11 Random real zeroes: no derivatives

11a Exponential concentration in general

11a1 Definition. 1 (a) A sequence \((x_n)_n\) of real numbers is exponentially decaying, if
\[
\exists \delta > 0, \ C < \infty \ \forall n \ |x_n| \leq Ce^{-\delta n}.
\]
(b) A sequence \((X_n)_n\) of random variables \(X_n : \Omega_n \to \mathbb{R}\) is exponentially concentrated at zero, if for every \(\varepsilon > 0\) the sequence of numbers \(P(|X_n| > \varepsilon)\) is exponentially decaying.
(c) A sequence \((X_n)_n\) of random variables \(X_n : \Omega_n \to \mathbb{R}\) is exponentially concentrated, if there exist \(x_n \in \mathbb{R}\) such that \((X_n - x_n)_n\) is exponentially concentrated at zero.

Notation:
\[(X_n)_n \in \text{ExpConZero} ; \quad (X_n)_n \in \text{ExpCon} .\]

Only the distributions of these \(X_n\) matter. For a sequence \((\mu_n)_n\) of probability measures on \(\mathbb{R}\) we define the relations \((\mu_n)_n \in \text{ExpConZero}\) and \((\mu_n)_n \in \text{ExpCon}\) evidently, getting \((X_n)_n \in \text{ExpConZero} \iff (\mu_n)_n \in \text{ExpConZero}\) where \(\mu_n\) is the distribution of \(X_n\); and the same for ExpCon. However, the language of random variables is more appropriate in many cases below.

11a2 Exercise. (a) All exponentially decaying sequences of real numbers are a linear space.
(b) ExpConZero is a linear space (for given \((\Omega_n)_n\)).

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1Not a standard definition.
(c) Let \((X_n)_n \in \text{ExpConZero}\) and \(x_n \in \mathbb{R}\). Then \((X_n - x_n)_n \in \text{ExpConZero}\) if and only if \(x_n \to 0\).
Prove it.

Thus, the condition \((X_n - x_n)_n \in \text{ExpConZero}\) determines \((x_n)_n\) up to \(o(1)\).

Recall that a number \(x\) is called a median of a random variable \(X\) if
\[
\mathbb{P}(X < x) \leq \frac{1}{2} \leq \mathbb{P}(X \leq x).
\]
All medians of \(X\) are in general a compact nonempty interval (often a single point). Also, \(x\) is a median of \(X\) if and only if \((-x)\) is a median of \((-X)\).

11a3 Exercise. The following three conditions are equivalent for every sequence of random variables \(X_n\):
(a) \((X_n)_n \in \text{ExpCon}\);
(b) there exist medians \(x_n\) of \(X_n\) such that \((X_n - x_n)_n \in \text{ExpConZero}\);
(c) all medians \(x_n\) of \(X_n\) satisfy \((X_n - x_n)_n \in \text{ExpConZero}\).
Prove it.

In this sense,
\[
(X_n)_n \in \text{ExpCon} \quad \text{if and only if} \quad (X_n - \text{Me}(X_n))_n \in \text{ExpConZero}.
\]
The median interval of \(X_n\) is of length \(o(1)\) whenever \((X_n)_n \in \text{ExpCon}\).
Medians cannot be replaced with expectations...

11a4 Exercise. (a) \(\text{ExpCon}\) is a linear space (for given \((\Omega_n)_n\)).
(b) Let \((X_n)_n, (Y_n)_n \in \text{ExpCon}\), then \(\text{Me}(X_n + Y_n) = \text{Me}(X_n) + \text{Me}(Y_n) + o(1)\).
Formulate it accurately, and prove.

11a5 Exercise. ("Sandwich") Let random variables \(Y_n : \Omega_n \to \mathbb{R}\) be such that for every \(r > 0\) there exist \(X_n, Z_n : \Omega_n \to \mathbb{R}\) satisfying
\[
(X_n)_n, (Z_n)_n \in \text{ExpCon},
\forall n \ (X_n \leq Y_n \leq Z_n \text{ a.s.}),
\forall n \ \text{Me}(Z_n) - \text{Me}(X_n) \leq r.
\]
Then \((Y_n)_n \in \text{ExpCon}.
Prove it.
Gaussian measures usually ensure $E|X_n| < \infty$ (integrability) and $\text{Me}(X_n) - EX_n \to 0$. Thus, we define $\text{ExpConInt}$ (for given $\Omega_n$) as the set of all sequences $(X_n)_n$ where $X_n : \Omega_n \to \mathbb{R}$ are integrable, and

$$(X_n - EX_n)_n \in \text{ExpConZero}.$$  

This is a linear space.

