23 Random real zeroes: no derivatives

23a Random element of $L_2[0,1]$  
Continuing Sect. 22d, we consider a Gaussian process

$$\Xi : [0,1] \to G \subset L_2(\Omega,P), \quad \Xi(t) = f_1(t)g_1 + f_2(t)g_2 + \ldots,$$

where $(g_1, g_2, \ldots)$ is an orthonormal basis of $G$, and $f_k(t) = \langle \Xi(t), g_k \rangle$ are measurable. Necessarily,

$$\forall t \in [0,1] \quad |f_1(t)|^2 + |f_2(t)|^2 + \cdots = \|\Xi(t)\|^2 < \infty.$$

We upgrade $\Xi$ to the corresponding random element of $L_0[0,1]$ (as explained in Sect. 22d), denoted by $X : \Omega \to L_0[0,1]$. In general, $\int_0^1 \|\Xi(t)\|^2 \, dt = \sum_k \int |f_k(t)|^2 \, dt$ need not be finite. From now on we assume that it is:

$$\int_0^1 \|\Xi(t)\|^2 \, dt < \infty;$$

then, by Tonelli’s theorem,

$$\mathbb{E} \int_0^1 |X(t)|^2 \, dt = \int_0^1 (\mathbb{E} |X(t)|^2) \, dt = \int_0^1 \|\Xi(t)\|^2 \, dt < \infty,$$

which shows that $X$ is in fact a random element of $L_2[0,1]$. We approximate $X$ by another random element $X_n$ of $L_2[0,1]$,

$$X_n(t) = g_1 f_1(t) + \cdots + g_n f_n(t).$$

We may also treat $X$ and $X_n$ as elements of $L_2([0,1] \times \Omega)$.

23a1 Exercise. $X_n \to X$ in $L_2([0,1] \times \Omega)$.  
Prove it.

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1This is also sufficient (think, why).
2In fact, almost surely the series converges in $L_2(0,1)$. 

23a2 Exercise. For every \( f \in L_2[0,1] \) the random variables \( \langle f, X_n \rangle = \langle f_1, f \rangle g_1 + \cdots + \langle f_n, f \rangle g_n \) converge (as \( n \to \infty \)) in \( L_2(\Omega) \) to the random variable \( \langle f, X \rangle = \int_0^1 f(t)X(t) \, dt \).
Prove it.
Thus,
\[
\text{Var}(f, X) = \sum_k |\langle f, f_k \rangle|^2 \leq C \|f\|^2
\]
for some \( C \leq \sum_k \|f_k\|^2 = \int_0^1 \|\Xi(t)\|^2 \, dt < \infty \).

23a3 Proposition. Let \( C \) be such that for all \( f \in L_2[0,1] \)
\[
\text{Var}(f, X) \leq C \|f\|^2.
\]
Let \( \psi : L_2[0,1] \to \mathbb{R} \) be a Lip(1) function. Then the random variable \( \psi(X) \) belongs to GaussLip(\( \sqrt{C} \)).

First, we need the duality argument used already in 11c3.

23a4 Lemma. \( \|a_1f_1 + a_2f_2 + \ldots \|^2 \leq C(a_1^2 + a_2^2 + \ldots) \) for all \( (a_1, a_2, \ldots) \in l_2 \).

Proof. We introduce a linear operator \( S : l_2 \to L_2[0,1] \) by \( Sa = \sum a_k f_k \); the series converges in \( L_2[0,1] \), since \( \sum \|a_kf_k\| = \sum |a_k| \cdot \|f_k\| \leq (\sum |a_k|^2)^{1/2}(\sum \|f_k\|^2)^{1/2} < \infty \). We have \( \forall a \in l_2 \, \forall f \in L_2[0,1] \, \langle f, Sa \rangle = \langle S^*f, a \rangle \), where \( S^* : L_2[0,1] \to l_2 \), \( S^*f = (\langle f, f_1 \rangle, \langle f, f_2 \rangle, \ldots) \).

