13 Random real zeroes: two derivatives

13a Restricting the class of functions

Let $X$ satisfy assumption $C_{M,n,L}$.

We have

$$
\mathbb{E} X'(0)X(t) = \mathbb{E} \left( \frac{d}{du} \bigg|_{u=0} X(u) \right) X(t) = \mathbb{E} \frac{d}{du} \bigg|_{u=0} (X(u)X(t)) = \\
= \frac{d}{dt} \mathbb{E} \left( X(u)X(t) \right),
$$

since for $u \in (-1,1)$,

$$
\left| \frac{X(u) - X(0)}{u} X(t) \right| \leq \max_{[-1,1]} |X'(\cdot)| : |X(t)|,
$$

the majorant being integrable. By stationarity,

$$
\mathbb{E} X'(0)X(t) = \frac{d}{du} \bigg|_{u=0} \mathbb{E} X(0)X(t-u) = -\frac{d}{dt} \mathbb{E} X(0)X(t). 
$$

Similarly,

$$
\mathbb{E} X'(0)X'(t) = \frac{d}{dt} \mathbb{E} X'(0)X(t) = -\frac{d^2}{dt^2} \mathbb{E} X(0)X(t).
$$

In particular,

$$
\mathbb{E} X(0)X''(0) = -\mathbb{E} |X'(0)|^2 = -1,
$$

therefore (think, why)

$$
\mathbb{E} |X''(0)|^2 \geq 1,
$$

and $M \geq 1$ (otherwise assumption $C_{M,n,L}$ is never satisfied). Further,

$$
\mathbb{E} X'(0)X'(t) = -\frac{d^2}{dt^2} \sum_{k=1}^{N} |a_k|^2 \cos \lambda_k t = \sum_{k=1}^{N} |\lambda_k a_k|^2 \cos \lambda_k t
$$
as well as
\[
\mathbb{E} X''(0)X''(t) = \sum_{k=1}^{N} |\lambda_k^2 a_k|^2 \cos \lambda_k t.
\]
We have \( \mathbb{E} |X''(0)|^2 = \sigma^2 \) for some \( \sigma \in [1, \sqrt{M}] \). Taking into account that \( \lambda_k^4 \leq (1 + \lambda_k^2)^2 \) and \( \sigma \geq 1 \) we see that assumption \( C_{M,n,L} \) for \( X \) implies assumption \( A_{n,L} \) for \( \frac{1}{\sigma} X'' \).

For the two-dimensional process \((X', X'')\) we get
\[
\mathbb{E} \langle (X'(0), X''(0)), (X'(t), X''(t)) \rangle = \mathbb{E} (X'(0)X'(t) + X''(0)X''(t)) = \sum_k (\lambda_k^2 + \lambda_k^4) a_k^2 \cos \lambda_k t.
\]
Taking into account that \( \lambda_k^2 + \lambda_k^4 \leq (1 + \lambda_k^2)^2 \) and \( \sigma \geq 1 \) we see that assumption \( C_{M,n,L} \) for \( X \) implies assumption \( A_{n,L} \) for \( (X', \frac{1}{\sigma} X'') \).

Applying 11c1-11c2 to \( X'' \) we get
\[
\frac{1}{L} \int_{0}^{L} |X''(t)|^2 \, dt \leq \frac{C}{n} (X_1^2 + \cdots + X_{2N}^2)
\]
for some absolute constant \( C \).

13a1 Exercise. Prove that
\[
\mathbb{P} \left( \frac{1}{L} \int_{0}^{L} |X''(t)|^2 \, dt > 2M \right) \leq C_M e^{-cn}
\]
for some \( c_M > 0, C_M < \infty \) (dependent on \( M \) only).

In this sense,
\[
\frac{1}{L} \int_{0}^{L} |X''(t)|^2 \, dt \leq 2M \quad \text{very probably.}
\]