**11a6 Lemma.** Let random variables $Y_n : \Omega_n \to \mathbb{R}$ be such that for every $\varepsilon > 0$ there exist $X_n, Z_n : \Omega_n \to \mathbb{R}$ satisfying

$$(X_n)_n, (Z_n)_n \in \text{ExpConInt},$$

$$\forall n \ (X_n \leq Y_n \leq Z_n \ a.s.),$$

$$\forall n \ E Z_n - EX_n \leq \varepsilon.$$  

Then $(Y_n)_n \in \text{ExpConInt}.$

It can be proved similarly to 11a5. However, we need a quantitative version.

First, we note that the relation $(X_n)_n \in \text{ExpConInt}$ may be reformulated as follows: there exist families $(\delta_\varepsilon)_\varepsilon$ and $(C_\varepsilon)_\varepsilon$ of numbers $\delta_\varepsilon > 0$, $C_\varepsilon < \infty$ given for $\varepsilon > 0$ such that for all $n$,

$$\forall \varepsilon > 0 \ \mathbb{P}\left( |X_n - EX_n| > \varepsilon \right) \leq C_\varepsilon e^{-\delta_\varepsilon n}.$$  

Second, in order to get $\mathbb{P}\left( |Y_n - EY_n| > \varepsilon \right) \leq C_\varepsilon e^{-\delta_\varepsilon n}$ in the conclusion of “Sandwich”, we require $\mathbb{P}\left( |X_n - EX_n| > \varepsilon \right) \leq C_{r,\varepsilon} e^{-\delta_{r,\varepsilon} n}$ (and the same for $Z_n$) in the assumption; here $r$ is the parameter denoted by $r$ in 11a5.

The lemma below constructs $\delta_\varepsilon$ and $C_\varepsilon$ for given $\delta_{r,\varepsilon}$ and $C_{r,\varepsilon}$. The formulas are simple, but will not be used; rather, their existence will be used.

**11a7 Lemma.** ("Sandwich") Let positive numbers $\delta_{r,\varepsilon}$ and $C_{r,\varepsilon}$ be given for all positive $r$ and $\varepsilon$. Let random variables $Y_n : \Omega_n \to \mathbb{R}$ be such that for every $r > 0$ there exist $X_n, Z_n : \Omega_n \to \mathbb{R}$ satisfying

$$\forall n, \varepsilon \ \mathbb{P}\left( |X_n - EX_n| > \varepsilon \right) \leq C_{r,\varepsilon} e^{-\delta_{r,\varepsilon} n},$$

$$\forall n, \varepsilon \ \mathbb{P}\left( |Z_n - EZ_n| > \varepsilon \right) \leq C_{r,\varepsilon} e^{-\delta_{r,\varepsilon} n},$$

$$\forall n \ (X_n \leq Y_n \leq Z_n \ a.s.),$$

$$\forall n \ E Z_n - EX_n \leq r.$$  

Then

$$\forall n, \varepsilon \ \mathbb{P}\left( |Y_n - EY_n| > \varepsilon \right) \leq C_\varepsilon e^{-\delta_\varepsilon n}$$  

where $\delta_\varepsilon = \delta_{\varepsilon/2,\varepsilon/2}$ and $C_\varepsilon = 2C_{\varepsilon/2,\varepsilon/2}$.  

11a8 Exercise. Prove Lemma 11a7.

11a9 Lemma. (“Approximation”) Let integrable random variables $X_n : \Omega_n \to \mathbb{R}$ be such that for every $\varepsilon > 0$ there exist $Y_n : \Omega_n \to \mathbb{R}$ satisfying

$$(Y_n)_n \in \text{ExpConInt},$$

the sequence of numbers $\mathbb{P}(|X_n - Y_n| > \varepsilon)$ is exponentially decaying,

$$\forall n \quad |\mathbb{E} X_n - \mathbb{E} Y_n| \leq \varepsilon.$$

Then $(X_n)_n \in \text{ExpConInt}$.

