14 Sensitivity and superconcentration

14a Variance and gradient; proving (3a4)

14a1 Exercise. (“Gaussian integration by parts”) Prove that
\[ \int x f(x) \gamma^1(dx) = \int f'(x) \gamma^1(dx) \]
for every continuously differentiable, compactly supported \( f : \mathbb{R} \to \mathbb{R} \).

14a2 Exercise. Prove that
\[ \int \int f(x \cos \varphi + y \sin \varphi, -x \sin \varphi + y \cos \varphi) \gamma^1(dx) \gamma^1(dy) = \int \int f(x, y) \gamma^1(dx) \gamma^1(dy) \]
for all bounded continuous \( f : \mathbb{R}^2 \to \mathbb{R} \) and \( \varphi \in \mathbb{R} \).

14a3 Exercise. Prove that
\[ \int \int f(x, y) g(x \cos \varphi - y \sin \varphi, x \sin \varphi + y \cos \varphi) \gamma^1(dx) \gamma^1(dy) = \int \int f(x \cos \varphi + y \sin \varphi, -x \sin \varphi + y \cos \varphi) g(x, y) \gamma^1(dx) \gamma^1(dy) \]
for all bounded continuous \( f, g : \mathbb{R}^2 \to \mathbb{R} \) and \( \varphi \in \mathbb{R} \).

14a4 Exercise. Prove that
\[ \frac{d}{d\varphi} \int \int f(x) g(x \cos \varphi - y \sin \varphi) \gamma^1(dx) \gamma^1(dy) = \]
\[ = - \sin \varphi \int \int f'(x) g'(x \cos \varphi - y \sin \varphi) \gamma^1(dx) \gamma^1(dy) \]
for all continuously differentiable, compactly supported \( f, g : \mathbb{R} \to \mathbb{R} \) and \( \varphi \in \mathbb{R} \).
14a5 Exercise. Prove that
\[ \frac{d}{dt} \int f(x)g(y) \, d\gamma^t_1(dx\,dy) = -e^{-t} \int f'(x)g'(y) \, d\gamma^t_1(dx\,dy) \]
for all continuously differentiable, compactly supported \( f, g : \mathbb{R} \to \mathbb{R} \) and \( t \in (0, \infty) \).

14a6 Exercise. Prove that
\[ \int fg \, d\gamma^1 - \left( \int f \, d\gamma^1 \right) \left( \int g \, d\gamma^1 \right) = \int_0^\infty dt \, e^{-t} \int f'(x)g'(y) \, d\gamma^t_1(dx\,dy) \]
for all continuously differentiable, compactly supported \( f, g : \mathbb{R} \to \mathbb{R} \).

14a7 Exercise. (Generalization of 14a1 to \( x \in \mathbb{R}^d \))
\[ \int \nabla f(x) \, \gamma^d(dx) = \int x f(x) \, \gamma^d(dx) \]
for every continuously differentiable, compactly supported \( f : \mathbb{R}^d \to \mathbb{R} \). That is,
\[ \int \frac{\partial}{\partial x_k} f(x) \, \gamma^d(dx) = \int x_k f(x) \, \gamma^d(dx) \]
for \( k = 1, \ldots, d \).
Prove it.

14a8 Exercise. Generalize 14a3 to \( x, y \in \mathbb{R}^d \).

14a9 Exercise. (Generalization of 14a4 to \( \mathbb{R}^d \))
\[ \frac{d}{d\varphi} \int f(x)g(x \cos \varphi - y \sin \varphi) \, \gamma^1(dx) \gamma^1(dy) = \]
\[ = -\sin \varphi \int (\nabla f(x), \nabla g(x \cos \varphi - y \sin \varphi)) \, \gamma^d(dx) \gamma^d(dy) \]
for all continuously differentiable, compactly supported \( f, g : \mathbb{R}^d \to \mathbb{R} \) and \( \varphi \in \mathbb{R} \).
Prove it.

Similarly to 14a5 14a6 we get
\[ (14a10) \quad \frac{d}{dt} \int f(x)g(y) \, \gamma^t_1(dx\,dy) = -e^{-t} \int \langle \nabla f(x), \nabla g(y) \rangle \, \gamma^t_1(dx\,dy) \]
and finally,
\[ (14a11) \int f g \, d\gamma^d - \left( \int f \, d\gamma^d \right) \left( \int g \, d\gamma^d \right) = \int_0^\infty \! dt \, e^{-t} \iint \langle \nabla f(x), \nabla g(y) \rangle \, \gamma^d(dx\,dy), \]
which may also be thought of as \( \iint \langle \nabla f, \nabla g \rangle \, d\nu \) where \( \nu = \int e^{-t} \gamma^d \, dt \).

