5 Localization

5a Heat equation localized

5a1 Lemma. Let $K \subset [0, \infty) \times \mathbb{R}$ be a compact set, and $T$ a stopping time such that

$$
P \left( \forall t \ (t \wedge T, B(t \wedge T)) \in K \right) = 1.
$$

Let $G \subset [0, \infty) \times \mathbb{R}$ be a relatively open set, $G \supset K$, and $u : G \to \mathbb{R}$ a continuous function having continuous derivatives $u_{1,0}, u_{0,1}, u_{0,2}$ and satisfying the PDE $u_{1,0} + \frac{1}{2} u_{0,2} = 0$. Then the following process is a martingale:

$$
M(t) = u(t \wedge T, B(t \wedge T)).
$$

The proof will be given after some preparation. If $T$ is a stopping time such that $\exists t \ P (T \leq t) = 1$, then (recall 3a5 and 4a10)

$$
\mathbb{E} B(T) = 0, \quad \mathbb{E} B^2(T) = \mathbb{E} T.
$$

5a3 Exercise. Let $T$ be a stopping time, $f \in C[0, \infty)$, and $t \in [0, T(f)]$. Then the function

$$
g \mapsto T(f \uplus g) - t
$$

is a stopping time.

Prove it.

---

1. It is evidently equivalent to $\forall t \ P \left( (t \wedge T, B(t \wedge T)) \in K \right) = 1$. As before, $t \wedge T$ means $\min(t, T)$.
2. Just the intersection of the closed half-plane $[0, \infty) \times \mathbb{R}$ and an open subset of the plane $\mathbb{R}^2$.
3. As before, $f_{i,j}(t, x) = \frac{\partial^{i+j}}{\partial t^i \partial x^j} f(t, x)$. 

5a4 Exercise. Let $T_1, T_2$ be stopping times, $\exists t \mathbb{P}(T_2 \leq t) = 1$. Then the equalities

\begin{align*}
(5a5) \quad & \mathbb{E}(B(T_2) - B(T_1) | \mathcal{F}_{T_1}) = 0, \\
(5a6) \quad & \mathbb{E}((B(T_2) - B(T_1))^2 | \mathcal{F}_{T_1}) = \mathbb{E}(T_2 - T_1 | \mathcal{F}_{T_1})
\end{align*}

hold almost surely on the event \{ $T_1 \leq T_2$ \}.

Prove it.

Proof of Lemma 5a4. Denote $M(t) = u(t \wedge T, B(t \wedge T))$. We have to prove that $\mathbb{E}(M(t) | \mathcal{F}_s) = M(s)$ for $s \leq t$. It is sufficient to prove that

\begin{equation}
(5a7) \quad \mathbb{E}(M(t) | \mathcal{F}_s) - M(s) = o(t - s) \quad \text{a.s. for } s \leq t;
\end{equation}

here and henceforth all $o(\ldots)$ are uniform (in everything; this time, in $s, t, \omega$). Here is why (5a7) is sufficient:

$$
\mathbb{E}(\mathbb{E}(M(t + \varepsilon) | \mathcal{F}_s) | \mathcal{F}_s) = \mathbb{E}(M(t + \varepsilon) | \mathcal{F}_s)
$$

(think, why), thus,

$$
|\mathbb{E}(M(t + \varepsilon) | \mathcal{F}_s) - \mathbb{E}(M(t) | \mathcal{F}_s)| = |\mathbb{E}(\mathbb{E}(M(t + \varepsilon) | \mathcal{F}_s) - M(t) | \mathcal{F}_s)| \leq
\leq \mathbb{E}(|\mathbb{E}(M(t + \varepsilon) | \mathcal{F}_s) - M(t) | \mathcal{F}_s) = \mathbb{E}(o(\varepsilon) | \mathcal{F}_s) = o(\varepsilon).
$$

It remains to prove (5a7).

On the event \{ $T < s$ \} we have

$$
\mathbb{E}(M(t) | \mathcal{F}_s) - M(s) = \mathbb{E}(u(T, B(T)) | \mathcal{F}_s) - u(T, B(T)) = 0 \quad \text{a.s.},
$$

thus, it is sufficient to prove (5a7) on the event \{ $T \geq s$ \}. From now on we assume $T \geq s$.