11a10 Exercise. Prove Lemma 11a9.

Here is a quantitative version. The assumption $(Y_n)_n \in \text{ExpConInt}$ is weakened (to a single $\varepsilon$ . . .). The same $\delta, C_\varepsilon$ are used in two assumptions, which is not a problem (just take the minimum of two $\delta_\varepsilon$ and the sum of two $C_\varepsilon$).

11a11 Lemma. (“Approximation”) Let positive numbers $\delta_\varepsilon$ and $C_\varepsilon$ be given for all positive $\varepsilon$. Let random variables $X_n : \Omega_n \to \mathbb{R}$ be such that for every $\varepsilon > 0$ there exist $Y_n : \Omega_n \to \mathbb{R}$ satisfying

$$\forall n \quad P \left( |Y_n - \mathbb{E} Y_n| > \varepsilon \right) \leq C_\varepsilon e^{-\delta_\varepsilon n},$$
$$\forall n \quad P \left( |X_n - Y_n| > \varepsilon \right) \leq C_\varepsilon e^{-\delta_\varepsilon n},$$
$$\forall n \quad |\mathbb{E} X_n - \mathbb{E} Y_n| \leq \varepsilon.$$

Then

$$\forall n, \varepsilon \quad P \left( |X_n - \mathbb{E} X_n| > \varepsilon \right) \leq 2C_\varepsilon e^{-\delta_\varepsilon/3n}.$$

11a12 Exercise. Prove Lemma 11a11.

11b Exponential concentration over Gaussian measures

If a function $\xi : \mathbb{R}^d \to \mathbb{R}$ is Lip($\sigma$) for a given $\sigma > 0$ then Theorem 1a2 gives $\xi[\gamma^d] = f[\gamma^1]$ for an increasing $f : \mathbb{R} \to \mathbb{R}$, $f \in \text{Lip}(\sigma)$. Let us denote by GaussLip($\sigma$) the set of all such random variables. Clearly, $f(0)$ is the only median of $\xi$, and \footnote{$\zeta \sim \gamma^1$ as before.}

$$P( |\xi - \mathbb{M} \xi| > \varepsilon ) = P( |f(\zeta) - f(0)| > \varepsilon ) \leq$$
$$\leq P( |\zeta| > \varepsilon/\sigma ) \leq C \exp \left( - \frac{\varepsilon^2}{2\sigma^2} \right)$$
for some absolute constant $C$.\(^1\) Also, $|\operatorname{Me}(\xi) - \mathbb{E} \xi| = |f(0) - \int f \, d\gamma^1| \leq C\sigma$ for another absolute constant $C$.\(^2\) It follows easily that

$$ (11b1) \quad \mathbb{P} \left( |\xi - \mathbb{E} \xi| > \varepsilon \right) \leq C \exp \left( - \frac{c \varepsilon^2}{2\sigma^2} \right) $$

for some absolute constants $c, C;\(^3\)\(^4\)$ and, of course,

$$ (11b2) \quad \mathbb{E} |\xi - \mathbb{E} \xi| \leq C\sigma $$

for some absolute constant $C$.

### 11c Using assumption $A_n$

We consider the Gaussian random function $X(\cdot)$ introduced in Sect. 2a as a linear function of the independent $N(0, 1)$ random variables $X_1, \ldots, X_{2n}$ (via $a_1, \ldots, a_N$ and $\lambda_1, \ldots, \lambda_N$) under the assumption $A_n$ (also introduced in Sect. 2a). Here is a non-probabilistic property of the linear operator $\mathbb{R}^{2n} \rightarrow L_2[0, 1].\(^5\)

#### 11c1 Proposition.

$$ \int_0^1 X^2(t) \, dt \leq \frac{C}{n} (X_1^2 + \cdots + X_{2n}^2) $$

for some absolute constant $C$.

#### 11c2 Remark. Assumption $A_n$ requires also assumption $A$, namely $\sum_k a_k^2 = 1$, but we do not need it here; we use only the assumption

$$ \forall \lambda \in [0, \infty) \quad \sum_{k: \lambda_k \in [\lambda, \lambda + 1]} a_k^2 \leq \frac{1}{n}. $$

Given $f \in L_2[0, 1]$, we consider the random variable

$$ \langle f, X \rangle = \int_0^1 f(t) X(t) \, dt; $$

this is a linear combination of $X_1, \ldots, X_{2n}$, thus $\langle f, X \rangle \sim N(0, \operatorname{Var}(f, X))$.

