4 Extrema

4a Extrema of a random function

4b Gaussian case

4c Natural parameter

4d Some integral geometry

4a Extrema of a random function

Let $\mu$ be a probability measure on the space $C^2[a, b]$ of twice continuously differentiable functions. Two assumptions on $\mu$ are introduced below (similarly to 3c).

The first assumption: for each $x \in [a, b]$ the joint distribution of $f(x)$, $f'(x)$, $f''(x)$ has a density $p_x$:

$$\int \varphi(y, y', y'') p_x(y, y', y'') \, dy' \, dy'' = \int_{C^2[a, b]} \varphi(f(x), f'(x), f''(x)) \, \mu(df)$$

for every bounded Borel function $\varphi : \mathbb{R}^3 \to \mathbb{R}$. (Once again, the function $(x, y, y', y'') \mapsto p_x(y, y', y'')$ on $[a, b] \times \mathbb{R}^3$ may be chosen to be measurable.)

The second assumption:

$$\int\int_{[a, b] \times C^2[a, b]} |f''(x)| \, dx \, \mu(df) < \infty.$$

Once again, $\mu$ can be an arbitrary nondegenerate Gaussian measure on the (finite-dimensional linear) space of trigonometric (or algebraic) polynomials of degree $n$ (provided that its dimension is at least 3; the toy model (3a1) does not fit, but see 4a7 and notes after it).

4a3 Exercise. For every bounded Borel functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ and every $f \in C^2[a, b]$,

$$\int \psi(y') \sum_{x: f'(x) = y'} \varphi(f(x)) \mathrm{sgn} f''(x) = \int_a^b dx \psi(f'(x)) \varphi(f(x)) f''(x).$$

Prove it.

Hint: (3b7) for $f'$ and $\psi(f'(\cdot)) \varphi(f(\cdot))$ instead of $f$ and $g$. 
4a4 Exercise. For all bounded Borel functions $\varphi, \psi : \mathbb{R} \rightarrow \mathbb{R},$

$$\int dy' \psi(y') E \sum_{x : f'(x) = y'} \varphi(f(x)) \text{sgn } f''(x) =$$

$$= \int_a^b dx \int \int dy dy'' p_x(y, y', y'') \psi(y') \varphi(y) y'' .$$

Prove it.

Hint: 4a3 and Fubini (and do not forget integrability).

4a5 Exercise. For all bounded Borel functions $\varphi : \mathbb{R} \rightarrow \mathbb{R},$

$$E \sum_{x : f'(x) = y'} \varphi(f(x)) \text{sgn } f''(x) = \int_a^b dx \int dy \varphi(y) \int dy'' p_x(y, y', y'') y''$$

for almost all $y' \in \mathbb{R}.$

Prove it.

Similarly, one may get (if needed)

$$E \sum_{x : f'(x) = y'} \varphi(f(x)) = \int_a^b dx \int dy \varphi(y) \int dy'' p_x(y, y', y'') |y''| .$$

In terms of marginal and conditional densities

$$p_x(y, y') = \int dy'' p_x(y, y', y''), \quad p_x(y'', y') = \frac{p_x(y, y', y'')}{p_x(y, y')} ,$$

$$p_x(y') = \int dy p_x(y, y'), \quad p_x(y | y') = \frac{p_x(y, y')}{p_x(y')}$$

and the conditional expectation

$$E \left( f''(x) \middle| f(x) = y, f'(x) = y' \right) = \int dy'' p_x(y'' | y, y') y''$$

we have

$$\int dy'' p_x(y, y', y'') y'' = p_x(y, y') E \left( f''(x) \middle| f(x) = y, f'(x) = y' \right) ;$$

4a5 becomes

$$4a6 \quad E \sum_{x : f'(x) = y'} \varphi(f(x)) \text{sgn } f''(x) =$$

$$= \int_a^b dx p_x(y') \int dy p_x(y | y') \varphi(y) E \left( f''(x) \middle| f(x) = y, f'(x) = y' \right)$$

for almost all $y'.$
4a7 Exercise. Prove (4a6) assuming less than (4a1), namely, existence of the joint density $p_x(y, y')$ of $f(x)$, $f'(x)$ and the regression function $(y, y') \mapsto \mathbb{E}\left(f''(x) \mid f(x) = y, f'(x) = y' \right)$ (for each $x$) such that

$$
\mathbb{E}\varphi(f(x))\psi(f'(x))f''(x) = \int \int dy' dy \; p_x(y, y') \varphi(y)\psi(y') \mathbb{E}\left(f''(x) \mid f(x) = y, f'(x) = y' \right)
$$

for all bounded Borel functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ and all $x \in [a, b]$.

