2 Markov and strong Markov

2a Restart at a nonrandom time

Let $X$ be any one of the three processes introduced in Sect. 1 (the Brownian motion, the Cauchy process, the special Lévy process) on a probability space $(\Omega, \mathcal{F}, P)$. We construct a random function $Y$ on the product $\Omega^2 = \Omega \times \Omega$ (that is, $(\Omega, \mathcal{F}, P) \times (\Omega, \mathcal{F}, P)$) by gluing together two independent sample functions as follows:

\begin{equation}
Y(t)(\omega_1, \omega_2) = \begin{cases}
X(t)(\omega_1) & \text{for } t \leq 1, \\
X(1)(\omega_1) + X(t-1)(\omega_2) & \text{for } t \geq 1.
\end{cases}
\end{equation}

Clearly, sample functions of $Y$ are right continuous.

2a2 Exercise. $Y$ is distributed like $X$.\footnote{Recall 1c, especially 1c3. See also \ref{footnote}.}

Prove it.

This is the Markov property: at the instant 1 the process $X$ forgets its past and retains only a single point, $X(1)$.\footnote{By the way, a process with differentiable sample functions cannot be Markov (unless it is nonrandom); it have to retain $X'(1)$.} Of course, the Markov property holds at every instant $t \in (0, \infty)$, not just 1.

We turn to the Brownian motion, $B$. Given $x \in (0, \infty)$, we define the hitting time $T_x : \Omega \to [0, \infty]$ by

\begin{equation}
T_x = \inf\{t : B(t) = x\}
\end{equation}

(as usual, $\inf\emptyset = \infty$).
2a4 Exercise. (a) $T_x$ is measurable (in $\omega$, for a fixed $x$); (b) the distribution of $T_x$ is uniquely determined, that is, does not depend on the choice of $(\Omega, \mathcal{F}, P)$ and $B$ as far as $B$ is a Brownian motion.\(^1\)

Prove it.

Such statements should be made every time we construct a random variable out of the Brownian motion;\(^2\) however, they will be usually omitted.

2a5 Exercise. $T_x$ is distributed like $x^2T_1$.

Prove it.

We introduce the random variable\(^3\)

\begin{equation}
L = \max\{t \in [0, 1] : B(t) = 0\},
\end{equation}

and want to calculate its distribution,

$$\mathbb{P}(L < t) = \mathbb{P}(\forall s \in [t, 1] \ B(s) \neq 0) = ?$$

Given $B(t) = x > 0$, the conditional probability of this event should be equal to

$$\mathbb{P}(\forall s \in [0, 1-t] \ B(s) \neq x) = \mathbb{P}(T_x > 1-t) = \mathbb{P}(T_1 > \frac{1-t}{x^2})$$

(think, why); for $x < 0$ the situation is similar. We guess that

\begin{equation}
\mathbb{P}(L < t) = \int_{-\infty}^{\infty} p_t(x) \mathbb{P}(T_1 > \frac{1-t}{x^2}) \, dx,
\end{equation}

where $p_t(x) = (2\pi t)^{-1/2} \exp\left(-\frac{x^2}{2t}\right)$.

The proof combines the Markov property of the Brownian motion with the Fubini theorem. We use $\omega_1$ on $[0, t]$, switch to $\omega_2$ on $[t, 1]$, substitute this combination for $B$ into $L$ and get

$$\mathbb{P}(L < t) = \mathbb{P}(\forall s \in [t, 1] \ B(s) \neq 0) =$$

$$= (P \times P)\{ (\omega_1, \omega_2) : \forall s \in [t, 1] \ B(t)(\omega_1) + B(s-t)(\omega_2) \neq 0 \} =$$

$$= \int_{\Omega} f(B(t)(\omega_1)) P(d\omega_1) = \mathbb{E} f(B(t)) = \int_{\mathbb{R}} p_t(x) f(x) \, dx,$$

\(^1\)See also 2a3, 2b4.

\(^2\)For instance, $L$ and $R$, see 2a6, 2a9.

\(^3\)‘$L$ is for left or last’ [1, Sect. 7.2, Exer. 2.2].
where

\[
    f(x) = P\{\omega_2 : \forall s \in [t, 1] \ x + B(s - t)(\omega_2) \neq 0\} = \\
    = \mathbb{P}\left(\forall s \in [0, 1-t] \ B(s) \neq -x\right) = \mathbb{P}(T_{|x|} > 1-t) = \mathbb{P}\left(T_1 > \frac{1-t}{x^2}\right);
\]

(2a7) follows.