---

\(^1\) $C = \sup_{t > 0} e^{t^2/2} \int_t^\infty (2\pi)^{-1/2} e^{-u^2/2} \, du = 2 \sup_{t > 0} (2\pi)^{-1/2} \int_0^\infty \exp(-\frac{s^2}{2} - ts) \, ds = 1$.

\(^2\) $C = (2\pi)^{-1/2} \int_0^\infty t e^{-t^2/2} \, dt = 1/\sqrt{2\pi}$.

\(^3\) Here and henceforth, constants $c$ and $C$ (possibly with indices) are positive. They may be different in different formulas.

\(^4\) In fact, $c = 1$ and $C = 2$. Moreover, $\mathbb{P} (|\xi - \mathbb{E} \xi| > \varepsilon) \leq 2 \mathbb{P} (\sigma \zeta > \varepsilon)$ (Cirel’son, Ibragimov, Sudakov 1976), thus, $\mathbb{P} (|\xi - \mathbb{E} \xi| > \varepsilon) \leq \exp(-\frac{\varepsilon^2}{2\sigma^2})$.

\(^5\) But under assumption $A$ only, the operator need not be of small norm; just try $N = 1$. 
11c3 Exercise. Deduce 11c1 from the following claim (to be proved soon):
\[
\text{Var}(f, X) \leq \frac{C}{n} \|f\|^2.
\]

11c4 Exercise. Prove that
\[
\text{Var}(f, X) = \sum_{k=1}^{N} a_k^2 |g(\lambda_k)|^2,
\]
where \(g(\lambda) = \int_0^1 e^{i\lambda t} f(t) \, dt\).

It is well-known that \(\|g\|_2^2 = 2\pi \|f\|_2^2\). Thus, the claim in 11c3 boils down to
\[
\sum_{k} a_k^2 |g(\lambda_k)|^2 \leq C \|g\|_2^2 \sup_{\lambda; \lambda_k \in [\lambda, \lambda + 1]} \sum_k a_k^2,
\]
which may be rewritten as
\[
\int |g|^2 \, d\mu \leq C \left( \int |g|^2 \, dm \right) \sup_{\lambda} \mu([\lambda, \lambda + 1])
\]
where \(\mu = \sum_k a_k^2 \delta_{\lambda_k}\) (a discrete measure), and \(m\) is the Lebesgue measure.

The idea is, roughly, that \(g\) cannot be nearly concentrated on a short interval, because \(f\) is concentrated on an interval of length 1. The proof, given below, uses Fourier transform \((\varphi \mapsto \hat{\varphi})\) and convolution \((\ast)\). If you are familiar with these, keep reading. Otherwise feel free to skip the rest of 11c.

11c6 Lemma. There exist even real-valued functions \(\varphi \in L_{\infty}[-0.5, 0.5] \subset L_1(\mathbb{R})\) and \(\psi \in L_1(\mathbb{R})\) such that \(\hat{\varphi}(x) \hat{\psi}(x) = 1\) for all \(x \in [-1, 1]\).

Proof. We take \(\varphi(t) = \text{const on } [-0.5, 0.5] \text{ (and 0 outside)}, \varphi(t) = \frac{1}{\pi} \sin \frac{\pi}{2} t\), note that \(\varphi(\cdot)\) does not vanish on \([-1, 1]\), \(1/\varphi(\cdot)\) is smooth on \([-1, 1]\) and therefore can be extended to a smooth compactly supported function \(\hat{\varphi}(\cdot)\); its Fourier transform is integrable, since it decays fast enough.

