This preparatory chapter aims at some acquaintance with unbounded operators and functions of them. Postponing the general theory, here we treat functions of the differentiation operator on \( L_2(\mathbb{R}) \) using the Fourier transform.

### 2a Introduction

For a diagonal matrix \( A = \text{diag}(a_1, \ldots, a_n) \) we have \( p(A) = \text{diag}(p(a_1), \ldots, p(a_n)) \) for every polynomial \( p \). For a diagonalizable matrix \( A \) we have \( FAF^{-1} = \text{diag}(a_1, \ldots, a_n) \) for some (invertible) matrix \( F \), and \( Fp(A)F^{-1} = p(FAF^{-1}) = \text{diag}(p(a_1), \ldots, p(a_n)) \). It is natural to define

\[
\varphi(A) = F^{-1} \text{diag}(\varphi(a_1), \ldots, \varphi(a_n)) F
\]

for every \( \varphi : \{a_1, \ldots, a_n\} \rightarrow \mathbb{C} \). The result does not depend on the choice of \( F \). The map \( \varphi \mapsto \varphi(A) \) is a homomorphism of algebras, that is,

- **linearity:** \((a\varphi + b\psi)(A) = a\varphi(A) + b\psi(A)\),
- **multiplicativity:** \((\varphi \cdot \psi)(A) = \varphi(A)\psi(A)\)

for all \( a, b \in \mathbb{C} \) and \( \varphi, \psi \in \mathbb{C}^{\{a_1, \ldots, a_n\}} \). Note also

- **unit preservation:** \( 1 \text{l}(A) = 1 \text{l} \).

Assume in addition that \( A^* = A \), then \( a_1, \ldots, a_n \in \mathbb{R} \), \( F \) can be chosen unitary, and the homomorphism is a \(*\)-homomorphism, that is,

- **involution preservation:** \( \overline{\varphi(A)} = (\varphi(A))^* \)

for all \( \varphi \in \mathbb{C}^{\{a_1, \ldots, a_n\}} \). In particular, \( \varphi(A) \) is self-adjoint for all \( \varphi \in \mathbb{R}^{\{a_1, \ldots, a_n\}} \). Note also
positivity: if $\varphi \geq 0$ then $\varphi(A) \geq 0$

for all $\varphi \in \mathbb{R}^{\{a_1, \ldots, a_n\}}$.

For a compact self-adjoint operator in a Hilbert space the situation is similar; a finite spectrum $\{a_1, \ldots, a_n\}$ is replaced with a sequence converging to 0.

For a bounded (not just compact) self-adjoint operator in a Hilbert space the situation is similar in principle, but more complicated technically, because of (possibly) continuous spectrum. Additional technical complications appear for unbounded self-adjoint operators.

In this chapter we consider mostly the (unbounded) differentiation operator in $L_2(\mathbb{R})$, which is rather easy due to its diagonalization by the Fourier transform.

2b Multiplication operators

All multiplication operators are functions of one important operator $Q$, the generator of the unitary group $(V(b))_{b \in \mathbb{R}}$.

We know that $L_\infty(\mathbb{R})$ acts on $L_2(\mathbb{R})$ by multiplication operators,

$$L_2 \ni f \mapsto \varphi \cdot f \in L_2, \quad \varphi \in L_\infty.$$

2b1 Exercise. Formulate and prove the five properties of this action:

linearity,
multiplicativity,
unit preservation,
involution preservation,
positivity.

What about multiplication

$$f \mapsto (q \mapsto qf(q))$$

by the unbounded function $q \mapsto q$? Surely it is not a bounded operator. We define

$$D_Q = \left\{ f \in L_2(\mathbb{R}) : \int q^2 |f(q)|^2 \, dq < \infty \right\},$$

$$Q : D_Q \to H,$$

$$Qf : q \mapsto qf(q) \quad \text{for } f \in D_Q;$$
Q is an example of so-called “densely defined unbounded linear operator”, and the dense linear set $D_Q$ is its domain. Similarly, for every $\varphi \in L_0(\mathbb{R})$ (just a measurable function $\mathbb{R} \to \mathbb{C}$) we define

$$D_\varphi = \{ f \in L_2(\mathbb{R}) : \varphi \cdot f \in L_2(\mathbb{R}) \},$$

$$A_\varphi : D_\varphi \to H, \quad A_\varphi f = \varphi \cdot f \quad \text{for } f \in D_\varphi;$$

$A_\varphi$ is a densely defined linear operator, unbounded unless $\varphi \in L_\infty$. The special case $\varphi = \text{id} : q \mapsto q$ leads to the operator $A_{\text{id}} = Q$.