2a8 Exercise. Let

\[
    R = \inf\{t \in [1, \infty) : B(t) = 0\}
\]

(possibly, \(\infty\).) Then

\[
    \mathbb{P}(R > 1 + t) = \int_{-\infty}^{\infty} p_1(x) \mathbb{P}\left(T_1 > \frac{t}{x^2}\right) dx.
\]

Prove it.

2b Hit and restart

Similarly to (2a1) we let (recall (2a3))

\[
    Y(t)(\omega_1, \omega_2) = \begin{cases} 
    B(t)(\omega_1) & \text{for } t \leq T_1(\omega_1), \\
    1 + B(t - T_1(\omega_1))(\omega_2) & \text{for } t \geq T_1(\omega_1).
    \end{cases}
\]

2b2 Proposition. \(Y\) is distributed like \(B\).

The proof will be given in 2c but do not hesitate to use 2b2 now.

This is a special case of strong Markov property.³

You see, the process \(B\) forgets the past when hitting the level 1. Of course, the same happens when hitting \(x\), for every \(x \in \mathbb{R}\), not just 1.

2b3 Exercise. Prove that

\[
    \mathbb{P}\left(\max_{[0,t]} B(\cdot) \geq 1\right) = 2 \mathbb{P}\left(B(t) \geq 1\right).
\]

Similarly, \(\mathbb{P}\left(\max_{[0,t]} B(\cdot) \geq x\right) = 2 \mathbb{P}\left(B(t) \geq x\right)\) for all \(x \in [0, \infty)\). Thus,

\[
    \max_{[0,t]} B(\cdot) \text{ is distributed like } |B(t)|.
\]

¹\(R\) is for right or return' \(\mathbb{I}\) Sect. 7.2, Exer. 2.1.
²But see 2b5.
³See also 2f8.
The distribution of $T_x$ is therefore 

$$\mathbb{P}(T_x \leq t) = \mathbb{P} \left( \max_{[0,t]} B(\cdot) \geq x \right) = 2 \mathbb{P}(B(t) \geq x) = 2 \mathbb{P} \left( B(1) \geq \frac{x}{\sqrt{t}} \right) = 2 \int_{x/\sqrt{t}}^{\infty} p_1(y) \, dy.$$ 

**2b5 Exercise.** Prove that 

$$\inf_{[0,\infty)} B(\cdot) = -\infty, \quad \sup_{[0,\infty)} B(\cdot) = \infty \quad \text{a.s.}$$

**2b6 Exercise.** Almost surely, 

$$\forall \varepsilon > 0 \left( \min_{[0,\varepsilon]} B(\cdot) < 0 \text{ and } \max_{[0,\varepsilon]} B(\cdot) > 0 \right).$$

Prove it.

**2b7 Exercise.** $B$ does not restart at the random time $L$ (defined by **2a6**). Prove it.

Now we are in position to finalize the calculation of the distribution of $L$ and $R$ started in **2a7**, **2a10**; the integrals need some effort, and give 

$$\mathbb{P}(L \leq t) = \frac{2}{\pi} \arcsin \sqrt{t} \quad \text{for } 0 \leq t \leq 1,$$

$$\mathbb{P}(R \leq t) = \frac{2}{\pi} \arctan \sqrt{t-1} \quad \text{for } 1 \leq t < \infty,$$

see [1, Sect. 7.4, Example 4.4].

Let us calculate the density:

$$\frac{d}{dt} \mathbb{P}(T_x \leq t) = 2 \frac{d}{dt} \int_{x/\sqrt{t}}^{\infty} p_1(y) \, dy = -2p_1 \left( \frac{x}{\sqrt{t}} \right) \cdot x \cdot \left( -\frac{1}{2} t^{-3/2} \right) = \frac{x}{t^{3/2}} p_1 \left( \frac{x}{\sqrt{t}} \right) = \frac{x}{\sqrt{2\pi}} \frac{x}{t^{3/2}} \exp \left( -\frac{x^2}{2t} \right);$$

the derivative is continuous on $[0,\infty)$ (in spite of $t$ in the denominator); we got the density (of the distribution) of $T_x$. Note that $\mathbb{E} T_x = \infty$. Interestingly, $T_x$ is distributed like the special Lévy process at time $x$. 

### Footnotes

2b10 Exercise. For all \( x, y \in (0, \infty) \),
\[
T_{x+y} - T_x \text{ is independent of } T_x \text{ and distributed like } T_y.
\]
Prove it.

The formula \( p_{s+t} = p_s * p_t \) for \( p_t(x) = \frac{t}{\sqrt{2\pi x^{3/2}}} \exp\left(-\frac{t^2}{2x}\right) \), claimed in 1a without proof, follows from 2b10. Similarly to 2b10, the process \((T_x)_{x \in [0, \infty)}\) has stationary independent increments. Also, its sample functions are continuous from the left (think, why).

The random function \((T_x)_{x \in [0, \infty)}\) is distributed as the left-continuous modification of the special Lévy process.

See also [1, Sect. 7.4].

But wait, we did not prove 2b2 yet...

2c Delayed restart

An important step toward the proof of Prop. 2b2 is made here. Instead of the random time \( T_1 \) taking on a continuum of values we introduce (for a given \( n \)) a random time \( \tau_n \) with a finite number of values,

\[
\tau_n = \frac{k}{2^n} \quad \text{whenever } \frac{k-1}{2^n} < T_1 \leq \frac{k}{2^n} \quad \text{for } k = 1, 2, \ldots, 2^n;
\]

\( \tau_n = \infty \) whenever \( T_1 > 2^n \).

Clearly, \( \tau_n \downarrow T_1 \) a.s., as \( n \to \infty \).

Similarly to 2b1, we restart at \( \tau_n \).

\[
Y_n(t)(\omega_1, \omega_2) = \begin{cases} B(t)(\omega_1) & \text{for } t \leq \tau_n(\omega_1), \\ B(\tau_n(\omega_1))(\omega_1) + B(t - \tau_n(\omega_1))(\omega_2) & \text{for } t \geq \tau_n(\omega_1). \end{cases}
\]

Similarly to 2b2, we claim the following.

2c3 Lemma. For every \( n \) the random function \( Y_n \) is distributed like \( B \).

The proof will be given in 2e. Now we’ll deduce 2b2 from 2c3.

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1. “It may be difficult for the novice to appreciate the fact that twenty five years ago a formal proof of the strong Markov property was a major event.” Kai Lai Chung, John B. Walsh, “Markov processes, Brownian motion, and time symmetry”, second edition, Springer (1982 and) 2005; see page 73.
Proof of (2b2) (assuming (2c3)). The random function $Y$ defined by (2b1) is evidently continuous. In order to prove that $Y$ is distributed like $B$ it is sufficient to check that $(Y(t_1), \ldots, Y(t_j)) \sim (B(t_1), \ldots, B(t_j))$ for all $j$ and $t_1, \ldots, t_j \in (0, \infty)$. To this end it is sufficient to check that

$$(2c4)\quad E\varphi(Y(t_1), \ldots, Y(t_j)) = E\varphi(B(t_1), \ldots, B(t_j))$$

for every $j$ and every bounded continuous $\varphi : \mathbb{R}^j \to \mathbb{R}$.

By (2c3),

$$(2c5)\quad E\varphi(Y_n(t_1), \ldots, Y_n(t_j)) = E\varphi(B(t_1), \ldots, B(t_j))$$

for all $n$. As $n \to \infty$, we have (for almost all $\omega_1, \omega_2$)

$$\tau_n(\omega_1) \downarrow T_1(\omega_1);$$
$$B(\tau_n(\omega_1))(\omega_1) \to B(T_1(\omega_1))(\omega_1) = 1;$$
$$t - \tau_n(\omega_1) \to t - T_1(\omega_1);$$
$$B(t - \tau_n(\omega_1))(\omega_2) \to B(t - T_1(\omega_1))(\omega_2);$$
$$Y_n(t)(\omega_1, \omega_2) \to Y(t)(\omega_1, \omega_2)$$

for $t \geq T_1(\omega_1)$. And clearly $Y_n(t)(\omega_1, \omega_2) = B(t)(\omega_1) = Y(t)(\omega_1, \omega_2)$ for $t < T_1(\omega_1)$, if $n$ is large enough. Thus, $Y_n(t) \to Y(t)$ a.s. (for each $t$); therefore

$$E\varphi(Y_n(t_1), \ldots, Y_n(t_j)) \to E\varphi(Y(t_1), \ldots, Y(t_j))$$

by the bounded convergence theorem. In combination with (2c5) it gives (2c4).