We note that \( \text{Var}(f, X) = \|S^*f\|^2 \); thus, \( \|S^*f\|^2 \leq C\|f\|^2 \) for all \( f \).
Finally,
\[
\|Sa\| = \sup_{\|f\| \leq 1} \langle f, Sa \rangle = \sup_{\|f\| \leq 1} \langle S^*f, a \rangle \leq \sup_{\|f\| \leq 1} \|S^*f\|||a|| \leq \sqrt{C}||a||.
\]

\( \square \)

Proof of the proposition. Similarly to the proof of 22d5 we assume that \( (\Omega, P) = (\mathbb{R}^\infty, \gamma^\infty) \), \( g_k \) are the coordinates, and will prove that \( \psi(X) \) is a Lip(\( \sqrt{C} \)) function on \( (\mathbb{R}^\infty, \gamma^\infty) \).

We take \( n_1 < n_2 < \ldots \) such that \( \sum_{i=1}^{n_k} f_i g_i \to X \) (as \( k \to \infty \)) almost everywhere on \([0,1] \times \Omega \).

\(^1\)In fact, \( n_k = k \) fit.
Given $a \in l^2$, we introduce $h = a_1 f_1 + a_2 f_2 + \cdots \in L^2[0,1]$; $\|h\|^2 \leq C\|a\|^2$ by $(23a4)$. For almost all $(t, x) \in [0, 1] \times (\mathbb{R}^\infty, \gamma^\infty)$ we have

$$X(x + a, t) - X(x, t) = \lim_{n_k} \sum_{i=1}^{n_k} (x_i + a_i) f_i(t) - \lim_{n_k} \sum_{i=1}^{n_k} x_i f_i(t) = \lim_{n_k} \sum_{i=1}^{n_k} a_i f_i(t) = h(t).$$

Thus, $X(x + a) - X(x) = h$ for almost all $x \in (\mathbb{R}^\infty, \gamma^\infty)$. Finally,

$$|\psi(X(x + a)) - \psi(X(x))| \leq \|X(x + a) - X(x)\| = \|h\| \leq \sqrt{C}\|a\|.$$ 

Here is a useful formula for the variance:

$$(23a5) \quad \text{Var}(f, X) = \int_0^1 \int_0^1 f(s)f(t) \left( \mathbb{E} \Xi(s)\Xi(t) \right) \, ds \, dt$$

for every $f \in L^2[0,1]$. Proof:

$$\mathbb{E} \left( \int f(t)X(t) \, dt \right)^2 = \mathbb{E} \int f(s)X(s)f(t)X(t) \, ds \, dt = \int \int \left( \mathbb{E} f(s)X(s)f(t)X(t) \right) \, ds \, dt,$$

since

$$\mathbb{E} \int \left| f(s)X(s)f(t)X(t) \right| \, ds \, dt = \mathbb{E} \left( \int |f(t)X(t)| \, dt \right)^2 \leq \mathbb{E} \left( \int |f(t)|^2 \, dt \right) \left( \int |X(t)|^2 \, dt \right) = \|f\|^2 \int_0^1 \|\Xi(t)\|^2 \, dt < \infty.$$ 

23b Using assumption $A_n$

Let $\Xi : \mathbb{R} \to G \subset L^2(\Omega, P)$ be a mean-square continuous stationary Gaussian random process on $\mathbb{R}$, and $\mu$ its spectral measure:

$$\mathbb{E} \Xi(0)\Xi(t) = \int_{-\infty}^{+\infty} e^{i\lambda t} \mu(d\lambda) = \int_{-\infty}^{+\infty} \cos \lambda t \mu(d\lambda).$$

1In fact, the distribution $X[\gamma^\infty]$ of $X$ is a Gaussian measure on $L^2[0,1]$, and $h$ is its admissible shift.
Here is another useful formula for the variance, this time in terms of the spectral measure (recall 11c4):

\[(23b1) \quad \text{Var}(f, X) = \int \left| \int_0^1 f(t)e^{i\lambda t} \, dt \right|^2 \mu(d\lambda) \]

for every \( f \in L_2[0,1] \). Proof:

\[
\text{Var}(f, X) = \iint f(s)f(t) \left( \int e^{i\lambda(t-s)} \mu(d\lambda) \right) \, ds \, dt = \int \mu(d\lambda) \left( \int f(s)e^{i\lambda s} \, ds \right) \left( \int f(t)e^{i\lambda t} \, dt \right),
\]

since

\[
\int \mu(d\lambda) \left( \int |f(s)f(t)e^{i\lambda(t-s)}| \, ds \right) \, dt = \mu(\mathbb{R}) \left( \int |f(t)| \, dt \right)^2 < \infty .
\]

We generalize assumptions \( A \) and \( A_n \) of Sect. 2 as follows.