**2b2 Exercise.** If $\varphi, \psi \in L_0$ satisfy $\varphi - \psi \in L_\infty$ then

$$D_\varphi = D_\psi,$$

$$A_\varphi f - A_\psi f = (\varphi - \psi) \cdot f \quad \text{for } f \in D_\varphi = D_\psi.$$

Prove it.

In particular, $D_{\text{id} + c1} = D_{\text{id}} = D_Q$ for each $c \in \mathbb{C}$, and $A_{\text{id} + c1} = Q + c\mathbb{1}$.

**2b3 Exercise.** Let $\varphi \in L_\infty$, $\psi \in L_0$, then

$$D_{\varphi \cdot \psi} = \{ f : \varphi \cdot f \in D_\psi \} \supset D_\psi,$$

$$(\varphi \cdot \psi) \cdot f = \psi \cdot (\varphi \cdot f) \quad \text{for } f \in D_{\varphi \cdot \psi}.$$

The relations $D_{\varphi \cdot \psi} = D_\psi$ and $(\varphi \cdot \psi) \cdot f = \varphi \cdot (\psi \cdot f)$ (for $f \in D_{\varphi \cdot \psi}$) are generally wrong; however, they hold if $|\varphi(\cdot)|$ is bounded away from 0.

Prove the positive claims, and find counterexamples to the negative claims.

An interesting special case is well-known as Cayley transform. Given $\psi \in L_0$ such that $\psi = \psi$, we introduce $\varphi \in L_\infty$ by

$$\varphi(x) = \frac{\psi(x) - i}{\psi(x) + i},$$

observe that $|\varphi(\cdot)| = 1$ and $\psi - i\mathbb{1} = \varphi \cdot (\psi + i\mathbb{1})$, therefore $A_\varphi$ is unitary and $(\psi - i\mathbb{1}) \cdot f = (\psi + i\mathbb{1}) \cdot (\varphi \cdot f)$, which leads to a remarkable relation between the unbounded\(^1\) self-adjoint operator $A = A_\psi$ and the unitary operator $U = A_\varphi$:

$$\quad (A - i\mathbb{1}) f = (A + i\mathbb{1}) U f \quad \text{for } f \in D_A$$

\(^1\)Here and henceforth I often write “unbounded” meaning “generally, unbounded”, that is, “not necessarily bounded”.
(which determines $U$ uniquely), and

$$(\mathbb{I} - U)Af = i(\mathbb{I} + U)f \quad \text{for } f \in D_A$$

(since $(1 - \varphi) \cdot \psi = i(1 + \varphi)$), which restores $A$ from $U$.

Postponing the general definition of a function of operator, for now we define

$$\varphi(Q) = A_\varphi \quad \text{for } \varphi \in L_0(\mathbb{R}).$$

In particular, $\varphi = \text{id} + c\mathbb{I} : q \mapsto q + c$ gives $\varphi(Q) = Q + c\mathbb{I}$; $\varphi : q \mapsto q^n$ gives $\varphi(Q) = Q^n$; also, $\varphi : q \mapsto e^{ibq}$ gives $\varphi(Q) = \exp(ibQ)$.

2b5 Exercise. Let $n \in \{2, 3, \ldots\}$.

(a) $Q^n f$ is defined if and only if $Q^{n-1} f$ is defined and belongs to $D_Q$;
(b) in this case $Q^n f = Q(Q^{n-1} f)$.

Prove it.

2b6 Exercise. (a) $Q^{-1} f$ is defined if and only if there exists $g \in D_Q$ such that $Qg = f$;
(b) in this case such $g$ is unique, and $Q^{-1} f = g$.

Prove it.

Recall the unitary operators $V(b)$ of (1b12) (denoted there by $V_1(b)$).

