. 5b, respectively).

Denoting by E the vector field that corresponds to ω and by H the vector field that corresponds to $d\omega$ we have

$$H = \operatorname{curl} E\,,$$

the curl being defined by

$$H_1 = D_2E_3 - D_3E_2\,, \quad H_2 = D_3E_1 - D_1E_3\,, \quad H_3 = D_1E_2 - D_2E_1$$

where $H = (H_1, H_2, H_3)$ and $E = (E_1, E_2, E_3)$. Compare it with the cross product of two 3-dimensional vectors (mentioned in 2b17(c), generalized in (4e5)):

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)\,.$$

Physicists like to say that $\operatorname{curl} E = \mathbf{D} \times E$ where $\mathbf{D}$ is the "vector" (D_1, D_2, D_3).

In particular, if $\omega = d\varphi$ for some $\varphi \in C^2(\mathbb{R}^3)$, then $E = \nabla\varphi$ and $\operatorname{curl} E = \operatorname{curl} \nabla\varphi = 0$ (since $D_iE_j - D_jE_i = D_iD_j\varphi - D_jD_i\varphi = 0$), which is a special case of 6b20.

If a 2-box $\Gamma : B \to \mathbb{R}^3$ is such that $\Gamma(B^\circ) \subset \mathbb{R}^3$ is a 2-manifold and $(B^\circ, \Gamma|_{B^\circ})$ is its chart, then $\int_\Gamma d\omega$ is the flux of H through $\Gamma(B^\circ)$ by (4e7). More generally, we call $\int_\Gamma d\omega$ the flux of H through the singular 2-box Γ.

Similarly to "vector calculus for $N = 2$", $\int_{\partial\Gamma} \omega$ is the circulation of E around $\partial\Gamma$. Stokes' theorem $\int_\Gamma d\omega = \int_{\partial\Gamma} \omega$ turns into the *"classical Stokes' theorem"* (also known as "Kelvin-Stokes theorem", "curl theorem" and "Stokes' formula"):

(6e1) the circulation of E around $\partial\Gamma$

is equal to the flux of $\operatorname{curl} E$ through Γ

for every vector field E (of class C^1) on $\mathbb{R}^3$ and every singular 2-box Γ in $\mathbb{R}^3$. In this sense, the curl is the circulation density, called also "vorticity" (and its flux is called also the net vorticity of E throughout Γ). A small paddle-wheel in the flow spins the fastest when its axle points in the direction of the curl vector (of the velocity field, recall "Facet 1" in Sect. 5b), and in this case its angular speed is half the length of the curl vector.[1]

It follows from (6e1) that

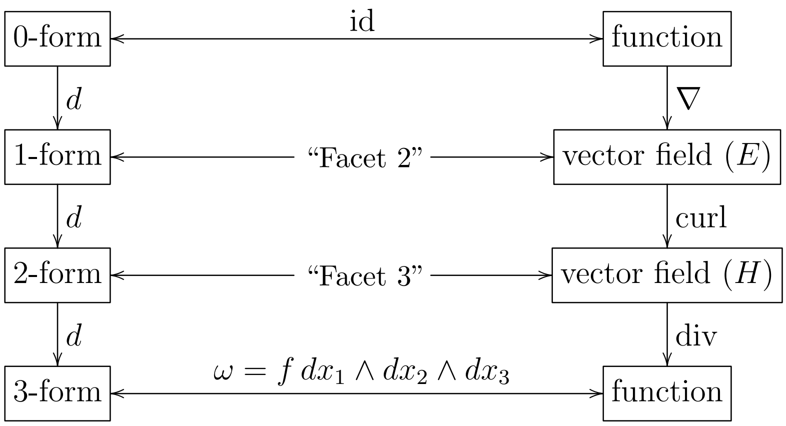
$$\left| \oint_{\partial\Gamma} E \right| \le \big(\max |\operatorname{curl} E|\big) \operatorname{area}(\Gamma)\,. \tag{6e2}$$

In addition, Stokes' theorem for 2-forms and 3-forms in $\mathbb{R}^3$, being a special case of Stokes' theorem for $(N-1)$-forms and N-forms in $\mathbb{R}^N$, boils down to

[1]Shifrin p. 394.

the divergence theorem for singular boxes, 5d8. We summarize.

(6e3)



The general formula $d(d\omega) = 0$ implies two less general formulas

(6e4) $$\operatorname{curl}(\nabla f) = 0\,, \qquad \operatorname{div}(\operatorname{curl} E) = 0\,.$$

6e5 Exercise. Let α, β be 1-forms on $\mathbb{R}^3$, and ω a 2-form on $\mathbb{R}^3$. Translate the relation $\alpha \wedge \beta = \omega$ (6b9) into the language of vector calculus (that is, of the vector fields E, F, H that correspond to α, β, ω).

6e6 Exercise. $d(\varphi\omega) = d\varphi \wedge \omega + \varphi\, d\omega$ for all $\varphi \in C^1(\mathbb{R}^N)$ and 1-forms ω of class C^1 on $\mathbb{R}^N$.

Prove it.

6e7 Exercise. For $N = 3$ translate 6e6 into the language of vector calculus.

