12 Random real zeroes: one derivative

12a Proving Theorem 2b1

For now, \( X \) satisfies just assumption \( A \).

12a1 Lemma. Let \( u \in \mathbb{R} \). Almost surely, no \( t \in \mathbb{R} \) satisfies both \( X(t) = u \) and \( X'(t) = 0 \).

12a2 Exercise. Assume the opposite: \( \mathbb{P} \left( \exists t \in \mathbb{R} (X(t) = u, X'(t) = 0) \right) > 0 \). Then

\[ \mathbb{P} \left( \exists t \in [0,1] (X(t) = u, X'(t) = 0) \text{ and } \forall t \in [0,1] |X''(t)| \leq A \right) > 0 \]

for some \( A < \infty \).

Prove it.

12a3 Exercise.

\[ \mathbb{E} \int_0^1 1_{(u-\varepsilon,u+\varepsilon)}(X(t)) \, dt \geq p \min \left( 1, \sqrt{\frac{2\varepsilon}{A}} \right), \]

where \( p = \mathbb{P} \left( \exists t \in [0,1] (X(t) = u, X'(t) = 0) \text{ and } \forall t \in [0,1] |X''(t)| \leq A \right) \).

Prove it.

On the other hand, by Lemma 2a4 applied to \( \varphi = 1_{(u-\varepsilon,u+\varepsilon)} \),

\[ \mathbb{E} \int_0^1 1_{(u-\varepsilon,u+\varepsilon)}(X(t)) \, dt = O(\varepsilon). \]

Thus, \( p \) must vanish, and so, Lemma \( \boxed{12a1} \) is proved. It means that a given number has no chance to be a critical value of \( X(\cdot) \). Then, \( \{ t \in [0,1] : X(t) = u \} \) is a finite set\(^1\) and

\[ \xi_v = \sum_{t \in [0,1], X(t) = v} \varphi(X'(t)) \]

treated as a function of \( v \) for a given \( X(\cdot) \) is continuous at \( u \). However, we cannot conclude that \( \mathbb{E} \xi_v \) is continuous in \( v \) unless we have an integrable majorant for these \( \xi_v \).

Now let \( X \) satisfy assumption \( B \).

\(^1\)Which also follows from the polynomial form of \( X(\cdot) \).
12a4 Exercise. Let $\varphi, \varphi_1, \varphi_2, \ldots : \mathbb{R} \to [0, \infty)$ be Borel functions such that either $\varphi_n \downarrow \varphi$ pointwise, or $\varphi_n \uparrow \varphi$ pointwise. If the equality
\[
\mathbb{E} \left\{ \frac{1}{L} \sum_{t \in [0, L], X(t) = u} \psi(X'(t)) \right\} = \frac{1}{2\pi} \int \psi(y)|y|e^{-y^2/2} \, dy \in [0, \infty]
\]
holds for $\psi = \varphi_1, \varphi_2, \ldots$ then it holds for $\psi = \varphi$.
Prove it.

Therefore it is sufficient to prove Theorem 2b1 under additional assumptions on $\varphi$:
\[
\varphi : \mathbb{R} \to [0, \infty) \text{ is continuous and bounded}, \quad \varphi(\cdot) = 0 \text{ on } [-a, a]
\]
for some $a > 0$.

If $\forall t \in [0, L] \ |X''(t)| \leq A$ then points $t \in [0, L]$ such that $X(t) = u, \ |X'(t)| \geq a$ are far apart at least $2a/A$ (think, why), and therefore the number of such points is at most $1 + \frac{AL}{2a}$. It follows that
\[
0 \leq \frac{1}{L} \sum_{t \in [0, L], X(t) = u} \varphi(X'(t)) \leq \left( \frac{1}{L} + \frac{1}{2a} \max_{[0, L]} |X''(\cdot)| \right) \sup_{\xi_u} \varphi(\cdot),
\]
which is an integrable majorant for the random variables $\xi_u$. Thus, convergence a.s. implies convergence of expectations, and we conclude that
\[
\mathbb{E} \xi_u \text{ is continuous in } u.
\]

Now we note that
\[
\int_{u-\varepsilon}^{u+\varepsilon} \xi_u \, dv = \frac{1}{L} \int_0^L |X'(t)| \varphi(X'(t)) 1_{(u-\varepsilon, u+\varepsilon)}(X(t)) \, dt
\]
(basically, $dv = X'(t) \, dt$, and $X(\cdot)$ is piecewise monotone). Thus,
\[
\frac{1}{2\varepsilon} \int_{u-\varepsilon}^{u+\varepsilon} \mathbb{E} \xi_u \, dv = \frac{1}{2\varepsilon} \int_0^L \left( \mathbb{E} |X'(t)| \varphi(X'(t)) 1_{(u-\varepsilon, u+\varepsilon)}(X(t)) \right) \, dt = \frac{1}{L} \int_0^L dt \left( \int |y| \varphi(y) \gamma^1(dy) \right) \frac{1}{2\varepsilon} \gamma^1((u-\varepsilon, u+\varepsilon)),
\]
since\(^1\) \((X(t), X'(t)) \sim \gamma^2\). The limit \(\varepsilon \to 0\) gives
\[
E \xi_u = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \int |y|\varphi(y)\gamma^1(dy)
\]
for all \(u\). In particular,
\[
E \xi_0 = \frac{1}{\sqrt{2\pi}} \int |y|\varphi(y)\gamma^1(dy),
\]
which proves Theorem 2b1 for \(\varphi\) satisfying (\ref{eq:12a1}), therefore, for all \(\varphi\).\(^2\)

\(^1\)\(0 = \frac{d}{dt} E X^2(t) = E 2X(t)X'(t)\).

\(^2\)And moreover, \(E \frac{1}{L} \sum_{t \in [0, L], X(t) = u} \psi(X'(t)) = e^{-u^2/2} \frac{1}{\sqrt{2\pi}} \int \psi(y)|y|e^{-y^2/2} dy\).