We define $R$ by

$$
u(t \wedge T, B(t \wedge T)) - u(s, B(s)) = u_{1,0}(s, B(s))(t \wedge T - s) +
+ u_{0,1}(s, B(s))(B(t \wedge T) - B(s)) + \frac{1}{2} u_{0,2}(s, B(s))(B(t \wedge T) - B(s))^2 + R,
$$

take $\mathbb{E}(\ldots | \mathcal{F}_s)$, use (5a5), (5a6) and get

\begin{align*}
\mathbb{E}(M(t) | \mathcal{F}_s) - M(s) &= u_{1,0}(s, B(s))\mathbb{E}(t \wedge T - s | \mathcal{F}_s) +
+ u_{0,1}(s, B(s)) \cdot 0 + \frac{1}{2} u_{0,2}(s, B(s)) \cdot \mathbb{E}(t \wedge T - s | \mathcal{F}_s) + \mathbb{E}(R | \mathcal{F}_s) =
= \left(u_{1,0} + \frac{1}{2} u_{0,2}\right)(s, B(s)) \cdot \mathbb{E}(t \wedge T - s | \mathcal{F}_s) + \mathbb{E}(R | \mathcal{F}_s) = \mathbb{E}(R | \mathcal{F}_s);
\end{align*}
it remains to check that $\mathbb{E}(R | \mathcal{F}_s) = o(t - s)$.

We have

$$R = o(t \wedge T - s) + o((B(t \wedge T) - B(s))^2);$$

$o(\ldots)$ are uniform in $s, t, \omega$ since $u_{1,0}$ and $u_{0,2}$ are uniformly continuous on $K$. Clearly, $t \wedge T - s \leq t - s$. It remains to prove that

$$\int_{-\infty}^{+\infty} (o(x^2) \wedge C) p_\varepsilon(x) \, dx = o(\varepsilon).$$

The integral over $\mathbb{R} \setminus [-\delta, \delta]$ is exponentially small. The integral over $[-\delta, \delta]$ is much smaller than $\int x^2 p_\varepsilon(x) \, dx = \varepsilon$ if $\delta$ is small enough.

We may generalize 5a1 in the spirit of 4b10.

**5a8 Lemma.** Let $K \subset [0, \infty) \times \mathbb{R}$ be a compact set, and $T$ a stopping time such that

$$\mathbb{P}(\forall t \quad (t \wedge T, B(t \wedge T)) \in K) = 1.$$

Let $G \subset [0, \infty) \times \mathbb{R}$ be a relatively open set, $G \supset K$, and $u : G \to \mathbb{R}$ a continuous function having continuous derivatives $u_{1,0}, u_{0,1}, u_{0,2}$. Then the following process is a martingale:

$$M(t) = u(t \wedge T, B(t \wedge T)) - \int_0^{t \wedge T} v(s, B(s)) \, ds,$$

where $v = u_{1,0} + \frac{1}{2} u_{0,2}$.

**Proof.** Similarly to the proof of 5a1, we get

$$\mathbb{E}(M(t) | \mathcal{F}_s) - M(s) =$$

$$= v(s, B(s)) \cdot \mathbb{E}(t \wedge T - s | \mathcal{F}_s) + \mathbb{E}(R | \mathcal{F}_s) - \mathbb{E}\left(\int_s^{t \wedge T} v(r, B(r)) \, dr | \mathcal{F}_s\right) =$$

$$= \mathbb{E}(R | \mathcal{F}_s) - \mathbb{E}\left(\int_s^{t \wedge T} (v(r, B(r)) - v(s, B(s))) \, dr | \mathcal{F}_s\right).$$

By the uniform continuity of $v$ on $K$, for every $\varepsilon$ there exists $\delta$ such that $|v(r, B(r)) - v(s, B(s))| \leq \varepsilon$ whenever $|r - s| \leq \delta$ and $|B(r) - B(s)| \leq \delta$. Assuming $t - s \leq \delta$ we have

$$\mathbb{E}\left(\int_s^{t \wedge T} |v(r, B(r)) - v(s, B(s))| \, dr | \mathcal{F}_s\right) \leq$$

$$\leq \varepsilon(t \wedge T - s) + 2 \max_K |v(\cdot)| \mathbb{P}\left(\max_{[s,t \wedge T]} |B(\cdot) - B(s)| > \delta | \mathcal{F}_s\right) \leq$$

$$\leq \varepsilon(t - s) + o(t - s).$$

Therefore it is $o(t - s).$
5b Local martingales

5b1 Definition. A Brownian local martingale is a random continuous function \((M_t)_{t \in [0, \infty)}\) (on a probability space carrying a Brownian motion \((B_t)_{t}\)) such that there exists a sequence of stopping times \(T_1, T_2, \ldots\) (so-called localizing sequence) satisfying

\[
T_n \uparrow +\infty \quad \text{a.s.;}
\]
\[
(M_{t \wedge T_n})_t \quad \text{is a Brownian martingale (for each } n).\]

5b2 Proposition. Let \(u : [0, \infty) \times \mathbb{R} \to \mathbb{R}\) be a continuous function having continuous derivatives \(u_{1,0}, u_{0,1}, u_{0,2}\). Then the following process is a Brownian local martingale:

\[
M(t) = u(t, B(t)) - \int_0^t v(s, B(s)) \, ds,
\]

where \(v = u_{1,0} + \frac{1}{2} u_{0,2}\).