6e8 Exercise. (a) If an n-box $\Gamma : B \to \mathbb{R}^N$ satisfies $\forall u \in \partial B \;\; \Gamma(u) = 0$, $1 \le n \le N$
then $\int_\Gamma d\omega = 0$ for all $(n-1)$-forms ω of class C^1 on $\mathbb{R}^N$.

(b) If $\varphi : C^1(\mathbb{R}^n \to \mathbb{R}^N)$ has a bounded support, then $\int_{\mathbb{R}^n} \varphi^*(d\omega) = 0$ for all $(n-1)$-forms ω of class C^1 on $\mathbb{R}^N$.

Prove it.

6e9 Remark. Applying 6e8(b) to $n = N$ and $\omega = x_1\, dx_2 \wedge \dots \wedge dx_n$ we get another proof[1] to (3a3) $\int \det(D\varphi) = 0$.

6e10 Exercise. Generalize 6e8 to Γ such that $\Gamma(\partial B) \subset M$ for some manifold $M \subset \mathbb{R}^N$ of dimension $n-2$ (or less), assuming $n \ge 2$.

6e11 Exercise. Let a vector field E (of class C^1) on $\mathbb{R}^3$ satisfy

$$\Big\langle \operatorname{curl} E\Big(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\Big), \Big(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\Big) \Big\rangle = 0 \quad \text{whenever } x^2 + y^2 + z^2 = 1,\ z > -0.9\,;$$

[1]For the first proof see 5c10.

prove that $\int_{\gamma_z} E = 0$ for all $z \in (-0.9, 1)$, where $\gamma_z(t) = \begin{pmatrix} \sqrt{1-z^2}\cos t \\ \sqrt{1-z^2}\sin t \\ z \end{pmatrix}$ for $t \in [0, 2\pi]$.

6e12 Exercise. Let a vector field E (of class C^1) on $\mathbb{R}^3$ satisfy

$$\Big\langle \operatorname{curl} E\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Big\rangle = 0 \quad \text{whenever } x^2 + y^2 + z^2 = 1, \ -0.9 < z < 0.9\,;$$

prove that $\int_{\gamma_z} E$ does not depend on $z \in (-0.9, 0.9)$; here γ_z is the same as in 6e11.

6e13 Exercise. Consider a vector field

$$E\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -yf\big(\sqrt{x^2+y^2}\big) \\ xf\big(\sqrt{x^2+y^2}\big) \\ 0 \end{pmatrix}, \quad \text{that is,} \quad E\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} = \begin{pmatrix} rf(r)\cos(\theta+\frac{\pi}{2}) \\ rf(r)\sin(\theta+\frac{\pi}{2}) \\ 0 \end{pmatrix}$$

for a function $f : [0, \infty) \to \mathbb{R}$ of class C^1.

(a) Check that E is of class C^1, and

$$\operatorname{curl} E\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{x^2+y^2}f'\big(\sqrt{x^2+y^2}\big)+2f\big(\sqrt{x^2+y^2}\big) \end{pmatrix}, \quad \text{that is,}$$

$$\operatorname{curl} E\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ rf'(r)+2f(r) \end{pmatrix}.$$

(b) Given $\varepsilon > 0$, construct f such that

$$\begin{aligned} rf'(r) + 2f(r) > 0 \quad &\text{for } r \in (0, \varepsilon)\,, \\ rf'(r) + 2f(r) = 0 \quad &\text{for } r \in [\varepsilon, \infty)\,. \end{aligned}$$

(c) Conclude that $\int_{\gamma_z} E$ in 6e12 need not vanish.

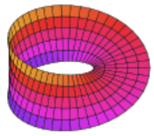
6e14 Exercise. Let a vector field E (of class C^1) on $\mathbb{R}^3$ satisfy

$$\operatorname{curl} E\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 1/r \end{pmatrix} = 0 \quad \text{for all } r > 0, \ \theta \in [0, 2\pi]\,,$$

and in addition, $|E(x, y, z)| = o\big(\sqrt{x^2 + y^2 + z^2}\big)$ as $x^2+y^2+z^2 \to \infty$. Prove that $\int_{\gamma_r} E = 0$ for all $r > 0$; here $\gamma_r(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ 1/r \end{pmatrix}$ for $t \in [0, 2\pi]$.

A Möbius strip may be defined as such a surface:

$$\left\{ \begin{pmatrix} (R+rs\cos\frac{\theta}{2})\cos\theta \\ (R+rs\cos\frac{\theta}{2})\sin\theta \\ rs\sin\frac{\theta}{2} \end{pmatrix} : s \in [-1,1],\, \theta \in [0,2\pi] \right\},$$



for given $R > r > 0$.[1] Its boundary is a curve $\{\gamma(t) : t \in [0, 4\pi]\}$,

$$\gamma(t) = \begin{pmatrix} (R+r\cos\frac{t}{2})\cos t \\ (R+r\cos\frac{t}{2})\sin t \\ r\sin\frac{t}{2} \end{pmatrix}.$$

6e15 Exercise. For γ as above and E of 6e13,

(a) check that

$$\langle E(\gamma(t)), \gamma'(t) \rangle = (R + r\cos\tfrac{t}{2}) f(R + r\cos\tfrac{t}{2});$$

(b) choose f such that $\operatorname{curl} E = 0$ on the Möbius strip, but $\int_\gamma E > 0$;

(c) does it contradict Stokes' theorem? Explain.

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[1] Images from Wikipedia.