Clearly,

$$\exp(ibQ) = V(b) \quad \text{for all } b \in \mathbb{R}.$$  

The operator $Q$ is the generator of the one-parameter unitary group $(V(b))_{b \in \mathbb{R}}$ in the following sense.

2b7 Exercise. (a) The following three conditions are equivalent for every $f \in L_2(\mathbb{R})$:

(a1) $\|f - \exp(i\lambda Q)f\| = O(\lambda)$ as $\lambda \to 0$;
(a2) $\frac{d}{d\lambda}\bigg|_{\lambda=0} \exp(i\lambda Q)f$ exists (in the norm);
(a3) $f \in D_Q$.

(b) In this case

$$Qf = -i\frac{d}{d\lambda}\bigg|_{\lambda=0} \exp(i\lambda Q)f.$$  

Prove it.

Hint: $|1 - e^{ibq}| \leq |bq|$; use Fatou’s lemma for (a1)$\implies$(a3), and the dominated convergence theorem for (a3)$\implies$(a2).
2c  Functions of the differentiation operator

All operators commuting with shifts are functions of one important operator $P$, the generator of the unitary group $(U(a))_{a \in \mathbb{R}}$ of shifts.

Recalling the general form of an operator commuting with shifts,

$$B_{\varphi}f = F^{-1}(\varphi \cdot Ff),$$

we observe another action $\varphi \mapsto B_{\varphi}$ of $L_\infty(\mathbb{R})$ on $L_2(\mathbb{R})$.

2c1 Exercise. Formulate and prove the five properties of this action:

- linearity,
- multiplicativity,
- unit preservation,
- involution preservation,
- positivity.

Hint: use 2b1 and unitarity of $F$.

We do the same for unbounded operators. Namely, for every $\varphi \in L_0(\mathbb{R})$ we define

$$D_{B_{\varphi}} = \{ f \in L_2(\mathbb{R}) : Ff \in D_{A_{\varphi}} \} = F^{-1}D_{A_{\varphi}},$$

$$B_{\varphi} : D_{B_{\varphi}} \to H,$$

$$B_{\varphi}f = F^{-1}(A_{\varphi}Ff) \quad \text{for } f \in D_{B_{\varphi}};$$

$B_{\varphi}$ is a densely defined linear operator (unbounded unless $\varphi \in L_\infty$) unitarily equivalent to $\varphi(Q)$,

$$B_{\varphi} = F^{-1}\varphi(Q)F,$$

and we treat it as a function of the operator $P = B_{\text{id}}$:

$$P = F^{-1}QF,$$

$$\varphi(P) = F^{-1}\varphi(Q)F.$$

Recall the unitary operators $U(a)$ of (1b11) (denoted there by $U_1(a)$). We have

$$U(a) = \exp(iaP) \quad \text{for all } a \in \mathbb{R}.$$

The operator $P$ is the generator of the one-parameter unitary group $(U(a))_{a \in \mathbb{R}}$ in the following sense.
2c2 Exercise. (a) The following three conditions are equivalent for every \( f \in L_2(\mathbb{R}) \):

(a1) \( \| f - U(a)f \| = O(a) \) as \( a \to 0 \);

(a2) \( \frac{d}{da} \bigg|_{a=0} U(a)f \) exists (in the norm);

(a3) \( f \in D_P \).

(b) In this case

\[
P f = -i \frac{d}{da} \bigg|_{a=0} U(a)f.
\]

Prove it.

Hint: use 2b7, unitarity of \( \mathcal{F} \), and the equality \( U(a) = \exp(iaP) \).

If \( f \) is nice enough, say, continuously differentiable and compactly supported, then clearly \( f' \in L_2 \) and

\[
U(a)f = f + af' + o(a) \quad \text{in the norm, as } a \to 0
\]

(since \( U(a)f : q \mapsto f(q + a) \)), thus \( f \in D_P \) and

\[
Pg = -if'.
\]

We see that in some sense \( iP \) is the differentiation operator \( f \mapsto f' \). However, what happens for not so nice functions?