Applying Theorem 2a2 (or rather, its two-dimensional generalization) to the two-dimensional process \((X', \frac{1}{\sigma} X'')\) we get for any a.e. continuous \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) of linear growth,
\[
\frac{1}{L} \int_{0}^{L} \varphi \left( X'(t), \frac{1}{\sigma} X''(t) \right) \, dt \in \text{ExpConInt}(n).
\]
Using also Lemma 2a1 (or rather, its two-dimensional generalization) we get for every \( \varepsilon > 0 \),
\[
\frac{1}{L} \int_{0}^{L} \varphi \left( X'(t), \frac{1}{\sigma} X''(t) \right) \, dt \leq \int \varphi \, d\gamma^2 + \varepsilon \quad \text{very probably.}
\]
In particular,
\[ \mathbb{E} \frac{1}{L} \int_0^L \frac{1}{\sigma} |X''(t)| \mathbf{1}_{[A, \infty)}(\|X'(t)\|) \, dt = \left( \int |u| \gamma_1^1(du) \right) \cdot 2 \gamma_1([A, \infty)) \leq Ce^{-A^2/2}, \]
therefore for every \( A > 0 \) (separately),
\[ \frac{1}{L} \int_0^L |X''(t)| \mathbf{1}_{[A, \infty)}(\|X'(t)\|) \, dt \leq C \sqrt{M} e^{-A^2/2} \quad \text{very probably,} \]
where \( C \) is an absolute constant.\(^1\) Similarly,
\[ \frac{1}{L} \int_0^L |X''(t)| \mathbf{1}_{[-a, a]}(X'(t)) \, dt \leq C \sqrt{Ma} \quad \text{very probably.} \]

Here is a non-probabilistic fact.

13a2 Lemma. Let \( f : [0, L] \to \mathbb{R} \) be twice continuously differentiable, and \( a > 0 \). Then
\[ \sum_{t \in [0, L], f(t) = 0, 0 < |f'(t)| \leq a} |f'(t)| \leq \frac{1}{2} \int_0^L |f''(s)| \mathbf{1}_{[-2a, 2a]}(f'(s)) \, ds + 2a. \]

**Proof.** For each \( t \in [0, L] \) such that \( f(t) = 0 \) and \( 0 < |f'(t)| \leq a \) we consider the set \( \{ s \in [0, L] : 0 < |f'(s)| < 2|f'(t)| \} \) and its connected component \( I_t \) containing \( t \). Clearly, \( I_t \) is an interval, \( f \) is strictly monotone on \( I_t \), and such intervals are pairwise disjoint. If \( I_t \subset (0, L) \) then
\[ \int_{I_t} |f''(s)| \, ds \geq 2|f'(t)| \]
(think, why). Taking into account that \( |f'(\cdot)| \leq 2a \) on \( I_t \) we have
\[ 2 \sum |f'(t)| \leq \sum \int_{I_t} |f''(s)| \, ds \leq \int_0^L |f''(s)| \mathbf{1}_{[-2a, 2a]}(f'(s)) \, ds, \]
where the sum is taken over \( t \) such that \( I_t \subset (0, L) \). Other \( t \) (at most two) contribute at most \( 2a \). \( \square \)

The random variable
\[ \xi_a = \frac{1}{L} \sum_{t \in [0, L], X(t) = 0, |X'(t)| \leq a} |X'(t)| \]
is a special case of \( \xi \) of Theorem 2c1, for \( \varphi(x) = \frac{1}{|x|} \mathbf{1}_{[-a, a]}(x) \).

\(^1\)Every \( C > \sqrt{2/\pi} \) fits.
13a3 Exercise. For every \( a > 0 \) (separately), \( \xi_a \leq C \sqrt{Ma} \) very probably; here \( C \) is an absolute constant. Prove it.

13a4 Exercise. \( \mathbb{E} \xi_a \leq \frac{1}{3\pi} a^3 \). Prove it.

13a5 Exercise. It is sufficient to prove Theorem 2c1 for functions \( \varphi \) that vanish on a neighborhood\(^1\) of 0. Prove it.

13a6 Lemma. Let \( f : [0, L] \to \mathbb{R} \) be twice continuously differentiable, and \( A > 2 \min_{[0, L]} |f'(\cdot)| \). Then

\[
\sum_{t \in [0, L], f(t) = 0, |f'(t)| \geq A} |f'(t)| \leq 2 \int_0^L |f''(s)| 1_{(A/2, \infty)}(|f'(s)|) \, ds.
\]

Proof. For each \( t \in [0, L] \) such that \( f(t) = 0 \) and \( |f'(t)| \geq A \) we consider the set \( \{ s \in [0, L] : |f'(s)| > 0.5|f'(t)| \} \) and its connected component \( I_t \) containing \( t \). Clearly, \( I_t \) is an interval, \( f \) is strictly monotone on \( I_t \), and such intervals are pairwise disjoint. It cannot happen that \( I_t = [0, L] \), since \( |f'(\cdot)| \geq 0.5A \) on \( I_t \). Thus,

\[
\int_{I_t} |f''(s)| \, ds \geq \frac{1}{2} |f'(t)|
\]

(think, why). We have

\[
\frac{1}{2} \sum |f'(t)| \leq \sum \int_{I_t} |f''(s)| \, ds \leq \int_0^L |f''(s)| 1_{(A/2, \infty)}(|f'(s)|) \, ds.
\]