2d Maybe restart, maybe not

Here we prove Lemma (2c3) for the simplest case, $n = 0$. (Be careful, mind (2b7)!) By (2c1),

$$\tau_0 = \begin{cases} 1 & \text{if } T_1 \leq 1, \\ \infty & \text{if } T_1 > 1. \end{cases}$$

By (2c2),

$$Y_0(t)(\omega_1, \omega_3) = \begin{cases} B(t)(\omega_1) & \text{if } t \leq 1, \\ B(t)(\omega_1) & \text{if } t \geq 1 \text{ and } \max_{[0,1]} B(\cdot)(\omega_1) < 1, \\ B(1)(\omega_1) + B(t - 1)(\omega_3) & \text{if } t \geq 1 \text{ and } \max_{[0,1]} B(\cdot)(\omega_1) \geq 1. \end{cases}$$

\footnote{Why $\omega_3$? Wait a little...}
We want to prove that $Y_0 \sim B$. The distribution of $Y_0$ does not change if we replace $B$ with another process $X$ distributed like $B$. We choose (recall (2a1))

\[ X(t)(\omega_1, \omega_2) = \begin{cases} B(t)(\omega_1) & \text{for } t \leq 1, \\ B(1)(\omega_1) + B(t - 1)(\omega_2) & \text{for } t \geq 1 \end{cases} \]

and consider

\[ X(t)(\omega_1, \omega_2, \omega_3) = \begin{cases} X(t)(\omega_1, \omega_2) & \text{if } t \leq 1, \\ X(t)(\omega_1, \omega_2) & \text{if } t \geq 1 \text{ and } \max_{0,1} X(\cdot)(\omega_1, \omega_2) < 1, \\ X(1)(\omega_1, \omega_2) + B(t - 1)(\omega_3) & \text{if } t \geq 1 \text{ and } \max_{0,1} X(\cdot)(\omega_1, \omega_2) \geq 1. \end{cases} \]

Similarly to 2a4\(^1\), $Y$ is distributed like $Y_0$. We have

\[ Y(t)(\omega_1, \omega_2, \omega_3) = \begin{cases} B(t)(\omega_1) & \text{if } t \leq 1, \\ B(1)(\omega_1) + B(t - 1)(\omega_2) & \text{if } t \geq 1 \text{ and } \omega_1 \in A, \\ B(1)(\omega_1) + B(t - 1)(\omega_3) & \text{if } t \geq 1 \text{ and } \omega_1 /\in A, \end{cases} \]

where $A = \{ \omega_1 : \max_{0,1} B(\cdot)(\omega_1) < 1 \}$.

2d1 Exercise. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $A \subset \Omega$ a measurable set, $f : \Omega^2 \to \mathbb{R}$ a bounded measurable function. Define $g : \Omega^3 \to \mathbb{R}$ by

\[ g(\omega_1, \omega_2, \omega_3) = \begin{cases} f(\omega_1, \omega_2) & \text{if } \omega_1 \in A, \\ f(\omega_1, \omega_3) & \text{if } \omega_1 /\in A. \end{cases} \]

Then

\[ \int\int\int_{\Omega^3} g \, d(P \times P \times P) = \int\int_{\Omega^2} f \, d(P \times P). \]

Prove it.

It follows that $Y$ is distributed like $X$, therefore, like $B$, which proves Lemma 2c3 for $n = 0$.

2e The proof, at last

If two non-overlapping changes are separately harmless, then they are jointly harmless in the following sense.

\(^1\)See also 2f4
2e1 Exercise. (a) Let $X, Y_1, Y_2$ be identically distributed random variables (on a probability space) such that $\mathbb{P}(Y_1 \neq X \text{ and } Y_2 \neq X) = 0$. Then the random variable $Z$ defined by

$$Z = \begin{cases} 
X & \text{if } Y_1 = X \text{ and } Y_2 = X, \\
Y_1 & \text{if } Y_1 \neq X \text{ and } Y_2 = X, \\
Y_2 & \text{if } Y_1 = X \text{ and } Y_2 \neq X
\end{cases}$$

is distributed like $X$.

(b) The same holds for random vectors and random continuous functions. Prove it.

The same holds for any finite (or countable) collection of pairwise non-overlapping changes.