2c3 Theorem. The following three conditions on \( f, g \in L_2(\mathbb{R}) \) are equivalent:

(a) \( f \in D_P \) and \( iP f = g \);

(b) there exist continuously differentiable compactly supported functions \( f_1, f_2, \ldots \) such that

\[
f_n \to f \quad \text{in } L_2, \quad f_n' \to g \quad \text{in } L_2;
\]

(c) for every \( a \in \mathbb{R} \),

\[
U(a)f = f + \int_0^a U(b)g \, db.
\]

(The latter is the Riemann integral of a continuous vector-function, recall 1g, especially 1g1.)
Proof (sketch). (a)⇒(b): we take \( f_n = (f \cdot \mathbb{1}_{(-n,n)}) * h_n \) where \( h_n \) are “triangles” \( q \mapsto \max(0, n - n^2|q|) \); then \( f_n \rightarrow f \) in \( L_2 \), and \( U(a)f_n = (U(a)f) * h_n \), thus \( f'_n = \frac{d}{da}\big|_{a=0} U(a)f_n = \frac{d}{da}\big|_{a=0} U(a)f \) * \( h_n = g * h_n \rightarrow g \) in \( L_2 \).

(b)⇒(c): \( \frac{d}{da}U(a)f_n = U(a)f'_n \), thus \( U(a)f_n = f_n + \int_0^a U(b)f'_n \, db \); we take the limit as \( n \rightarrow \infty \).

(c)⇒(a): \( \frac{d}{da}\big|_{a=0} U(a)f = \frac{d}{da}\big|_{a=0} \int_0^a U(b)g \, db = g \).

According to 2c4(b), \( f_n \rightarrow f \) in the so-called Sobolev space \( W^1_2(\mathbb{R}) \), and so, \( D_P = W^1_2(\mathbb{R}) \). Two more equivalent condition (without proof):

(d) \( \langle f, h' \rangle = -\langle g, h \rangle \) for all continuously differentiable compactly supported functions \( h \);

(e) \( f(x) = \lim_{y \to -\infty} f^x_a g(y) \, dy \) for almost all \( x \).

So, the Fourier transform diagonalizes also the differentiation operator: if \( f' = g \) in the generalized sense described above (namely, \( iPf = g \)) then \( i\rho(\mathcal{F}f)(p) = (\mathcal{F}g)(p) \) for almost all \( p \) (namely, \( iQ\mathcal{F}f = \mathcal{F}g \)). The converse is also true.

The relation \( \varphi(P) = \mathcal{F}^{-1}\varphi(Q)\mathcal{F} \) gives in particular operators \( P^n = \mathcal{F}^{-1}Q^n\mathcal{F} \).

**2c4 Exercise.** Let \( n \in \{2, 3, \ldots \} \).

(a) \( P^nf \) is defined if and only if \( P^{n-1}f \) is defined and belongs to \( D_P \);

(b) in this case \( P^n f = P(P^{n-1}f) \).

Prove it.

Hint: use 2b5.

For an infinitely differentiable compactly supported function \( f \) we have \((iP)^n f = f^{(n)} \). It is tempting to conclude that

\[
 f(q + a) = \sum_{n=0}^{\infty} \frac{a^n}{n!} f^{(n)}(q), \quad \text{since} \quad \exp(i\rho P) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!}(i\rho)^n,
\]

but this conclusion is evidently wrong (unless \( f = 0 \)). A series of unbounded operators is a more delicate matter!

**2c5 Exercise.** (a) \( P^{-1}f \) is defined if and only if there exists \( g \in D_P \) such that \( Pg = f \);

(b) in this case such \( g \) is unique, and \( P^{-1}f = g \).

Prove it.

Hint: use 2b6.

The Cayley transform of \( P \) (recall 2b4) is the unitary operator \( \varphi(P) = \mathcal{F}^{-1}\varphi(Q)\mathcal{F} \) where \( \varphi : p \mapsto \frac{p-1}{p+i} \). It satisfies