If \( f, g \) are Lip(1) functions then \( |\langle \nabla f, \nabla g \rangle| \leq 1 \) and so, \( |\int f g \, d\gamma^d - \left( \int f \, d\gamma^d \right) \left( \int g \, d\gamma^d \right)| \leq 1 \). In particular,
\[ (14a12) \int \! f^2 \, d\gamma^d - \left( \int f \, d\gamma^d \right)^2 \leq 1 \quad \text{for } f \in \text{Lip}(1). \]

14a13 Exercise. Deduce (14a12) from Theorem 1a2.

Moreover,
\[ \left| \iint \langle \nabla f(x), \nabla f(y) \rangle \, \gamma^d(dx\,dy) \right| \leq \left( \iint |\nabla f(x)|^2 \, \gamma^d(dx\,dy) \right)^{1/2} \left( \iint |\nabla f(y)|^2 \, \gamma^d(dx\,dy) \right)^{1/2} = \int |\nabla f|^2 \, d\gamma^d; \]
in combination with (14a11) (for \( f = g \)) it gives
\[ (14a14) \int f^2 \, d\gamma^d - \left( \int f \, d\gamma^d \right)^2 \leq \int |\nabla f|^2 \, d\gamma^d, \]
the Poincare inequality\(^1\) for Gaussian measure. It is evidently stronger than (14a11).

We cannot just apply (14a11) to \( f = g = \xi, \, \xi(x) = \max_{a \in A} \langle x, a \rangle \), since \( \xi \) is neither continuously differentiable nor compactly supported. However, the needed generalizations are easy. First, (14a11) holds for a piecewise continuously differentiable Lipschitz function \( f : \mathbb{R} \to \mathbb{R} \) (think, why).\(^2\) Second, (14a7) holds for \( f = \xi \), since the restriction of \( \xi \) to a straight line is the maximum of finitely many linear functions. Thus, (14a11) applies to \( f = g = \xi \); taking into account that \( \nabla \xi = \alpha \) we get (3a4).

\(^1\)The simplest classical Poincare inequality: \( \int_0^1 f^2(x) \, dx - \left( \int_0^1 f(x) \, dx \right)^2 \leq \frac{1}{\pi^2} \int_0^1 f''(x) \, dx \); the equality holds for \( f(x) = \cos \pi x \).

\(^2\)Wider generalization is possible, but we do not need it.
14b Proving Lemma 3a2

14b1 Exercise. Prove that
\[ \int \int f(x, y, x \cos \varphi + y \sin \varphi, -x \sin \varphi + y \cos \varphi) \gamma^1(dx)\gamma^1(dy) = \]
\[ = \int \int f(x \cos \varphi - y \sin \varphi, x \sin \varphi + y \cos \varphi, x, y) \gamma^1(dx)\gamma^1(dy) \]
for all bounded continuous \( f : \mathbb{R}^4 \rightarrow \mathbb{R} \) and \( \varphi \in \mathbb{R} \).

But do not think that
\[ \int \int f(x, y, x \cos \varphi - y \sin \varphi, x \sin \varphi + y \cos \varphi) \gamma^1(dx)\gamma^1(dy) = \]
\[ = \int \int f(x \cos \varphi - y \sin \varphi, x \sin \varphi + y \cos \varphi, x, y) \gamma^1(dx)\gamma^1(dy), \]
this is generally wrong (think, why).

14b2 Exercise. The measure \( \gamma^1_t \) is symmetric. That is,
\[ \int \int f(x, y) \gamma^1_t(dx, dy) = \int \int f(y, x) \gamma^1_t(dx, dy) \]
for all bounded continuous \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( t \in [0, \infty) \).
Prove it.

Thus,
\[ \int \int f(x, e^{-t}x + \sqrt{1 - e^{-2t}}u) \gamma^1(dx)\gamma^1(du) = \]
\[ = \int \int f(e^{-t}y + \sqrt{1 - e^{-2t}}v, y) \gamma^1(dx)\gamma^1(dy). \]

14b3 Lemma. For every bounded continuous \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( t \in [0, \infty) \),
\[ \int \int f(x) f(z) \gamma^2_t(dx, dz) = \int \left( \int f(e^{-t}y + \sqrt{1 - e^{-2t}}u) \gamma^1(du) \right)^2 \gamma^1(dy). \]