Proof. Let \(T_n = \inf\{t : (t, B(t)) \notin [0, n) \times (-n, n)\}\), then clearly \(T_n \uparrow \infty\), and \((M(t \wedge T_n))_t\) is a martingale by Lemma 5a8. \(\square\)

5b3 Corollary. Let \(u\) satisfy the conditions of Prop. 5b2 and the PDE \(u_{1,0} + \frac{1}{2} u_{0,2} = 0\). Then the process \(M(t) = u(t, B(t))\) is a local martingale.

Recall Tychonoff’s counterexample mentioned in 4a (after 4a12); it is a function that satisfies the PDE (4a4) but violates (4a1). By 5b3 it leads to a local martingale that is not a martingale. Somehow, the expectation escapes to the spatial infinity when \(t \to 1\). \(\square\)

5b4 Exercise. The following is a local martingale but not a martingale:

\[
M(t) = \begin{cases} 
  p_{1-t}(B(t)) & \text{for } t \in [0, 1), \\
  0 & \text{for } t \in [1, \infty).
\end{cases}
\]

Prove it.

---

1This is a local martingale w.r.t. the Brownian filtration \((\mathcal{F}_t)_t\). Generally, a local martingale w.r.t. a given filtration is defined similarly, but need not be continuous (rather, r.c.l.l.). I often omit the word ‘Brownian’.

2In reversed time, heat comes from the spatial infinity by a giant fast oscillating heat wave. A terrible spectacle!

3As before, \(p_t(x) = (2\pi t)^{-1/2} \exp(-\frac{x^2}{2t})\).
5b5 Proposition. Let \((M_t)_t\) be a local martingale, \((T_n)_n\) a localizing sequence, and
\[\sup_n \mathbb{E} M^2_{t \wedge T_n} < \infty\] for all \(t\).
Then \((M_t)_t\) is a martingale.

5b6 Corollary. A local martingale \((M_t)_t\) satisfying
\[\mathbb{E} \max_{s \in [0,t]} M^2_s < \infty\] for all \(t\)
is a martingale.

5b7 Exercise. Prove that
\[\|M_{t \wedge T_n + k} - M_{t \wedge T_n}\|_1 \leq 2\sqrt{\mathbb{P}(T_n < t)} \left(\|M_{t \wedge T_n + k}\|_2 + \|M_{t \wedge T_n}\|_2\right)\].

5b8 Exercise. Prove that \(M_t \in L_1\) and \(M_{t \wedge T_n} \to M_t\) in \(L_1\) as \(n \to \infty\).

5b9 Exercise. Prove Prop. 5b5.

The condition \(\mathbb{E} M^2_t < \infty\) on a local martingale does not guarantee that it is a martingale! This condition fails for 5a10 (and Tychonoff’s counterexample), however, later (in Sect. 6c) we’ll see a local martingale \(M(\cdot)\) satisfying
\[\sup_{t \in [0,\infty]} \mathbb{E} e^{\|M(t)\|} < \infty\] but still not a martingale.\(^1\)

5c Heat equation revisited

5c1 Theorem. \(^2\) Let \(u\) satisfy the conditions of Prop. 5b2. Assume that
\[\frac{1}{x^2} \ln^+ |u(t,x)| \to 0 \quad \text{as } x \to \pm \infty,\]
\[\frac{1}{x^2} \ln^+ |v(t,x)| \to 0 \quad \text{as } x \to \pm \infty\]
uniformly in \(t \in [0,b]\) for every \(b\); here \(v = u_{1,0} + \frac{1}{2} u_{0,2}\), that is,
\[v(t,x) = \left(\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2}\right) u(t,x)\].
Then the following process is a Brownian martingale:
\[M(t) = u(t, B(t)) - \int_0^t v(s, B(s)) \, ds\].

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\(^1\) "... we stress the fact that local martingales are \textit{much more general} than martingales and warn the reader against the common mistaken belief that local martingales need only be integrable in order to be martingales." [1] page 117.

