Solutions to selected exercises

2b24 Exercise (surface of revolution or body of revolution). Let $M_1$ be an $n$-manifold in $\mathbb{R}^3$ (here $n = 1$ or $n = 2$) such that

$$\forall(x, y, z) \in M_1 \ (0, -z, y) \notin T_{(x,y,z)}M_1.$$ 

Consider the set

$$M = \{(x, cy - sz, sy + cz): (x, y, z) \in M_1, (c, s) \in S\}$$

where $S = \{(c, s) \in \mathbb{R}^2: c^2 + s^2 = 1\}$ (the circle). Assume that the mapping $\psi : S \to M$. Then

(a) $M$ is an $(n + 1)$-manifold in $\mathbb{R}^3$;

(b) if $(G_1, \psi_1)$ is a chart of $M_1$ and $(G_2, \psi_2)$ is a chart of $S$, then $(G_1 \times G_2, \psi)$ is a chart of $M$; here $\psi(u_1, u_2) = (x, cy - sz, sy + cz)$ whenever $\psi_1(u_1) = (x, y, z)$ and $\psi_2(u_2) = (c, s)$.

Prove it.

Solution. Item (a) follows from (b); here is (b).

We have $\psi = F \circ (\psi_1, \psi_2)$ where $F : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$, $F((x, y, z), (c, s)) = (x, cy - sz, sy + cz)$ and $(\psi_1, \psi_2) : G_1 \times G_2 \to M_1 \times S \subset \mathbb{R}^3 \times \mathbb{R}^2$, $(\psi_1, \psi_2)(u_1, u_2) = (\psi_1(u_1), \psi_2(u_2))$.

Similarly to 2b13 it is sufficient to consider a chart of $S$ around the point $(1, 0)$ only, since the rotation $(x, y, z) \mapsto (x, cy - sz, sy + cz)$ of $\mathbb{R}^3$ sends $F((x, y, z), (1, 0))$ to $F((x, y, z), (c, s))$.

By 2b9, $M_1 \times S \subset \mathbb{R}^3 \times \mathbb{R}^2$ is an $(n + 1)$-manifold; and (from the solution of 2b9), $(G_1 \times G_2, (\psi_1, \psi_2))$ is a chart of $M_1 \times S$. Thus, $(\psi_1, \psi_2)$ is a homeomorphism from $G_1 \times G_2$ onto the relatively open (in $M_1 \times S$) chart $\psi_1(G_1) \times \psi_2(G_2)$. It is given that $F|_{M_1 \times S}$ is a homeomorphism $M_1 \times S \to M$. Thus, $\psi \circ F \circ (\psi_1, \psi_2)$ is a homeomorphism from $G_1 \times G_2$ onto the relatively open (in $M$) set $\psi(G_1 \times G_2)$. Also, $\psi \in C^1(G_1 \times G_2 \to \mathbb{R}^3)$, since $\psi_1 \in C^1(G_1 \to \mathbb{R}^3)$, $\psi_2 \in C^1(G_2 \to \mathbb{R}^2)$, and $F \in C^1(\mathbb{R}^5 \to \mathbb{R}^3)$.

It remains to check that the linear operator $(D\psi)(u_1, u_2) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^3$ is one-to-one, assuming that $\psi_2(u_2) = (1, 0)$. By the chain rule, $(D\psi)(u_1, u_2) = (DF)_{(x,y,z),(1,0)} \circ (D\psi_1)(u_1, u_2)$ where $(x, y, z) = \psi_1(u_1)$. Also, $(D\psi_1)_{u_1} : \mathbb{R}^n \to T_{(x,y,z)}M_1$ is one-to-one, and $(D\psi_2)_{u_2} : \mathbb{R} \to T_{(1,0)}S$ is one-to-one. It remains to check that

$$(DF)_{(x,y,z),(1,0)}(h, k) = 0 \quad \Rightarrow \quad (h = 0, k = 0)$$

for $h \in T_{(x,y,z)}M_1$, $k \in T_{(1,0)}S$. 