\[
 (P - i\mathbb{1})f = (P + i\mathbb{1})Uf \quad \text{for} \ f \in D_P,
\]
which means just \( f' + f = g' - g \) where \( g = Uf \), provided that \( f \) and \( g \) are nice enough (otherwise the derivatives are generalized). Can we calculate \( U \) more explicitly? Yes, we can! First we note that \( \varphi = 1L_{(\infty,0)}(q) \); \( \varphi : p \mapsto \frac{1}{p} = i \). Recalling Sect. 1h we observe that we can get \( Uf = f - 2f \ast g \) if we find \( g \in L_1 \) such that \((2\pi)^{1/2}Fg = \psi\). Clearly, \( g = (2\pi)^{-1/2}F^{-1}\psi \in L_2 \); but does \( g \) belong to \( L_1 \), and can we calculate it explicitly? Fortunately, such a function is well-known:

\[
g(q) = e^q \mathbb{1}_{(\infty,0)}(q) ;
\]

\[
\int_{-\infty}^{0} e^q e^{-ipq} dq = \int_{-\infty}^{0} e^{(1-ip)q} dq = \frac{1}{1-ip} = \frac{i}{p+1}.
\]

So,

\[
Uf = f - 2f \ast g;
\]

\[
Uf : q \mapsto f(q) - 2 \int_{0}^{\infty} e^{-u} f(q + u) \, du.
\]

2d Frequency bands, spectral projections

The operators \( Q \) and \( P \) have no eigenvectors but still have many invariant subspaces. The corresponding projections are instrumental in signal processing and quantum mechanics.

Indicator functions \( \varphi = \mathbb{1}_{(a,b)} \in L_\infty(\mathbb{R}) \) satisfy \( \varphi^2 = \varphi \) and \( \varphi' = \varphi \), therefore the operators

\[
E_{a,b} = E^{(Q)}_{a,b} = \varphi(Q) = \mathbb{1}_{(a,b)}(Q)
\]

are self-adjoint (that is, orthogonal) projections \( L_2(\mathbb{R}) \rightarrow L_2(a,b) \subset L_2(\mathbb{R}) \).

The relation \( \mathbb{1}_{(a,b)} + \mathbb{1}_{(b,c)} = \mathbb{1}_{(a,c)} \) in \( L_\infty \) (for \( a < b < c \)) implies the relation \( E_{a,b} + E_{b,c} = E_{a,c} \) between operators, and the corresponding direct sum relation \( L_2(a,b) \oplus L_2(b,c) = L_2(a,c) \) between subspaces. These subspaces are invariant under \( Q \) (and all \( \varphi(Q) \)). Note that

\[
\|E^{(Q)}_{a,b} f\|^2 = \langle E^{(Q)}_{a,b} f, f \rangle = \int_{a}^{b} |f(q)|^2 \, dq.
\]

In signal processing, \( \|f\|^2 \) is (proportional to) the energy of the signal \( f \); \( |f(t)|^2 \) is the energy density at the time \( t \); and \( \langle E^{(Q)}_{a,b} f, f \rangle \) is the energy within the time interval \((a, b)\).

In quantum mechanics, \( |f(q)|^2 \) is the probability density (at the point \( q \)) of the coordinate of a one-dimensional particle with the wave function \( f \).
(\|f\| = 1 \text{ is required}), and \langle E_{a,b}^{(Q)} f, f \rangle \text{ is the probability of finding the particle within the spatial interval } (a, b) \text{ (provided that the coordinate is measured).}^1

Accordingly, the operators

\[ E_{a,b}^{(P)} = \mathbb{1}_{(a,b)}(P) = \mathcal{F}^{-1} E_{a,b}^{(Q)} \mathcal{F} \]

are orthogonal projections satisfying \( E_{a,b}^{(P)} + E_{b,c}^{(P)} = E_{a,c}^{(P)} \) (for \( a < b < c \)). The corresponding subspaces (“frequency bands”) satisfy the direct sum relation, and are invariant under \( P \) (and all \( \varphi(P) \)).

**2d1 Exercise.**

\[
\|E_{a,b}^{(P)} f\|^2 = \langle E_{a,b}^{(P)} f, f \rangle = \int_a^b |(\mathcal{F} f)(p)|^2 \, dp.
\]
Prove it.

Hint: \( \mathcal{F}^{-1} = \mathcal{F}^* \).

In signal processing, \( \|\mathcal{F} f(\omega)\|^2 \) is the spectral density of the signal energy at the frequency \( \omega \); and \( \langle E_{a,b}^{(P)} f, f \rangle \) is the energy within the frequency band \( (a, b) \).