Proof of 2e1. We consider random continuous functions

$$Y_{n,k}(t)(\omega_1, \omega_2) = \begin{cases} 
B(t)(\omega_1) & \text{if } t \leq k \cdot 2^{-n}, \\
B(t)(\omega_1) & \text{if } t \geq k \cdot 2^{-n} \text{ and } \tau_n(\omega_1) \neq k \cdot 2^{-n}, \\
B(k \cdot 2^{-n})(\omega_1) + B(t - k \cdot 2^{-n})(\omega_2) & \text{if } t \geq k \cdot 2^{-n} \text{ and } \tau_n(\omega_1) = k \cdot 2^{-n}.
\end{cases}$$

Each $Y_{n,k}$ is distributed like $B$ by the argument of 2d. It remains to apply 2e1. □

2f Technicalities: sigma-fields and stopping times

The Borel $\sigma$-field\(^1\) $\mathcal{B}$ on the space $C[0, 1]$ of all continuous functions $[0, 1] \to \mathbb{R}$ can be defined in many equivalent ways; here is the best one for our purposes:

$$\mathcal{B} \text{ is generated by the functions}$$

(2f1) $C[0, 1] \ni f \mapsto f(t) \in \mathbb{R}$

where $t$ runs over $[0, 1]$.

2f2 Exercise. Prove that each of the following four sets of functions $C[0, 1] \to \mathbb{R}$ generates the Borel $\sigma$-field:

(a) $f \mapsto f(t)$ for rational $t \in [0, 1]$;
(b) $f \mapsto \max_{[a,b]} f(\cdot)$ for $[a, b] \subset [0, 1]$;
(c) $f \mapsto \int_a^b f(x) \, dx$ for $[a, b] \subset [0, 1]$;
(d) $f \mapsto \|f - g\|$ for $g \in C[0, 1]$.

Prove it.

\(^1\)In other words, “$\sigma$-algebra”.

It follows easily from (d) that the Borel $\sigma$-field is generated by open (or closed) balls, as well as by open (or closed) sets.

For any $t \in [0, \infty)$ the Borel $\sigma$-field on $C[0, t]$ is defined similarly.

Now, for a given $t \in [0, \infty)$ we define a $\sigma$-field $\mathcal{B}_t$ on the set $C[0, \infty)$ of all continuous (not necessarily bounded) functions $[0, \infty) \to \mathbb{R}$ as consisting of inverse images of all Borel subsets of $C[0, t]$ under the restriction map

$$C[0, \infty) \ni f \mapsto f|_{[0,t]} \in C[0, t].$$

Clearly, $\mathcal{B}_t$ is generated by the functions

$$C[0, \infty) \ni f \mapsto f(s) \in \mathbb{R}$$

for $s \in [0, t]$.

The $\sigma$-field generated by $\bigcup_t \mathcal{B}_t$ will be denoted by $\mathcal{B}_\infty$ and called the Borel $\sigma$-field of $C[0, \infty)$. Clearly, $\mathcal{B}_\infty$ is generated by the functions

$$C[0, \infty) \ni f \mapsto f(t) \in \mathbb{R}$$

for $t \in [0, \infty)$.

Here are two equivalent definitions of a random continuous function.

**Exercise.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then the following two conditions on a function $X : \Omega \to C[0, \infty)$ are equivalent:

(a) for each $t \in [0, \infty)$ the function

$$\Omega \ni \omega \mapsto X(t)(\omega)$$

is $\mathcal{F}$-measurable;

(b) for each $\mathcal{B}_\infty$-measurable function $\varphi : C[0, \infty) \to \mathbb{R}$, the function

$$\Omega \ni \omega \mapsto \varphi(X(\cdot)(\omega))$$

is $\mathcal{F}$-measurable.

Prove it.

For the next exercise you need something like the monotone class theorem or Dynkin’s $\pi – \lambda$ theorem; see [1, Appendix A2, (2.1) and (2.2)].

Here are two equivalent definitions of identically distributed random continuous functions.

**Exercise.** The following two conditions on random continuous functions$^1$

$X, Y$ are equivalent:

$^1$Maybe, on different probability spaces.
(a) for every $n$ and $t_1, \ldots, t_n \in [0, \infty)$ the random vectors $(X(t_1), \ldots, X(t_n))$ and $(Y(t_1), \ldots, Y(t_n))$ are identically distributed;

(b) for every $\mathcal{B}_\infty$-measurable function $\varphi : C[0, \infty) \to \mathbb{R}$ the random variables $\varphi(X(\cdot))$ and $\varphi(Y(\cdot))$ are identically distributed.