Now we may apply 4a6 to the toy model (3a1). Here $p_x(y, y', y'')$ does not exist, since $y'' = -y$ always. However, $\mathbb{E}\left(f''(x) \mid f(x) = y, f'(x) = y' \right) = -y$; also, both $p_x(y')$ and $p_x(y|y')$ is just the standard normal density; we get

$$
\mathbb{E} \sum_{x: f'(x) = y'} \varphi(f(x)) \text{sgn} f''(x) = \frac{1}{2\pi} \int_0^{2\pi} dx e^{-x^2/2} \int dy e^{-y^2/2} \varphi(y) \cdot (-y) = -e^{-y^2/2} \int ye^{-y^2/2} \varphi(y) dy;
$$

for $y' = 0$ it means

$$
\mathbb{E} \sum_{x: f'(x) = 0} \varphi(f(x)) \text{sgn} f''(x) = -\int ye^{-y^2/2} \varphi(y) dy.
$$

In fact, $f'(\cdot)$ vanishes at two points, the minimum and the maximum. Here $f(x) = \pm M$ and $f''(x) = -f(x)$, thus $\sum_{x: f'(x) = 0} \varphi(f(x)) \text{sgn} f''(x) = \varphi(-M) - \varphi(M)$, and the expectation is $\int_0^\infty (\varphi(-u) - \varphi(u)) f_M(u) du$; recall (3a5).

4b Gaussian case

Let $\gamma$ be a (centered) Gaussian measure on $C^2[a, b]$ such that for every $x \in [a, b]$

$$
(4b1) \quad \int_{C^2[a, b]} |f(x)|^2 \gamma(df) = 1, \\
(4b2) \quad \int_{C^2[a, b]} |f'(x)|^2 \gamma(df) = \sigma^2(x) > 0
$$

for some $\sigma : [a, b] \to (0, \infty)$. We know (recall 3d3) that the function $\sigma(\cdot)$ is continuous. Similarly, the function $x \mapsto \int |f''(x)|^2 \gamma(df)$ is continuous, therefore bounded, which ensures (Ba2). Also (recall (3d5)),

$$
(4b3) \quad p_x(y, y') = \frac{1}{2\pi \sigma(x)} \exp \left(-\frac{y^2}{2} - \frac{y'^2}{2\sigma^2(x)}\right)
$$
is the joint density of \( f(x) \) and \( f'(x) \).

The joint distribution of \( f(x), f'(x), f''(x) \) is a Gaussian measure on \( \mathbb{R}^3 \) (maybe, degenerate). The normal correlation theorem (recall 1c) gives us a linear regression function (for each \( x \))

\[
(4b4) \quad (y, y') \mapsto \mathbb{E}\left( f''(x) \bigg| f(x) = y, f'(x) = y' \right) = A(x)y + B(x)y'.
\]

By 4a7 we may use 4b6:

\[
(4b5) \quad \mathbb{E} \sum_{x : f'(x) = y'} \varphi(f(x)) \text{ sgn } f''(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\sigma(x)} \exp \left( -\frac{y'^2}{2\sigma^2(x)} \right) \int_{\mathbb{R}} dy e^{-y'^2/2} \varphi(y)(A(x)y + B(x)y')
\]

for almost all \( y' \); here \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) is an arbitrary bounded Borel function.

The right-hand side of (4b5) is continuous in \( y' \). Similarly to 3d12, in order to prove (4b5) for all \( y' \) we will prove (assuming continuity of \( \varphi \)) that the left-hand side is also continuous in \( y' \). Similarly to 3d11, it is sufficient to check continuity of the function\(^1\)

\[
y' \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} du e^{-u'^2/2} \sum_{x : f'(x) = y'} \varphi(f_u(x)) \text{ sgn } f''_u(x),
\]

where \( f_u(\cdot) = g(\cdot) + uh(\cdot); g, h \in C^2[a, b] \) and \( h'(x) \neq 0 \) for all \( x \in [a, b] \). To this end we transform the integral in \( u \) into an integral in \( x \):

\[
(4b6) \quad \int_{\mathbb{R}} (d\Phi(u)) \sum_{x : f'(x) = y'} \varphi(f_u(x)) \text{ sgn } f''_u(x) = \pm \int_{\mathbb{R}} \varphi(f_{U(x)}(x)) d\Phi(U(x));
\]