Taking into account that

\[
\min_{[0, L]} |X'(\cdot)| \leq \left( \frac{1}{L} \int_0^L |X'(t)|^2 \, dt \right)^{1/2}
\]

we see (similarly to 13a1) that

\[
\min_{[0, L]} |X'(\cdot)| \leq 2 \quad \text{very probably.}
\]

\(^1\)The neighborhood depends on \( \varphi \), of course.
Similarly to 13a3, for every $A > 4$ (separately), the random variable 

$$\xi_A = \frac{1}{L} \sum_{t \in [0,L], X(t) = 0, |X'(t)| \geq A} |X'(t)|$$

satisfies 

$$\xi_A \leq C \sqrt{M} e^{-A^2/8}$$

very probably, as well as $E \xi_A \leq C \sqrt{M} e^{-A^2/8}$. Here is the conclusion.

13a7 Proposition. It is sufficient to prove Theorem 2c1 for functions $\varphi$ such that

$$\exists a, A \in (0, \infty) \forall x \in \mathbb{R} \quad (\varphi(x) \neq 0 \implies a < |x| < A).$$

The condition $\sup(|\varphi(x)|/|x|) < \infty$ becomes just boundedness of $\varphi$.

13a8 Exercise. It is sufficient to prove Theorem 2c1 for Lipschitz functions $\varphi$ (satisfying 13a7).

Prove it.

13b Getting rid of randomness

According to 13a8, we consider a function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying\(^1\)

- $\forall x \in \mathbb{R} \quad |\varphi(x)| \leq 1$,
- $\forall x, y \in \mathbb{R} \quad |\varphi(x) - \varphi(y)| \leq |x - y|$,
- $\forall x \in \mathbb{R} \quad (\varphi(x) \neq 0 \implies |x| > a$).

We approximate the random variable of Theorem 2c1,

$$\xi = \frac{1}{L} \sum_{t \in [0,L], X(t) = 0} \varphi(X'(t)),$$

by another random variable (for $\varepsilon \to 0+$)

$$\eta_\varepsilon = \frac{1}{2\varepsilon L} \int_0^L \varphi(X'(t))|X'(t)|1_{(-\varepsilon,\varepsilon)}(X(t)) \, dt.$$ 

We know that\(^2\)

$$E \xi = \frac{1}{2\pi} \int \varphi(y)|y|e^{-y^2/2} \, dy$$

\(^1\)The first two conditions can be enforced multiplying $\varphi$ by a small number. The condition about $|x| < A$ is not needed.

\(^2\)By Theorem 2b1.
\[ E \eta_\varepsilon = \frac{1}{2\varepsilon} \iint \varphi(y)|y|\mathbf{1}_{(-\varepsilon,\varepsilon)}(x) \gamma^2(dx dy) = \left( \int \varphi(y)y|y| \gamma^1(dy) \right) \frac{1}{2\varepsilon} \gamma^1((-\varepsilon,\varepsilon)) \rightarrow \frac{1}{\sqrt{2\pi}} \int \varphi(y)y|y| \gamma^1(dy) , \]

thus, \( |E \eta_\varepsilon - E \xi| \rightarrow 0 \) (as \( \varepsilon \rightarrow 0^+ \)). We also know that

\[ \eta_\varepsilon \in \text{ExpConInt}(n) \quad \text{for every} \quad \varepsilon \quad \text{(separately)}. \]

In order to prove Theorem 2c1 we have to prove that

\[ \xi \in \text{ExpConInt}(n) ; \]

by the approximation lemma (11a9, 11a11) it is sufficient to prove that

\[ |\xi - \eta_\varepsilon| \leq \varepsilon_0 \quad \text{very probably} \]

if \( \varepsilon \) is small enough (for a given \( \varepsilon_0 \)).

Here is a non-probabilistic fact, to be proved in Sect. 13c.

13b1 Proposition. Let a twice continuously differentiable function \( f : [0, L] \rightarrow \mathbb{R} \) and a number \( B > 0 \) satisfy

\[ \frac{1}{L} \int_0^L |f''(t)|^2 \leq B^2 \]

and \( B\varepsilon < \min(1, a^3) \). Then

\[ \left| \frac{1}{L} \sum_{t \in [0, L], f(t)=0, f'(t) \neq 0} \varphi(f'(t)) - \frac{1}{2\varepsilon L} \int_0^L \varphi(f'(t))|f'(t)|\mathbf{1}_{(-\varepsilon,\varepsilon)}(f(t)) \, dt \right| \leq C \left( \frac{\varepsilon^{1/3}B^{4/3}}{a} + \frac{1}{L} \right) \]

for some absolute constant \( C \).