Prove it.

\textbf{2f5 Definition.} A \textit{stopping time} is a function $T : C[0, \infty) \to [0, \infty]$ such that

$$\{ f \in C[0, \infty) : T(f) \leq t \} \in \mathcal{B}_t$$

for all $t \in [0, \infty)$.

\textbf{2f6 Exercise.} The hitting time $T_1$ defined by

$$T_1(f) = \inf\{ t : f(t) = 1 \}$$

($\infty$, if the set is empty) is a stopping time.

Prove it.

\textbf{2f7 Exercise.} The function $L$ defined by

$$L(f) = \sup\{ t \in [0, 1] : f(t) = 0 \}$$

(0, if the set is empty) is not a stopping time.

Prove it.

Here is the strong Markov property of the Brownian motion.

\textbf{2f8 Theorem.} If $T$ is a stopping time then the random function

$$Y(t)(\omega_1, \omega_2) = \begin{cases} B(t)(\omega_1) & \text{for } t \leq T(\omega_1), \\ B(T(\omega_1))(\omega_1) + B(t - T(\omega_1))(\omega_2) & \text{for } t \geq T(\omega_1) \end{cases}$$

on $\Omega \times \Omega$ is distributed like the Brownian motion $B$.

The proof is quite similar to the proof of \textbf{2f2}.

\textbf{2f9 Remark.} A weaker (than \textbf{2f5}) assumption

$$\{ f \in C[0, \infty) : T(f) < t \} \in \mathcal{B}_t \quad \text{for all } t \in [0, \infty)$$

is still sufficient for Theorem \textbf{2f8} to hold.
Anyway, a delay is stipulated by the proof, recall 2c. Such $T$ is called a stopping time of the (right-continuous) filtration $(B_{t+})_t$, where $B_{t+} = \cap_{\varepsilon>0} B_{t+\varepsilon}$. In contrast, 2f5 defines a stopping time of the filtration $(B_t)_t$. (Generally, a filtration is defined as an increasing family of $\sigma$-fields.)

Here is an example of a stopping time of $(B_{t+})_t$ but not $(B_t)_t$:

$$T_{1+} = \inf \{ t : B(t) > 1 \}.$$ 

Note that $T_t \downarrow T_{1+}$ as $t \downarrow 1$. Similarly, $T_{x+}$ are introduced for all $x \in [0, \infty)$. Due to 2f9, all said in 2b about the process $(T_x)_{x \in [0, \infty)}$ holds also for $(T_{x+})_{x \in [0, \infty)}$, except for the left continuity; this time we get right continuity.

The random function $(T_{x+})_{x \in [0, \infty)}$ is distributed as the special Lévy process.

2f10 Exercise. $\mathbb{P}(T_x = T_{x+}) = 1$ for each $x \in [0, \infty)$. Prove it.

2g Hints to exercises

2a2 Calculate the joint distribution of $Y(t_1), Y(t_2) - Y(t_1), \ldots, Y(t_n) - Y(t_{n-1})$ assuming that $t_1 < \cdots < t_n$ and $1 \in \{t_1, \ldots, t_n\}$.

2a4 $\{\omega : T_x > t\} = \{\omega : \sup_{[0,t]} B(\cdot) < x\}$.

2a5 Use 1c2.

2b3 $\mathbb{P}\left(\max_{[0,t]} B(\cdot) \geq 1\right) = \mathbb{P}(T_1 \leq t)$; use 2b2.

2b5 $\mathbb{P}(T_x < \infty) = 1$.

2b6 $\lim_{x \to 0^+} \mathbb{P}(T_x < \varepsilon) = ?$

2b7 Use 2b6.

2b10 Use 2b5.

2d1 Fubini theorem.

2a1 (a), (b), (c): if $\varphi_n$ are measurable (w.r.t. a given $\sigma$-field) and $\varphi_n \to \varphi$ pointwise, then $\varphi$ is also measurable. (d): take a sequence $(g_n)_n$ dense in $C[0,1]$ and note that $\sup_{n: \|f - g_n\|<1} g_n(t) = f(t) + 1$.

2b8 (a) $\implies$ (b): all sets $A \subset C[0, \infty)$ such that $X^{-1}(A) \in \mathcal{F}$ are a $\sigma$-field.

2f10 $T_x$ and $T_{x+}$ are identically distributed, and $T_x \leq T_{x+}$.

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