here \( \Phi \) is the cumulative distribution function of \( N(0, 1) \); \( U(x) = (y' - g'(x))/h'(x) \); and the sign is ‘−’ if \( h'(\cdot) > 0 \) on \([a, b]\), but ‘+’ if \( h'(\cdot) < 0 \) on \([a, b]\). Clearly, the latter integral is continuous in \( y' \) (assuming continuity of \( \varphi \)). The equality (4b6) follows from (3b7) applied to \( U(x) \) instead of \( f(x) \) and \( \varphi(f_{U(x)}(x))\Phi'(U(x)) \) instead of \( g(x) \):

\[
\int_{\mathbb{R}} \sum_{x \in U^{-1}(u)} \varphi(f_{U(x)}(x))\Phi'(U(x)) \text{ sgn } U'(x) = \int_{\mathbb{R}} dx U'(x)\varphi(f_{U(x)}(x))\Phi'(U(x));
\]

taking into account that \( x \in U^{-1}(u) \iff f'_u(x) = y' \) we get

\[
\int_{\mathbb{R}} \sum_{x : f'_u(x) = y'} \varphi(f_u(x)) \text{ sgn } U'(x) = \int_a^b \varphi(f_{U(x)}(x))\Phi'(U(x))U'(x) dx .
\]

\(^1\)And in addition, integrability of its supremum in \( y' \) (running over a bounded interval).
It remains to note that \( f''(x) = -h'(x)U'(x) \), which follows from the equality \( f'_U(x) = y' \) by differentiation (in \( x \)).

Thus, (4b5) holds for all \( y' \), especially, for \( y' = 0 \):

\[
\mathbb{E} \sum_{x: f'(x) = 0} \varphi(f(x)) \text{sgn} f''(x) = \frac{1}{2\pi} \left( \int_a^b \frac{dx}{\sigma(x)} A(x) \right) \left( \int_{\mathbb{R}} dy \, e^{-y^2/2} \varphi(y) y \right);
\]

here \( A(x) \) is defined by the Gaussian regression, \( \mathbb{E} \left( f''(x) \mid f(x) = y, f'(x) = 0 \right) = A(x)y \). Being proved for bounded continuous \( \varphi \), (4b7) holds for all bounded Borel functions \( \varphi \), since it is in fact an equality between (finite) measures,

\[
\mathbb{E} \sum_{x: f'(x) = 0} (\text{sgn} f''(x)) \delta_{f(x)} = \frac{1}{2\pi} \left( \int_a^b \frac{dx}{\sigma(x)} A(x) \right) \left( \int_{\mathbb{R}} dy \, e^{-y^2/2} y \delta_y \right);
\]

you see, \( \mathbb{E} \#\{x : f'(x) = 0\} = \frac{1}{\sqrt{2\pi}} \mathbb{E} \int_a^b |f''(x)| \, dx < \infty \) by (3d13) and (4a4).

Especially, the case \( \varphi = 1_{(y, \infty)} \) gives

\[
\mathbb{E} \sum_{x: f'(x) = 0, f(x) > y} \text{sgn} f''(x) = \frac{1}{2\pi} e^{-y^2/2} \int_a^b \frac{dx}{\sigma(x)} A(x)
\]

for all \( y \in \mathbb{R} \).

4c Natural parameter

The general case of (4b5) may be reduced to the special case \( \sigma(\cdot) = 1 \), that is,

\[
\int_{C^2[a,b]} |f'(x)|^2 \gamma(df) = 1 \quad \text{for all } x,
\]

by a change of variable, \( x_{\text{new}} = \int_0^x \sigma(x_1) \, dx_1 \). Clearly, the left-hand side of (4b9) is invariant under such change of variable. Now we assume (4c1).

4c2 Exercise. \( \mathbb{E} \left( f(x)f''(x) \right) = -1 \), that is,

\[
\int_{C^2[a,b]} f(x)f''(x) \gamma(df) = -1 \quad \text{for all } x.
\]

Prove it.