Note that \( n \) does not occur in 13b1 but \( L \) does.

13b2 Exercise. \( L \geq cn - C \) for some absolute constants \( c, C \).

Prove it.

Given \( \varepsilon_0 > 0 \), we choose \( \varepsilon \) such that \( C \frac{\varepsilon^{1/3}}{a}(2M)^{2/3} \leq \varepsilon_0/2 \) and \( \varepsilon \sqrt{2M} < \min(1, a^3) \); then, assuming that \( C/L \leq \varepsilon_0/2 \) (which holds for all \( n \) large enough) we get \( |\xi - \eta_\varepsilon| \leq \varepsilon_0 \) whenever \( \int \frac{1}{L} \int_0^L |X''(t)|^2 \, dt \leq 2M \), which happens very probably. Thus, in order to prove Theorem 2c1 it is sufficient to prove Proposition 13b1.\footnote{By the two-dimensional generalization of Lemma 2a1.}

\footnote{By the two-dimensional generalization of Theorem 2a2.}
13c Estimating the error

Here we prove the non-probabilistic Proposition 13b1.

We consider the set \( \{ t \in [0, L] : |f(t)| < \varepsilon, f'(t) \neq 0 \} \) and its connected components \( I \) such that
\[
\sup_{I} |f'(\cdot)| > \alpha .
\]
Clearly, \( I \) is an interval, \( f \) is strictly monotone on \( I \), and of course, such intervals are pairwise disjoint.

13c1 Exercise. The set \( I \) of all these intervals \( I \) is finite.

Prove it.

Assume that \( \delta \in (0, \alpha) \) is given (it will be chosen later). We say that an interval \( I \in I \) is good, if
\[
\sup_I f'(\cdot) - \inf_I f'(\cdot) \leq \delta ;
\]
otherwise \( I \) is bad. Denote by \( G \subset I \) the set of all good intervals.

Denoting \(^1\)
\[
\xi = \frac{1}{L} \sum_{I \in I} \sum_{t \in I, f(t)=0, f'(t)\neq 0} \varphi(f'(t)) , \quad \eta_\varepsilon = \frac{1}{2\varepsilon L} \int_0^L \varphi(f'(t))|f'(t)|1_{(-\varepsilon,\varepsilon)}(f(t)) \, dt
\]
we have
\[
\xi = \sum_{I \in I} \frac{1}{L} \sum_{t \in I, f(t)=0} \varphi(f'(t)) , \quad \eta_\varepsilon = \sum_{I \in I} \frac{1}{2\varepsilon L} \int_I \varphi(f'(t))|f'(t)| \, dt .
\]
Taking into account that \( |\varphi(\cdot)| \leq 1 \) we get
\[
\forall I \in I \quad |\xi_I| \leq \frac{1}{L} ,
\]
since the sum contains no more than one summand; and
\[
\forall I \in I \quad |\eta_{I,\varepsilon}| \leq \frac{1}{L} ,
\]
since \( \int_I |f'(t)| \, dt = |f(t) - f(s)| \) for \( I = (s, t) \).

At most two \( I \in I \) may violate \( I \subset (0, L) \); their contribution to \( |\xi - \eta_\varepsilon| \) cannot exceed \( 4/L \), that is harmless. From now on we assume that
\[
I \subset (0, L)
\]
for all considered \( I \in I \).

\(^1\)We thus redefine \( \xi \) and \( \eta_\varepsilon \), which should not be too confusing since the probabilistic context is no more needed.
13c2 Lemma. For every good $I \subset (0, L)$,

$$|\xi_I - \eta_{I, \varepsilon}| \leq \frac{\delta}{L}.$$  

Proof. Let $I = (r, t)$. We have $\min_I |f'(\cdot)| \geq a - \delta > 0$, therefore either $f(r) = -\varepsilon$, $f(t) = \varepsilon$ or $f(r) = \varepsilon$, $f(t) = -\varepsilon$; in every case, $\int_I |f'(u)| \, du = 2\varepsilon$.