Proof of the proposition. The function \(|g(\cdot)|^2\) is the Fourier transform of a function supported on \([-1, 1]\) and therefore invariant under multiplication by \(\hat{\varphi} \hat{\psi}\). It means that \(|g|^2 = |g|^2 \ast \varphi \ast \psi\). Thus,
\[
\int |g|^2 \, d\mu = \langle |g|^2 \ast \psi, \mu \ast \varphi \rangle \leq \| |g|^2 \ast \psi \|_1 \| \mu \ast \varphi \|_{\infty};
\]
\[
\| |g|^2 \ast \psi \|_1 \leq \| |g|^2 \|_1 \| \psi \|_1 = \| g \|_2^2 \| \psi \|_1 \leq C \| g \|_2^2;
\]
\[
\| \mu \ast \varphi \|_{\infty} \leq \| \varphi \|_{\infty} \sup_{\lambda} \mu([\lambda - 0.5, \lambda + 0.5]) \leq C \sup_{\lambda} \mu([\lambda, \lambda + 1]),
\]
which gives \((11c5)\). \(\square\)

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\(\square\) Do not forget that \(C\) may be different in different formulas.
11d  Proving Theorem 2a2

If \( \xi : L_2[0,1] \to \mathbb{R} \) is Lip(1) then \( \xi(X) \), treated as a function of \( X_1, \ldots, X_{2N} \), is a Lip\((C/\sqrt{n})\) function \( \mathbb{R}^{2N} \to \mathbb{R} \) (by (11c1)). Thus, \( \xi(X) \in \text{GaussLip}(C/\sqrt{n}) \).

By (11b1),
\[
\Pr\left( |\xi - \mathbb{E}\xi| > \varepsilon \right) \leq C \exp(-c\varepsilon^2 n)
\]

for some absolute constants \( c, C \). In this sense, abusing the language, we write (under assumption \( A_n \))
\[
\xi \in \text{ExpConInt}(n)
\]
whenever \( \xi \) is Lip(1) on \( L_2[0,1] \), or Lip\((C)\) for some \( C \) not depending on \( n \). Usually, a stronger condition will be satisfied: \( \xi \) is Lip\((C)\) on \( L_1[0,1] \).

11d2 Exercise. Prove Lemma 2a1.

11d3 Exercise. Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be Lip(1). Then the function \( \xi : L_1[0,1] \to \mathbb{R}, \)
\[
\xi(x) = \int_0^1 \varphi(x(t)) \, dt,
\]
is well-defined and Lip(1).

Prove it.

Thus, for such \( \varphi \) the random variable
\[
\xi = \int_0^1 \varphi(X(t)) \, dt
\]
satisfies
\[
\xi \in \text{GaussLip}(C/\sqrt{n}) ; \quad \xi \in \text{ExpConInt}(n)
\]
with absolute constants (as in (11d1)).

Now let \( \varphi \) be as in Theorem 2a2 (continuous a.e., of linear growth). We introduce for every \( k \)
\[
\varphi_k^-(x) = \inf_y (\varphi(y) + k|y - x|) , \quad \varphi_k^+(x) = \sup_y (\varphi(y) - k|y - x|).
\]

11d4 Exercise. (a) \( \varphi_k^- \), \( \varphi_k^+ \) are Lip\((k)\) functions \( \mathbb{R} \to \mathbb{R} \) for all \( k \) large enough;\(^2\)
(b) \( \varphi_k^- \uparrow \varphi \) and \( \varphi_k^+ \downarrow \varphi \) almost everywhere;

\(^1\)I often write just \( \xi \) instead of \( \xi(X) \).
\(^2\)Do you understand why not just “for all \( k \)?"
(c) there exists $C_\varphi$ such that for all $k$ large enough and all $x$

$$-C_\varphi(1 + |x|) \leq \varphi^-(x) \leq \varphi^+(x) \leq C_\varphi(1 + |x|).$$

Prove it.

It follows (using Fubini and the dominated convergence theorem) that $E\xi - k \uparrow E\xi$ and $E\xi - k \downarrow E\xi$ a.s., where $\xi = \int_0^1 \varphi^- (X(t)) dt$. We have a “sandwich”; and so, Theorem 2a2 follows by 11a7. (The upper bound $2e^{-c_\varphi n}$ is not stronger than $C_\varphi e^{-c_\varphi n}$ since $c_\varphi$ can be made smaller.)

11e Proving Theorem 2a3

The function $T$ was defined in Sect. 2a on $C[0, 1]$, but the same definition works on $L^1[0, 1]$ and evidently gives a Lip(1) function $T : L^1[0, 1] \to [0, \infty)$. It follows that $T(X) \in \text{ExpConInt}(n)$. However, Theorem 2a3 states that $T(X) \in \text{ExpConZero}(n)$. Thus, it is sufficient to prove that $E T(X) \leq \varepsilon_n \to 0$.