\(^2\) See also [2], p. 36.
Theorem 4b10 is thus generalized; the condition $\frac{1}{x} \ln^+ |u_{i,j}(t,x)| \to 0$ appears to be unnecessary (unless $(i,j) = (0,0)$).

Note especially the case $v = 0$.

Also Prop. 4a9 is now generalized: Condition (4a1) is equivalent to the PDE (4a4) for all functions satisfying the conditions of Theorem 5c1.

Proof of Theorem 5c1. Let $T_n = \inf \{ t : (t, B(t)) \notin [0, n) \times (-n,n) \}$ (as in the proof of 5b2). We have

$$\mathbb{P} \left( \max_{[0,t]} |B(\cdot)| \geq c \right) \leq 2 \mathbb{P} \left( \max_{[0,t]} B(\cdot) \geq c \right) = 2 \mathbb{P} \left( |B(t)| \geq c \right);$$

and we may choose $\delta > 0$ at will. Thus,

$$\mathbb{E} \max_{[0,t]} |B(\cdot)| \leq C_\delta \exp(\delta B^2(t)).$$

5d Finite lifetime

5d1 Definition. Let $T$ be a stopping time. A random continuous function on $[0, T)$ is a function $^1X : \{(t, \omega) \in [0, \infty) \times \Omega : t < T(\omega) \} \to \mathbb{R}$ such that for every $t$ the function $X(t, \cdot)$ on $\{\omega : T(\omega) > t \}$ is measurable, and for almost every $\omega$ the function $X(\cdot, \omega)$ on $[0, T(\omega))$ is continuous.

5d2 Definition. A random continuous function on $[0, T)$ is a Brownian local martingale$^2$ on $[0, T)$ if there exists a sequence of stopping times $T_1, T_2, \ldots$ (called localizing sequence) satisfying

$$T_n < T \quad \text{and} \quad T_n \uparrow T \quad \text{a.s.};$$

$$(M_{1\wedge T_n})_t \quad \text{is a Brownian martingale (for each } n).$$

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$^1$Or rather, equivalence class.

$^2$A martingale, in contrast to a local martingale, is defined on the whole $[0, \infty)$. 
**5d3 Proposition.** Let \( G \subset [0, \infty) \times \mathbb{R} \) be a relatively open set, \( T \) a stopping time,
\[
\mathbb{P}\left( \forall t \in [0, T) \ (t, B(t)) \in G \right) = 1,
\]
\( u : G \to \mathbb{R} \) a continuous function having continuous derivatives \( u_{1,0}, u_{0,1}, u_{0,2} \).
Then the following process is a Brownian local martingale on \([0, T)\):
\[
M(t) = u(t, B(t)) - \int_0^t v(s, B(s)) \, ds \quad \text{for } t \in [0, T),
\]
where \( v = u_{1,0} + \frac{1}{2} u_{0,2} \).

**Proof.** We take relatively open sets \( G_1 \subset G_2 \subset \cdots \subset G \) such that \((0, 0) \in G_1, G_1 \cup G_2 \cup \cdots = G \) and the closure \( \overline{G}_n \) of \( G_n \) is a compact subset of \( G \) (for each \( n \)).\(^1\) We define stopping times \( T_n = \inf\{ t : t \geq T \text{ or } (t, B(t)) \notin G_n \} \) and observe that \( T_n \uparrow T \) a.s. (since otherwise a compact curve is included in \( G \) but not in any \( G_n \)). By Lemma 5a8 (applied to \( \overline{G}_n \) and \( T_n \)) the process \( t \mapsto M(t \wedge T_n) \) is a martingale. \( \square \)

### 5a3 Hints to exercises

- **5a3** recall Def. 2f5.
- **5a4** use 5a3.
- **5b4** The closed set \( \{(t, B(t)) : t \in [0, \infty)\} \) a.s. does not contain \((1, 0)\).
- **5b7** \( \|I_{T_n \leq t}\|_2 = \sqrt{\mathbb{P}(T_n \leq t)} \).
- **5b8** \( M_{t \wedge T_n} \) converges to something in \( L_1 \), and to \( M_t \) a.s.
- **5b9** \( \mathbb{E}(M_{t \wedge T_n} | \mathcal{F}_s) \to \mathbb{E}(M_t | \mathcal{F}_s) \) in \( L_1 \).

### References


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\(^1\)For example: \( G_n \) consists of all points \((t, x) \in G \) such that \( t < n, |x| < n \), and the closed \( 1/n \)-neighborhood of \((t, x) \) in \([0, \infty) \times \mathbb{R} \) is contained in \( G \).