Proof.
\[ I_2 = \int \left( \int f(e^{-t}y + \sqrt{1 - e^{-2t}}u) \gamma^1(du) \right)^2 \gamma^1(dy) = \]
\[ = \int \int \int f(e^{-t}y + \sqrt{1 - e^{-2t}}u) f(e^{-t}y + \sqrt{1 - e^{-2t}}v) \gamma^1(dy)\gamma^1(du)\gamma^1(dv); \]
for every $v$ we have

$$\int \int f(e^{-t}y + \sqrt{1 - e^{-2t}u})f(e^{-t}y + \sqrt{1 - e^{-2t}v}) \gamma^1(du) \gamma^1(dv) =$$

$$= \int \int f(x)f(e^{-t}x + \sqrt{1 - e^{-2t}w} + \sqrt{1 - e^{-2t}v}) \gamma^1(dx) \gamma^1(dw);$$

thus,

$$I_2 = \int \gamma^1(dx)f(x) \int \gamma^1(dw)f(e^{-2t}x + e^{-t}\sqrt{1 - e^{-2t}w} + \sqrt{1 - e^{-2t}v}) =$$

$$= \int \gamma^1(dx)f(x) \int \gamma^1(dx)f(e^{-2t}x + \sqrt{1 - e^{-4t}u}) = I_1,$$

since $e^{-2t}(1 - e^{-2t}) + 1 - e^{-2t} = 1 - e^{-4t}$.

The same holds for $\gamma^d$, and we get

$$\int \int f(x)f(y) \gamma^d_t(dx dy) \geq 0$$

for every bounded continuous $f : \mathbb{R}^d \rightarrow \mathbb{R}$. By approximation it holds for all $f \in L_2(\gamma^d)$, which proves a half of Lemma 3a2.

The same holds for vector-functions (think, why). In particular,

$$\int \int \langle \nabla f(x), \nabla f(y) \rangle \gamma^d_t(dx dy) \geq 0$$

whenever $\int |\nabla f|^2 d\gamma^d < \infty$.

By (14a10), $\int \int f(x)f(y) \gamma^d_t(dx dy)$ decreases in $t$ for good functions $f$.

By approximation it holds for all $f \in L_2(\gamma^d)$, which completes the proof of Lemma 3a2.¹

14c Proving Theorem 3a3

**Correction.** Item (a) of Theorem 3a3 should be: assumption $D_{2n2}$ implies assumption $E_n$.

The function

$$\varphi(t) = \mathbb{E} \langle \alpha(X), \alpha(X_t) \rangle$$

1In fact, Lemma 3a2 is not “Gaussian”: it holds for every time-symmetric Markov process. Here is its translation into the language of functional analysis. Let $(U_t)_{t \geq 0}$ be a one-parameter semigroup of Hermitian operators in a Hilbert space, satisfying $||U_t|| \leq 1$ for all $t$. Then the function $t \mapsto \langle U_t \psi, \psi \rangle$ is nonnegative and decreasing on $[0, \infty)$ for every vector $\psi$ of the Hilbert space. (The proof is quite simple.)
satisfies

(14c1) \[ \forall t \ 0 \leq \varphi(t) \leq 1, \]
\[ \varphi \text{ is decreasing on } [0, \infty) \]
(think, why).

**14c2 Exercise.** For every \( \varphi \) satisfying (14c1) and every \( x \in (0, \infty) \),
(a) \[ \int_{0}^{\infty} e^{-t \varphi(t)} dt \leq x + \varphi(x); \]
(b) \[ \varphi(x) \leq \frac{e^{x}}{x} \int_{0}^{\infty} e^{-t \varphi(t)} dt. \]
Prove it.

By (3a4), \( \int_{0}^{\infty} e^{-t \varphi(t)} dt = \text{Var}(\xi) \). Thus, \( D_{2n} \) means \( \int e^{-t \varphi(t)} dt \leq \frac{1}{2n^{2}} \) and implies \( \varphi(1/n) \leq \frac{e^{1/n}}{1/n} \cdot \frac{1}{2n^{2}} = \frac{1}{n} \) for \( n \geq 2 \), which proves 3a3(a).

On the other hand, \( E_{2n} \) means \( \varphi(\frac{1}{2n}) \leq \frac{1}{2n} \) and implies \( \int e^{-t \varphi(t)} dt \leq \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} \), which is \( D_{n}; 3a3(b) \) is thus proved.

**14d Hints to exercises**

**14a3** this is a generalization of 14a2 and nevertheless, it is a special case of 14a2.

14a2 \[ \int f(x \cos \varphi + y \sin \varphi) y \gamma_{1}(dy) = \sin \varphi \int f'(x \cos \varphi + y \sin \varphi) \gamma_{1}(dy). \]
14a3 \[ e^{-t} = \cos \varphi. \]
14a13 \[ \frac{1}{2} \int \int |f(x) - f(y)|^{2} \mu(dx)\mu(dy) = \int f^{2} d\mu - (\int f d\mu)^{2}. \]
14b1 this is, again, a generalization of 14a2 and nevertheless, a special case of 14a2.

14b2 apply 14b1 to \( f(x, y, u, v) = g(x, u) \).
14c2 \[ \int_{0}^{\infty} = \int_{0}^{x} + \int_{x}^{\infty}. \]