We modify $T$ as follows:

$$T_k(f) = \inf_g \|\psi_k(f(\cdot)) - \psi_k(g(\cdot))\|_1,$$

where $g$ is as before (distributed $\gamma^1$), and $\psi_k(x) = \text{mid}(-k, x, k)$, that is, $-k$ for $x \in (-\infty, -k]$; $x$ for $x \in [-k, k]$; and $k$ for $x \in [k, \infty)$. We have

$$E |T_k(X(\cdot)) - T(X(\cdot))| \leq E \|\psi_k(X(\cdot)) - X(\cdot)\|_1 + \|\psi_k(g(\cdot)) - g(\cdot)\|_1 = 2 \int |\psi_k(x) - x| \gamma^1(dx) \to 0$$

as $k \to \infty$. It remains to prove that $E T_k(X) \leq \varepsilon_{k,n} \to 0$ as $n \to \infty$.

11e1 Exercise. For every $f \in L^1[0, 1]$ and every Lip(1) function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\left| \int_0^1 \varphi(f(t)) dt - \int \varphi d\gamma^1 \right| \leq T(f).$$

Prove it.

It is well-known that

$$\sup_{\varphi} \left| \int_0^1 \varphi(f(t)) dt - \int \varphi d\gamma^1 \right| = T(f),$$

where the supremum is taken over all Lip(1) functions $\mathbb{R} \to \mathbb{R}$. (I use this fact without proof.) Clearly we may demand $\varphi(0) = 0$.

\footnote{Kantorovich-Rubinstein theorem. This $T(f)$ is nothing but the transportation distance between $\gamma^1$ and the distribution of $f$. This fact is evident when $f$ is a step function. It extends to the whole $L^1[0, 1]$ by continuity.}
**11e2 Exercise.** For every $k$ and $\varepsilon$ there exists a finite set of $\text{Lip}(1)$ functions $\varphi_1, \ldots, \varphi_N : [-k, k] \to \mathbb{R}$ such that $\varphi_1(0) = 0, \ldots, \varphi_N(0) = 0$, and every $\text{Lip}(1)$ function $\varphi : [-k, k] \to \mathbb{R}$ such that $\varphi(0) = 0$ is $\varepsilon$-close to some $\varphi_i$ uniformly on $[-k, k]$.

Prove it.

**11e3 Exercise.** Prove that

$$T_k(f) \leq 2\varepsilon + \max_{i=1,\ldots,N} \left| \int_0^1 \varphi_i(\psi_k(f(t))) \, dt - \int \varphi_i(\psi_k(\cdot)) \, d\gamma \right|.$$ 

The function $\varphi_i(\psi_k(\cdot))$ is $\text{Lip}(1)$, thus the random variable $\xi_{i,k} = \int_0^1 \varphi_i(\psi_k(X(t))) \, dt$ belongs to $\text{GaussLip}(C/\sqrt{n})$. By (11b2), $E|\xi_{i,k} - E\xi_{i,k}| \leq C/\sqrt{n}$. Thus,

$$E T_k(X) \leq 2\varepsilon + \max_{i=1,\ldots,N} |\xi_{i,k} - E\xi_{i,k}| \leq 2\varepsilon + N_{k,\varepsilon} \cdot \frac{C}{\sqrt{n}},$$

which can be made small enough by choosing $\varepsilon$ first and $n$ afterwards. That is, $E T_k(X) \leq \varepsilon_{k,n} \to 0$ as $n \to \infty$, which completes the proof.\(^1\)