In quantum mechanics, \( |(\mathcal{F} f)(p)|^2 \) is the probability density (at the point \( p \)) of the momentum of a one-dimensional particle with the wave function \( f \) (\( \|f\| = 1 \text{ is required} \)), and \( \langle E_{a,b}^{(P)} f, f \rangle \) is the probability of finding the momentum within the interval \( (a, b) \) (provided that the momentum is measured).^2

**2d2 Exercise.** For every \( f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \),

\[
E_{a,b}^{(P)} f = g_{a,b} * f, \quad \text{where} \quad g_{a,b}(q) = \frac{1}{2\pi i} \frac{e^{ibq} - e^{iaq}}{q}.
\]

Prove it.

Hint: \( \mathcal{F}(g * f) = \ldots \)

Especially, \( g_{-b,b}(q) = \frac{\sin bq}{\pi q} \).

Be careful: \( g_{a,b} \) belongs to \( L_2(\mathbb{R}) \) but not \( L_1(\mathbb{R}) \). Nevertheless the convolution operator \( f \mapsto g_{a,b} * f \) is well-defined on a dense set of functions \( f \) and extends by continuity to all \( f \in L_2 \).^3

---

1The *ideal* measurement of the coordinate is meant. Do not take it too seriously. It is rather a toy model of a quantum measurement. The infinite resolution is unfeasible.

2Once again, the *ideal* measurement of the momentum is meant.

3Which cannot be said about \( |g_{a,b}(\cdot)| \ldots \)
2e  List of formulas

Multiplication operators:

(2e1) \[ Qf : q \mapsto qf(q) \text{ for } f \in D_Q; \]
(2e2) \[ \varphi(Q)f = \varphi \cdot f : q \mapsto \varphi(q)f(q) \text{ for } f \in D_{\varphi(Q)}; \]
(2e3) \[ \exp(ibQ) = V(b); \]
(2e4) \[ Qf = -i \frac{d}{db}\bigg|_{b=0} V(b)f \text{ for } f \in D_Q; \]
(2e5) \[ E_{a,b}(Q) = \mathbb{1}_{(a,b)}(Q); \]
(2e6) \[ \|E_{a,b}^{(Q)}f\|^2 = \langle E_{a,b}^{(Q)}f, f \rangle = \int_a^b |f(q)|^2 \, dq. \]

Operators commuting with shifts:

(2e7) \[ P = \mathcal{F}^{-1}Q\mathcal{F}; \]
(2e8) \[ Pf : q \mapsto -if'(q) \text{ for nice } f; \]
(2e9) \[ \varphi(P) = \mathcal{F}^{-1}\varphi(Q)\mathcal{F} : f \mapsto \mathcal{F}^{-1}(\varphi \cdot \mathcal{F}f); \]
(2e10) \[ \exp(iaP) = U(a); \]
(2e11) \[ Pf = -i \frac{d}{da}\bigg|_{a=0} U(a)f \text{ for } f \in D_P; \]
(2e12) \[ E_{a,b}^{(P)} = \mathbb{1}_{(a,b)}(P) = \mathcal{F}^{-1}E_{a,b}^{(Q)}\mathcal{F}; \]
(2e13) \[ \|E_{a,b}^{(P)}f\|^2 = \langle E_{a,b}^{(P)}f, f \rangle = \int_a^b |(\mathcal{F}f)(p)|^2 \, dp; \]
(2e14) \[ E_{a,b}^{(P)}f = \left(q \mapsto \frac{1}{2\pi i} \frac{e^{ibq} - e^{iaq}}{q} \right) * f \text{ for } f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}). \]

Index

- Cayley transform, 20
- densely defined, 20
- domain, 20
- generator, 21
- involution preservation, 18
- linearity, 18
- multiplicativity, 18
- positivity, 19
- unbounded operator, 20
- unit preservation, 18
- $D_P$, 23
- $D_Q$, 20
- $E_{a,b}^{(P)}$, 20
- $E_{a,b}^{(Q)}$, 25
- exp$(iaP)$, 22
- exp$(ibQ)$, 21
- $P^n$, 24
- $Q^n$, 21