Define $s \in I$ by $f(s) = 0$, then $\xi_I = \frac{1}{L} \varphi(f'(s)) = \frac{1}{2L} \int_I \varphi(f'(s)) |f'(u)| \, du$ and

$$|\xi_I - \eta_{I, \varepsilon}| \leq \frac{1}{2\varepsilon L} \int_I |\varphi(f'(s)) - \varphi(f'(u))| |f'(u)| \, du \leq \frac{1}{2\varepsilon L} \int_I \left| \frac{f'(s) - f'(u)}{\delta} \right|^2 |f'(u)| \, du = \frac{\delta}{L}.$$

\[ \Box \]

13c3 Exercise. Prove that

$$\sum |f'(t)| \leq \frac{1}{2} \int_0^L |f''(s)| \, ds,$$

where the sum is taken over $t$ such that $f(t) = 0$ and $f'(t) \neq 0$, except for the least and the greatest of these $t$.

We have $\frac{1}{L} \int_0^L |f''(t)| \, dt \leq B$, thus,

$$|G| \leq \frac{BL}{2a} + 2,$$

and so,

$$\sum_{I \in G} |\xi_I - \eta_{I, \varepsilon}| \leq \left( \frac{B}{2a} + \frac{2}{L} \right) \delta.$$

13c4 Lemma. For every bad interval $I \subset (0, L)$,

$$\int_I |f''(t)|^2 \, dt \geq \frac{a\delta^2}{16\varepsilon}.$$  

Proof. We take $s \in I$ such that $|f'(s)| > a$, note that $\sup_I |f'(\cdot) - f'(s)| > \delta/2$ and take the closest to $s$ point $t \in I$ such that $|f'(t) - f'(s)| = \delta/2$. Assume that $s < t$ (the case $t < s$ is similar). We have $\min_{[s,t]} |f'(\cdot)| \geq a - \frac{\delta}{2} \geq \frac{a}{2}$.
and \( \int_s^t |f'(u)| \, du = |f(t) - f(s)| \leq 2\varepsilon \), thus \( t - s \leq \frac{4\varepsilon}{a} \). Also, \( \int_s^t |f''(u)| \, du \geq |f'(t) - f'(s)| = \delta/2 \). Thus,

\[
\frac{\delta}{2} \leq \int_s^t |f''(u)| \, du \leq \left( \int_s^t |f''(u)|^2 \, du \right)^{1/2} \left( \int_s^t 1^2 \, du \right)^{1/2}
\]

\[
\int_s^t |f''(u)|^2 \, du \geq \int_s^t |f''(u)|^2 \, du \geq \frac{(\delta/2)^2}{t - s} \geq \frac{a\delta^2}{16\varepsilon}.
\]

Thus, the number of bad intervals \( I \subset (0, L) \) does not exceed

\[
\frac{16\varepsilon B^2 L}{a\delta^2},
\]

and so,

\[
\sum_{I \in \mathcal{I} \cap G, I \subset (0, L)} (|\xi_I| + |\eta_{I, \varepsilon}|) \leq \frac{16\varepsilon B^2 L}{a\delta^2} \left( \frac{1}{L} + \frac{1}{L} \right) = \frac{32\varepsilon B^2}{a\delta^2};
\]

\[
\sum_{I \in \mathcal{I} \cap G} |\xi_I - \eta_{I, \varepsilon}| \leq \frac{32\varepsilon B^2}{a\delta^2} + \frac{4}{L};
\]

\[
|\xi - \eta_\varepsilon| \leq \sum_{I \in \mathcal{I}} |\xi_I - \eta_{I, \varepsilon}| \leq \left( \frac{B}{2a} + \frac{2}{L} \right) \delta + \frac{32\varepsilon B^2}{a\delta^2} + \frac{4}{L}.
\]

Finally we choose

\[
\delta = (B\varepsilon)^{1/3},
\]

note that \( \delta < a \) and \( \delta < 1 \), and get

\[
|\xi - \eta_\varepsilon| \leq \frac{B}{2a}\delta + \frac{32\varepsilon B^2}{a\delta^2} + \frac{6}{L} \leq C \left( \frac{\varepsilon^{1/3} B^{4/3}}{a} + \frac{1}{L} \right),
\]

which completes the proof of Proposition 13b1 and ultimately, Theorem 2c1.

13d Hints to exercises

13a1 the random variable \( \xi = \left( \frac{1}{L} \int_0^L |X''(t)|^2 \, dt \right)^{1/2} \) belongs to GaussLip\( (C/\sqrt{n}) \), and \( \mathbb{E} \xi \leq (\mathbb{E} \xi^2)^{1/2} \leq \sqrt{M} \).

13a8 recall 11d4 and the paragraph after it.

13b2 \( \int_{[0, 2]} (1 + \lambda^2) \, \mu(d\lambda) \geq \mu([0, 2]) \geq 3/4 \).

13c8 Hint: similar to 13b2.
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