**11f Dimension two, and higher**

Returning to the definition of $X(\cdot)$ given in Sect. 2a via $a_1, \ldots, a_N$ and $\lambda_1, \ldots, \lambda_N$, we replace the numbers $a_1, \ldots, a_N > 0$ with vectors $a_1, \ldots, a_N \in \mathbb{R}^2$, thus getting $X : \mathbb{R} \to \mathbb{R}^2$; we endow $\mathbb{R}^2$ with the Euclidean norm $x \mapsto |x|$. Further, all occurrences of $a_k^2$ (in assumptions $A$ and $A_n$, and everywhere) turn into $|a_k|^2$, and all occurrences of $X^2(t)$ (in Prop. 11c1 and everywhere) into $|X(t)|^2$. We also replace the requirement $0 < \lambda_1 < \cdots < \lambda_N < \infty$ with a weaker requirement $0 < \lambda_1 \leq \cdots \leq \lambda_N < \infty$, thus allowing a single frequency to cover more than one dimension.\(^2\) The distribution of the process $X$ fails to determine uniquely the vectors $a_k$, but still determines the measure $\sum k |a_k|^2 \delta_{\lambda_k}$, since

$$E \langle X(0), X(t) \rangle = \sum_{k=1}^N |a_k|^2 \cos \lambda_k t .$$

\(^1\)In fact, $P(T(X) \geq \varepsilon) \leq \exp(-c((\varepsilon - \alpha_n)^+)^2 n)$ for some absolute constant $c$ and some $\alpha_n \to 0$ (depending on $n$ only). It is like the large deviations principle with the rate function $I(\varepsilon) \geq \varepsilon^2$.

\(^2\)Think, what does it change in the one-dimensional case.
Still, \textbf{11c3} and \textbf{11c4} hold, but $f \in L_2[0,1]$ turns into $f \in L_2([0,1] \to \mathbb{R}^2)$, and \textbf{11c4} becomes

$$\text{Var}\langle f, X \rangle = \sum_{k=1}^{N} |(a_k, g(\lambda_k))|^2 \leq \sum_{k=1}^{N} |a_k|^2 |g(\lambda_k)|^2.$$  

Nothing changes in the rest of Sect. \textbf{11c} (it is about the measure $\mu = \sum_k |a_k|^2 \delta_{\lambda_k}$).

Thus, \textbf{11c1} gives us a linear operator $\mathbb{R}^{2N} \to L_2([0,1] \to \mathbb{R}^2)$ of norm $\leq C/\sqrt{n}$. If $\xi : L_2([0,1] \to \mathbb{R}^2) \to \mathbb{R}$ is Lip(1) then $\xi(X) \in \text{GaussLip}(C/\sqrt{n})$.

The function $\varphi : \mathbb{R} \to \mathbb{R}$ in \textbf{2a1, 2a2, 11d3, 11d4} (as well as $\varphi_k^\pm$ in \textbf{11d4}) turns into $\varphi : \mathbb{R}^2 \to \mathbb{R}$; $\gamma^1$ in \textbf{2a1} turns into $\gamma^2$. And of course, $L_1[0,1]$ in \textbf{11d3} turns into $L_1([0,1] \to \mathbb{R}^2)$.

Theorem \textbf{2a2} is thus generalized.

About Theorem \textbf{2a3}. The definition of $T(f)$ is generalized evidently ($\gamma^1$ turns into $\gamma^2$); now $T$ is a Lip(1) function $L_1([0,1] \to \mathbb{R}^2) \to [0,\infty)$. The functions $\psi_k : \mathbb{R}^2 \to \mathbb{R}^2$ may be defined by $\psi_k(x) = x$ if $|x| \leq k$, otherwise $\psi_k(x) = kx/|x|$. The Kantorovich-Rubinstein theorem holds for all metric spaces, in particular $\mathbb{R}^2$. Exercise \textbf{11e2} generalizes for a disk of $\mathbb{R}^2$ (and in fact for every precompact metric space). Exercise \textbf{11e3} and the rest of the proof remain valid.

Theorem \textbf{2a3} is thus generalized.

All said about $\mathbb{R}^2$ holds equally well for $\mathbb{R}^d$, $d = 3, 4, \ldots$.

\textbf{11g} Hints to exercises

\textbf{11d2} Fubini.

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\item $T$, 8
\item $T_k$, 8
\item $X(\cdot)$, 5
\item $X_1, \ldots, X_2n$, 5
\end{itemize}

1 And so, the absolute constant $C$ in \textbf{11c1} remains intact.

2 Still, $P(T(X) \geq \varepsilon) \leq \exp(-c((\varepsilon - \alpha_n)^+)^2n)$ for the same absolute constant $c$ as in dimension one, and another (worse) sequence $\alpha_n \to 0$. 

\text{Tel Aviv University, 2010} \quad Gaussian measures : proofs and more