Un fortunately, the formulation of Exercise 2c33 is erroneous: the factor \( J_{\psi_2} \) is missing in the formula for \( J_\psi \). I am sorry. Here is the corrected formulation.

**2c33 Exercise (Surface of Revolution or Body of Revolution).** Let \( M_1, n, M, S, (G_1, \psi_1), (G_2, \psi_2), (G_1 \times G_2, \psi) \) be as in 2b24(b). Then

\[
J_\psi(u_1, u_2) = J_{\psi_1}(u_1)J_{\psi_2}(u_2) \text{ dist}((0, -z, y), T_{(x,y,z)}M_1) \quad \text{where } (x, y, z) = \psi_1(u_1).
\]

In particular, if \( M_1 \subset \mathbb{R}^2 \times \{0\} \), then also \( T_{(x,y,z)}M_1 \subset \mathbb{R}^2 \times \{0\} \); \( (0, -z, y) = (0, 0, y) \perp \mathbb{R}^2 \times \{0\} \); thus,

\[
J_\psi(u_1, u_2) = |y|J_{\psi_1}(u_1)J_{\psi_2}(u_2) \quad \text{where } (x, y, 0) = \psi_1(u_1).
\]

Prove it.

**Solution.** Once again, it is sufficient to consider a chart of \( S \) around the point \((1, 0)\) only. Thus we assume that \( \psi_2(0) = (1, 0) \) and calculate \( J_\psi(u_1, 0) \).

Still, \( \psi = F \circ (\psi_1, \psi_2) \) and \( (DF)_{((x,y,z),(1,0))}(h, (0, \lambda)) = h + \lambda(0, -z, y) \).

Also, \( \psi_{\lambda}(0) = (0, \lambda) \) for some \( \lambda \) (since \( T_{(1,0)}S = \{(0, \lambda) : \lambda \in \mathbb{R}\} \), still); and \( J_{\psi_{\lambda}}(0) = |\lambda| \).

By the chain rule (of Analysis-3),

\[
(DF)_{(u_1, 0)} = (DF)_{((x,y,z),(1,0))} \circ (D(\psi_1, \psi_2))_{(u_1, 0)} : (h, k) \mapsto (DF)_{((x,y,z),(1,0))}(D(\psi_1)_{u_1}h, (D\psi_2)_{u_1}k) = (D\psi_1)_{u_1}h + \lambda k(0, -z, y)
\]

for \( h \in \mathbb{R}^n \) and \( k \in \mathbb{R} \); here \( (x, y, z) = \psi_1(u_1) \). In particular, \( (D_k\psi)_{(u_1, 0)} = (D_k\psi_1)_{u_1} \) for \( 1 \leq k \leq n \), and \( (D_{n+1}\psi)_{(u_1, 0)} = \lambda(0, -z, y) \).

The \((n + 1)\)-dimensional volume of the parallelootope spanned by these \( n + 1 \) vectors \( (D_k\psi)_{(u_1, 0)}, \ldots, (D_{n+1}\psi)_{(u_1, 0)} \) is \( J_\psi(u_1, 0) \), while the \( n \)-dimensional volume of its base, the parallelootope spanned by the first \( n \) vectors, is
$J_{\psi_1}(u_1)$. The latter parallelotope (the base) spans $T_{(x,y,z)}M_1$; thus, the height of the $(n+1)$-dimensional parallelotope is $|\lambda| \text{dist}((0, -z, y), T_{(x,y,z)}M_1)$. Taking into account that $J_{\psi_2}(0) = |\lambda|$ we get

$$J_{\psi}(u_1, u_2) = J_{\psi_1}(u_1)J_{\psi_2}(u_2) \text{dist}((0, -z, y), T_{(x,y,z)}M_1).$$