3 Sensitivity and superconcentration

3a Theory

3a1 Definition. The Gaussian measure $\gamma_t^d$ on $\mathbb{R}^{2d}$, for $t \in [0, \infty)$, is the joint distribution of two random vectors $X$ and

$$X_t = e^{-t}X + \sqrt{1-e^{-2t}}X'$$

where $X$ and $X'$ are independent random vectors distributed $\gamma^d$ each.

That is, $\gamma_t^d$ is the image of $\gamma^{2d}$ under the linear map $(x, x') \mapsto (x, e^{-t}x + \sqrt{1-e^{-2t}}x')$ for $x, x' \in \mathbb{R}^d$. Note that $\gamma_t^d = (\gamma_t^1)^d$.

For small $t$, $X_t$ is treated as a small perturbation of $X$. Whether $f(X_t)$ is a small perturbation of $f(X)$ or not, is a matter of sensitivity of a function $f$.

3a2 Lemma. For every $f \in L^2(\mathbb{R}^d, \gamma^d)$ the function

$$t \mapsto \int \int f(x)f(y) \gamma_t^d(dx dy) = \mathbb{E} \left( f(X) f(X_t) \right)$$

is nonnegative and decreasing on $[0, \infty)$.

The same holds for a vector-function $f : \mathbb{R}^d \to \mathbb{R}^d$ with the scalar product $\langle f(x), f(y) \rangle$.

We consider a random variable of the form (1c1):

$$\xi(X) = \max_{a \in A} \langle X, a \rangle,$$

where $X \sim \gamma^d$, and $A \subset \mathbb{R}^d$ is a given finite set, $\forall a \in A \ |a| = 1$. We also introduce the $A$-valued random vector $\alpha(X)$ where $\alpha : \mathbb{R}^d \to A$ is defined (almost everywhere) by

$$\langle X, \alpha(X) \rangle = \max_{a \in A} \langle X, a \rangle = \xi(X).$$

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1 This section is a lightweight introduction into a work of Chatterjee [1].
Assumption $D_n$ ("superconcentration"): \[ \text{Var}(\xi) \leq \frac{1}{n}. \]

Let $(X, X_t) \sim \gamma^d_t$; we compare $\alpha(X)$ and $\alpha(X_t)$ as follows.

Assumption $E_n$ ("sensitivity"): \[ \mathbb{E} \langle \alpha(X), \alpha(X_{1/n}) \rangle \leq \frac{1}{n}. \]

3a3 Theorem. For every $n$ large enough,
   (a) assumption $D_{n^2}$ implies assumption $E_n$;
   (b) assumption $E_{2n}$ implies assumption $D_n$.

Thus, sensitivity and superconcentration are equivalent. The key to the proof of 3a3 is a wonderful equality

\[ \text{(3a4) Var}(\xi) = \int_0^\infty \left( \mathbb{E} \langle \alpha(X), \alpha(X_t) \rangle \right) e^{-t} \, dt. \]

3b Application: first-time percolation

Recall the first-time percolation of Sect. 1d:

\[ X_L = \sum_k X_{k,l,k} \quad \text{and} \quad \xi_m = \frac{1}{\sqrt{m}} \max_L X_L. \]

It appears that \[ \text{Var}(\xi_m) \to 0 \quad \text{as} \quad m \to \infty \]
(superconcentration). Sensitivity follows! Note that $\langle \alpha(X), \alpha(X_t) \rangle$ is the overlap $\frac{1}{m}|L \cap L_t|$ between two optimal paths $L$ and $L_t$.

References