**ASSUMPTION A:**

\[ \mu(\mathbb{R}) = 1 . \]

That is, \( X(0) \sim N(0,1) \). Otherwise we may rescale \( X \).

**ASSUMPTION \( A_n \):** assumption \( A \) holds, and in addition,\(^1\)

\[ \forall \lambda \in [0,\infty) \quad \mu([\lambda, \lambda + 1]) \leq \frac{1}{n} . \]

The argument of Sect. 11c still applies, recall (11c5): for every \( f \in L_2[0,1] \),

\[ \int |g|^2 \, d\mu \leq C \left( \int |g(\lambda)|^2 \, d\lambda \right) \sup_\lambda \mu([\lambda, \lambda + 1]) ; \]

as before, \( g(\lambda) = \int_0^1 e^{i\lambda t} f(t) \, dt \), \( \|g\|^2 = 2\pi \|f\|^2 \), and

\[ \text{Var}(f, X) = \int |g|^2 \, d\mu . \]

Thus, assumption \( A_n \) implies (recall 11c3)

\[ \text{Var}(f, X) \leq \frac{C}{n} \|f\|^2 , \]

\(^1\)Alternatively you may take \( \lambda \in \mathbb{R} \); it is the same up to a factor 2 absorbed by an absolute constant.
and, by 23b3,
\[ \psi(X) \in \text{GaussLip}(C/\sqrt{n}) \]
whenever \( \psi : L_2[0, 1] \to \mathbb{R} \) is a Lip(1) function.

Now all arguments of 11d, 11e apply, and so, Theorems 2a2, 2a3 are generalized as follows.

Let \( X \) be a jointly measurable modification of a mean-square continuous stationary Gaussian random process on \( \mathbb{R} \), satisfying assumption \( A_n \).

23b2 Proposition. Let a function \( \varphi : \mathbb{R} \to \mathbb{R} \) be continuous almost everywhere, and
\[ \sup_x \frac{|\varphi(x)|}{1 + |x|} < \infty. \]
Then the random variable
\[ \xi = \int_0^1 \varphi(X(t)) \, dt \]
is integrable, \( E \xi = \int \varphi \, d\gamma^1 \), and for every \( \varepsilon > 0 \),
\[ P \left( \left| \xi - E \xi \right| \geq \varepsilon \right) \leq 2e^{-c_{\varepsilon, \varphi}n} \]
for some \( c_{\varepsilon, \varphi} > 0 \) (dependent on \( \varepsilon \) and \( \varphi \) only, not on \( n \)).

23b3 Proposition.
\[ P \left( T(X(\cdot)) \geq \varepsilon \right) \leq 2e^{-c_{\varepsilon}n} \]
for some \( c_{\varepsilon} > 0 \) dependent on \( \varepsilon \) only.

As before, for \( f \in L_1[0, 1] \),
\[ T(f) = \inf_g \int_0^1 |f(t) - g(t)| \, dt \]
where the infimum is taken over all measurable \( g : (0, 1) \to \mathbb{R} \) that send Lebesgue measure to \( \gamma^1 \).

A trivial rescaling of \( t \) by arbitrary \( L > 0 \) turns assumption \( A_n \) and Proposition 23b2 into the following.

**Assumption \( A_{n,L} \):** assumption \( A \) holds, and in addition,
\[ \forall \lambda \in [0, \infty) \quad \mu \left( \left[ \lambda, \lambda + \frac{1}{L} \right] \right) \leq \frac{1}{n}. \]
23b4 Corollary. Let $X$ satisfy $A_{n,L}$ and $\varphi$ be as in 23b2. Then the random variable
\[
\xi = \frac{1}{L} \int_0^L \varphi(X(t)) \, dt
\]
is integrable, $E\xi = \int \varphi \, d\gamma^1$, and for every $\varepsilon > 0$,
\[
P\left(|\xi - E\xi| \geq \varepsilon\right) \leq 2e^{-c_{\varepsilon,\varphi,n}}
\]
for some $c_{\varepsilon,\varphi} > 0$.

Now, at last, we can deal with a single process, getting rid of assumption $A_{n,L}$.

23b5 Theorem. Let $X$ be a jointly measurable$^1$ modification of a mean-square continuous stationary Gaussian random process on $\mathbb{R}$ whose spectral measure has a bounded density.$^2$ Let a function $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous almost everywhere, and
\[
\sup_x \frac{|\varphi(x)|}{1 + |x|} < \infty.
\]
Then random variables
\[
\xi_L = \frac{1}{L} \int_0^L \varphi(X(t)) \, dt \quad \text{for } L \in (0, \infty)
\]
are integrable, $E\xi_L = \int \varphi \, d\gamma^1$, and for every $\varepsilon > 0$,
\[
P\left(|\xi_L - E\xi_L| \geq \varepsilon\right) \leq 2e^{-c_{\varepsilon,\varphi,M}L}
\]
for some $c_{\varepsilon,\varphi,M} > 0$ (dependent only on $\varepsilon$, $\varphi$ and the supremum $M$ of the spectral density, not on $L$).

23b6 Exercise. Prove Theorem 23b5.

23b7 Exercise. Formulate and prove a single-process counterpart of 23b3.

23c Dimension two, and higher

A two-component (in other words, $\mathbb{R}^2$-valued) Gaussian random process on a set $T$ may be defined as a pair $(\Xi_1, \Xi_2)$ of Gaussian processes $\Xi_1, \Xi_2 : T \to G \subset L_2(\Omega, P)$. Or equivalently, as a Gaussian process $\Xi : T \times \{1, 2\} \to$ $\mu(\{1, 2\}) < \infty$.

$^1$Sample continuity is of course sufficient (by 22d3).
$^2$Equivalently, $\sup_{a<b} \frac{\mu([a,b])}{b-a} < \infty$. 

Similarly, a two-component random function $\xi$ on $T$ is a pair $(\xi_1, \xi_2)$ of random functions $\xi_1, \xi_2 : \Omega \to \mathbb{R}^T$, or a random function $\xi : \Omega \to \mathbb{R}^T \times \{1, 2\} = \mathbb{R}^T \times \{1, 2\}$. Clearly, $(\xi_1, \xi_2)$ is a modification of $(\Xi_1, \Xi_2)$ if and only if both $\xi_1$ is a modification of $\Xi_1$ and $\xi_2$ is a modification of $\Xi_2$. Continuity and measurability properties are defined evidently.

The covariance function of $\Xi : T \times \{1, 2\} \to G$ is $(s, k; t, l) \mapsto \mathbb{E}(\Xi(s, k)\Xi(t, l)) = \mathbb{E}\Xi_k(s)\Xi_l(t)$. Stationarity (assuming $T = \mathbb{R}$) is, by definition (recall 21e1),

$$\forall s, t \in \mathbb{R} \ \forall k, l \in \{1, 2\} \ \mathbb{E}\Xi_k(s)\Xi_l(t) = \mathbb{E}\Xi_k(0)\Xi_l(t - s).$$

For a stationarity $\Xi : \mathbb{R} \times \{1, 2\} \to G$ the covariance function $R : \mathbb{R} \times \{1, 2\} \times \{1, 2\} \to \mathbb{R}$ is, by definition,

$$R(t, k, l) = R_{k,l}(t) = \mathbb{E}\Xi_k(0)\Xi_l(t);$$

it determines the process up to isometry. Another function $r : \mathbb{R} \to \mathbb{R}$,

$$r(t) = \mathbb{E}\langle \Xi(0), \Xi(t) \rangle = \mathbb{E}(\Xi_1(0)\Xi_1(t) + \Xi_2(0)\Xi_2(t)) = R_{1,1}(t) + R_{2,2}(t),$$

containing only a partial information about $R$, will be called the traced covariance function. Normalizing the process to $r(0) = 1$ one may call $r$ the correlation function. However, such normalization is sometimes inconvenient, since the case $\Xi(0) \sim \gamma^2$ leads to $r(0) = 2$.

Clearly, the function $r$ is positive definite. Assuming mean square continuity of $\Xi$ we apply Bochner’s theorem and get the traced spectral measure,\(^2\) — a symmetric measure $\mu$ on $\mathbb{R}$ such that

$$\mathbb{E}\langle \Xi(0), \Xi(t) \rangle = r(t) = \int e^{i\lambda t} \mu(d\lambda).$$

In the finite-dimensional case treated in 11f, $r(t) = \sum_k |a_k|^2 \cos \lambda_k t$ ($a_k$ being vectors), thus, $\mu = \sum_k |a_k|^2(\delta_{\lambda_k} + \delta_{-\lambda_k})/2$.

Similarly to 28(3) we upgrade a two-component process $\Xi$ to the corresponding random element\(^3\) $X$ of $L_2([0, 1] \to \mathbb{R}^2)$ and consider

$$\langle f, X \rangle = \langle f_1, X_1 \rangle + \langle f_2, X_2 \rangle$$

\(^1\)A coordinate-free definition of a $E$-valued Gaussian process on $T$, for a finite-dimensional linear space $E$, may be given as follows: it is a linear map from $E^*$ to $G^T$.

\(^2\)The full (non-traced) spectral measure may be treated as a matrix-valued measure on $\mathbb{R}$, or equivalently, a $2 \times 2$ matrix whose elements are (signed) measures on $\mathbb{R}$. For an $E$-valued process one gets a “scalar product” on $E^*$ whose values are (signed) measures on $\mathbb{R}$.

\(^3\)Just upgrade $\Xi_1$ to $X_1$, $\Xi_2$ to $X_2$, and take $X = (X_1, X_2)$.\)
for $f = (f_1, f_2) \in L_2([0, 1] \to \mathbb{R}^2)$. We cannot calculate $\text{Var}(f, X)$ in terms of the traced spectral measure $\mu$ (like (23b1)), but we can bound it:

$$\text{Var}(f, X) \leq 2 \int \left| \int_0^1 f(t)e^{i\lambda t} \, dt \right|^2 \mu(d\lambda) = 2 \int \left( \left| \int_0^1 f_1(t)e^{i\lambda t} \, dt \right|^2 + \left| \int_0^1 f_2(t)e^{i\lambda t} \, dt \right|^2 \right) \mu(d\lambda).$$

Proof:

$$\text{Var}(f, X) = \|\langle f, X \rangle\|^2 = \|\langle f_1, X_1 \rangle + \langle f_2, X_2 \rangle\|^2 \leq 2\|\langle f_1, X_1 \rangle\|^2 + 2\|\langle f_2, X_2 \rangle\|^2$$

$$= 2 \int \left| \int_0^1 f_1(t)e^{i\lambda t} \, dt \right|^2 \mu_{1,1}(d\lambda) + 2 \int \left| \int_0^1 f_2(t)e^{i\lambda t} \, dt \right|^2 \mu_{2,2}(d\lambda),$$

where $\mu_{1,1}$ is the spectral measure for $X_1$, and $\mu_{2,2}$ --- for $X_2$; it remains to note that $\mu = \mu_{1,1} + \mu_{2,2}$ (think, why).

Assumption $A$ is replaced with

$$\Xi(0) \sim \gamma^2$$

(which implies $\mu(\mathbb{R}) = 2$); assumption $A_n$ still adds

$$\forall \lambda \in [0, \infty) \quad \mu([\lambda, \lambda + 1]) \leq \frac{1}{n}$$

where $\mu$ is the traced spectral measure. As before we get

$$\forall f \in L_2([0, 1] \to \mathbb{R}^2) \quad \text{Var}(f, X) \leq \frac{C}{n}\|f\|^2;$$

$$\psi(X) \in \text{GaussLip}(C/\sqrt{n})$$

whenever $\psi : L_2([0, 1] \to \mathbb{R}^2) \to \mathbb{R}$ is a Lip(1) function. Similarly to 11f, Propositions 23b2 and 23b3 generalize to two-component processes satisfying assumption $A_n$. Also Theorem 23b5 generalizes to two-component processes whose traced spectral measures have bounded densities.

All said about $\mathbb{R}^2$ holds equally well for $\mathbb{R}^d$, $d = 3, 4, \ldots$

23d Hints to exercises

23b6: $L = Cn$.  

$^1$In fact, the coefficient “2” is superfluous (see 11f for the discrete case); however, the stronger inequality is harder to prove.