\( f(x)f''(x)' = f'(x)f'(x) + f(x)f''(x) \); recall (3d4).
By \((3d4)\) applied to \(f\) and also to \(f'\),
\[
(4c3) \quad \mathbb{E}(f(x)f'(x)) = 0 \quad \text{and} \quad \mathbb{E}(f'(x)f''(x)) = 0.
\]
We see that the three random variables
\[
(4c4) \quad f(x), f'(x), f(x) + f''(x)
\]
are orthogonal. Therefore \(\mathbb{E}(f(x) + f''(x) | f(x) = y, f'(x) = y') = 0\), and
\[
(4c5) \quad \mathbb{E}(f''(x) | f(x) = y, f'(x) = y') = -y;
\]
in terms of \((4b4)\), it means that \(A(x) = -1\), \(B(x) = 0\). Now \((4b9)\) becomes
\[
(4c6) \quad \mathbb{E} \sum_{x: f(x) = 0, f(x) > y} \text{sgn} f''(x) = -\frac{b - a}{2\pi} e^{-y^2/2}.
\]
On the other hand, Rice’s formula \((3d6)\) gives
\[
\mathbb{E}(\# f^{-1}(y)) = \frac{b - a}{\pi} e^{-y^2/2},
\]
and we see that
\[
(4c7) \quad \mathbb{E}(\# f^{-1}(y)) = -2 \mathbb{E} \sum_{x: f'(x) = 0, f(x) > y} \text{sgn} f''(x).
\]
Here is a simple explanation of \((4c7)\). First (irrespective of any randomness), for every \(f \in C^2[a, b]\),
\[
\# f^{-1}(y) + 2 \sum_{x: f'(x) = 0, f(x) > y} \text{sgn} f''(x) =
\]
\[
= 1_{(y, \infty)}(f(b)) \text{sgn} f'(b) - 1_{(y, \infty)}(f(a)) \text{sgn} f'(a)
\]
(think, why), provided that the following degenerate cases are excluded:
\[
f'(a) = 0; \quad f'(b) = 0; \quad f'(x) = f''(x) = 0 \quad \text{for some} \quad x \in [a, b].
\]
Second, the expectation of the right-hand side vanishes, since \(f'(a)\) is independent of \(f(a)\) (and the same holds for \(b\)).

\footnote{The right-hand side disappears on the circle, that is, for \(2\pi\)-periodic functions restricted to \([0, 2\pi]\).}
4c8 Exercise. The degenerate cases are excluded for \(\gamma\)-almost all \(f\).

Prove it.

Hint: consider again \(f_u(\cdot) = g(\cdot) + uh(\cdot)\) for \(g, h \in C^2[a, b]\) and \(h'(x) \neq 0\) for all \(x \in [a, b]\); if \(f'_u(x) = f''_u(x) = 0\) for some \(x\) then \(u\) is a critical value of \(-g'(\cdot)/h'(\cdot)\); use Sard’s theorem.

Note that (4b1) is essential for (4c7).

We see that (4c6) follows easily from Rice’s formula. However, the approach of Sect. 4 is important in dimension two (and higher).

4d Some integral geometry

Similarly to 3e we consider a curve on \(S^{n-1} = \{z \in \mathbb{R}^n : |z| = 1\}\) parameterized by some \([a, b]\);

\[
Z \in C^2([a, b], \mathbb{R}^n), \quad Z([a, b]) \subset S^{n-1}, \quad Z'(\cdot) \neq 0.
\]

It leads to a Gaussian random vector in \(C^2[a, b]\),

\[
f(x) = \langle Z(x), \xi \rangle,
\]

where \(\xi\) is distributed \(\gamma^n\).

Extrema of \(f(\cdot)\) are extrema of the distance between a point of the curve and the random hyperplane \(\{z \in \mathbb{R}^n : \langle z, \xi \rangle = 0\}\). The (unsigned) distance is maximal when \(f'(x) = 0\) and \(\sgn f(x) \sgn f''(x) < 0\); it is minimal when \(f'(x) = 0\) and \(\sgn f(x) \sgn f''(x) > 0\). (Degenerate cases, \(f'(x) = f(x)f''(x) = 0\), are excluded almost surely, recall 4c8.) Using the natural parameter we have

\[
\mathbb{E} \sum_{x : f'(x) = 0} (-\sgn f(x) \sgn f''(x)) = -2 \mathbb{E} \sum_{x : f'(x) = 0, f(x) > 0} \sgn f''(x) = \frac{b - a}{\pi}
\]

by (4c6); and \(b - a\) is the length of the curve. Thus,

\[
(4d1) \quad \frac{\text{the mean number of maxima} - \text{the mean number of minima}}{\text{the length of the curve}} = \frac{1}{\pi}.
\]

Think, what happens for such a curve: