9 Beyond compactness: basic notions

A simple example of a non-compact space is $\mathbb{R}$. Here is an instructive example of a seminorm on the Banach space $C_{\mathrm{b}}(\mathbb{R})$ of all bounded continuous functions on $\mathbb{R}$;

$$\|f\| = \limsup_{|x|\to\infty} |f(x)| \, .$$

It satisfies (2a1), (2a2) and (2a6), however, it is not of the form $\sup(|f|\Pi)$. We feel that it is situated at the points $\pm\infty$ of the extension $[-\infty,\infty]$ of $\mathbb{R}$, not on $\mathbb{R}$ itself. We exclude such seminorms by requiring

$$f_k \downarrow 0 \text{ pointwise} \quad \Longrightarrow \quad \|f_k\| \to 0$$

for all $f_1, f_2, \dots \in C_{\mathrm{b}}(\mathbb{R})$.

9a Large deviations principle (LDP)

Let $\mathcal{X}$ be a Polish space, that is, a separable topological space metrizable by a complete metric.

All bounded continuous functions $\mathcal{X} \to \mathbb{R}$ are a (generally, nonseparable) Banach space $C_{\mathrm{b}}(\mathcal{X})$.

All (Borel) probability measures on $\mathcal{X}$ are a set $P(\mathcal{X})$. Every $\mu \in P(\mathcal{X})$ gives us a linear functional $C_{\mathrm{b}}(\mathcal{X}) \to \mathbb{R}$,

$$f \mapsto \int f \, \mathrm{d}\mu \, .$$

The linear functional determines μ uniquely.

Let numbers $p_1, p_2, \dots \in [1,\infty)$ be given such that $p_n \to \infty$.

9a1 Definition. (a) A sequence $(\mu_n)_n$ of probability measures on a Polish space $\mathcal{X}$ is *LD-convergent* with rate $(p_n)_n$, if the limit

$$\|f\|_{\lim} = \lim_{n\to\infty} \biggl(\int |f|^{p_n} \, \mathrm{d}\mu_n \biggr)^{1/p_n}$$

exists for all $f \in C_{\mathrm{b}}(\mathcal{X})$, and

$$f_k \downarrow 0 \text{ pointwise} \quad \Longrightarrow \quad \|f_k\|_{\lim} \to 0 \tag{9a2}$$

for all $f_k \in C_{\mathrm{b}}(\mathcal{X})$.

(b) The sequence $(\mu_n)_n$ satisfies *LDP* with rate $(p_n)_n$ and rate function I (a function $\mathcal{X} \to [0,\infty]$ such that $I^{-1}([0,c])$ is compact for every $c<\infty$), if

$$\lim_{n\to\infty} \biggl(\int |f|^{p_n} \, \mathrm{d}\mu_n \biggr)^{1/p_n} = \max_{x\in\mathcal{X}} \bigl(|f(x)| \mathrm{e}^{-I(x)} \bigr)$$

for all $f \in C_{\mathrm{b}}(\mathcal{X})$.

If $\mathcal{X}$ is compact then (9a2) is satisfied automatically (since it holds for the sup-norm). If $\mathcal{X}$ is not compact then (9a2) is violated by the sup-norm (see 9a4 below).

By a rate function (on $\mathcal{X}$) we mean just a function $I : \mathcal{X} \to [0,\infty]$ such that $I^{-1}([0,c])$ is compact for every $c < \infty$. A compact set being always closed, a rate function is lower semicontinuous. (See also 9c1.)

On $\mathbb{R}$ (or $\mathbb{R}^d$), a lower semicontinuous function I is a rate function if and only if $I(x) \to \infty$ as $x \to \pm\infty$ (think, why).

9a3 Exercise. Let I be a rate function (on $\mathcal{X}$). Then

(a) I reaches its minimum on every closed set;

(b) the maximum of $|f|\mathrm{e}^{-I}$ on $\mathcal{X}$ is reached for every $f \in C_{\mathrm{b}}(\mathcal{X})$;

(c) the seminorm $\|\cdot\|_I$ on $C_{\mathrm{b}}(\mathcal{X})$ defined by

$$\|f\|_I = \max_{\mathcal{X}} (|f|\mathrm{e}^{-I})$$

satisfies (2a1), (2a2), (2a6) and (9a2).

Prove it.

9a4 Exercise. Let $I : \mathcal{X} \to [0,\infty]$ be a lower semicontinuous function. If the seminorm $f \mapsto \sup_{\mathcal{X}}(|f|\mathrm{e}^{-I})$ satisfies (9a2) then I is a rate function.

Prove it.

Hint: otherwise, take ε and $x_1, x_2, \dots \in \mathcal{X}$ such that $I(x_k) \le \varepsilon$ and $\operatorname{dist}(x_k, x_l) \ge 2\varepsilon$ whenever $k \ne l$; consider $f_n(x) = \bigl(1 - \frac{1}{\varepsilon}\operatorname{dist}(x, \{x_n, x_{n+1}, \dots\})\bigr)^+$.

9a5 Exercise. Let $I_1, I_2 : \mathcal{X} \to [0,\infty]$ be lower semicontinuous. If $\sup_{\mathcal{X}}(|f|\mathrm{e}^{-I_1}) = \sup_{\mathcal{X}}(|f|\mathrm{e}^{-I_2})$ for all $f \in C_{\mathrm{b}}(\mathcal{X})$ then $I_1 = I_2$.

Prove it.

Hint: similar to 2a12.

We generalize 2a11 and 2a14 as follows.

9a6 Proposition. Let a seminorm $\|\cdot\|$ on $C_{\mathrm{b}}(\mathcal{X})$ satisfy (2a1), (2a2), (2a6) and (9a2). Then the function $I : \mathcal{X} \to [0,\infty]$ defined by

$$\mathrm{e}^{I(x)} = \sup\{ f(x) : \|f\| \le 1 \}$$

is a rate function, and

$$\|f\| = \max_{\mathcal{X}} \bigl(|f|\mathrm{e}^{-I} \bigr) \quad \text{for all } f \in C_{\mathrm{b}}(\mathcal{X})\,.$$

9a7 Exercise. Prove Proposition 9a6.

Hint: recall 4a. Given f and ε, find $g_1, g_2, \dots$ such that $\|g_k\| \le 1$ and $g_1 \vee g_2 \vee \dots > |f| - \varepsilon$ on $\mathcal{X}$. Apply (9a2) to the functions $(|f| - \varepsilon - g_1 \vee \dots \vee g_n)^+$.

9a8 Corollary. If $(\mu_n)_n$ is LD-convergent (with rate $(p_n)_n$) then $(\mu_n)_n$ satisfies LDP (with rate $(p_n)_n$) with one and only one rate function I, namely,

$$\mathrm{e}^{I(x)} = \sup\{ f(x) : \lim_{n\to\infty} \|f\|_{L_{p_n}(\mu_n)} \le 1 \}\,.$$

9a9 Exercise. Prove Corollary 9a8.

Similarly to 2a19,

$$\min_{x \in \mathcal{X}} I(x) = 0\,. \tag{9a10}$$

9b Contraction principle, and 'tilted LDP'

Let $\mathcal{X}_1, \mathcal{X}_2$ be Polish spaces, $F : \mathcal{X}_1 \to \mathcal{X}_2$ a continuous map, $(\mu_n)_n$ a sequence of probability measures on $\mathcal{X}_1$, and $(\nu_n)_n$ its image on $\mathcal{X}_2$ (that is, $\nu_n(B) = \mu_n(F^{-1}(B))$ for Borel sets $B \subset \mathcal{X}_2$).

9b1 Theorem. (a) If $(\mu_n)_n$ is LD-convergent (with rate $(p_n)_n$), then $(\nu_n)_n$ is LD-convergent (with rate $(p_n)_n$).

(b) If $(\mu_n)_n$ satisfies LDP with rate $(p_n)_n$ and rate function I_1, then $(\nu_n)_n$ satisfies LDP with rate $(p_n)_n$ and rate function I_2 defined by

$$I_2(y) = \min\{ I_1(x) : x \in \mathcal{X}_1, F(x) = y \}\,.$$

If $F^{-1}(\{y\}) = \emptyset$ then the minimum is $+\infty$ by definition. Otherwise, the minimum is reached by 9a3(a).

9b2 Exercise. Prove Theorem 9b1.

Hint: similar to 2b2. And do not forget to prove that I_2 is a rate function.

9b3 Exercise. Generalize Theorem 2c1 to Polish spaces.

9c The probability decay rate

First, the notion of semicontinuity.

9c1 Exercise. Generalize 2a8, 2a9 to Polish spaces.

Hint: when proving (a)$\Longrightarrow$(d), enforce $\varphi(\cdot) > 0$ by a transformation (say, $\mathrm{e}^{\varphi(\cdot)}$), and then consider $f_n(x) = \max\{c : \forall y \; (\operatorname{dist}(x,y) < c/n \implies \varphi(y) \ge c)\,\}$. You get continuous (but generally unbounded) functions.

Let $(\mu_n)_n$ satisfy LDP with rate $(p_n)_n$ and rate function I.

9c2 Exercise. Let $f : \mathcal{X} \to \mathbb{R}$.

(a) If $|f|$ is lower semicontinuous then

$$\liminf_n \|f\|_{L_{p_n}(\mu_n)} \ge \sup_{\mathcal{X}} (|f|\mathrm{e}^{-I})\,;$$

(b) if $|f|$ is bounded and upper semicontinuous then

$$\limsup_n \|f\|_{L_{p_n}(\mu_n)} \le \max_{\mathcal{X}} (|f|\mathrm{e}^{-I})\,.$$

Prove it.

Hints: (a): similar to 4b3(a);
(b): compactness is essential for 4b1(b), but the relation $\max_{\mathcal{X}}(f_j\mathrm{e}^{-I}) \downarrow \max_{\mathcal{X}}(|f|\mathrm{e}^{-I})$ holds provided that f_1 is bounded.

9c3 Exercise. Generalize Corollaries 4b4 and 4b6 to Polish spaces.

9d Exponential tightness

First, the usual tightness.

9d1 Exercise. Let μ be a probability measure on $\mathcal{X}$. Then for every $\varepsilon > 0$ there exists a finite set $S \subset \mathcal{X}$ such that $\mu(S_{+\varepsilon}) \ge 1 - \varepsilon$. (Recall (4b9).)

Prove it.

Hint: take $x_1, x_2, \dots$ dense in $\mathcal{X}$ and observe that $\mu\big(\{x_1,\dots,x_n\}_{+\varepsilon}\big) \to 1$ as $n \to \infty$.

9d2 Exercise. Let μ be a probability measure on $\mathcal{X}$. Then for every $\varepsilon > 0$ there exists a compact set $K \subset \mathcal{X}$ such that $\mu(K) \ge 1 - \varepsilon$.

Prove it.

Hint: take finite sets S_n such that $\sum_n \big(1 - \mu\big((S_n)_{+1/n}\big)\big) \le \varepsilon$ and consider $K = \cap_n (S_n)_{+1/n}$.

9d3 Exercise. The following three conditions on probability measures $\mu_1, \mu_2, \dots$ on $\mathcal{X}$ are equivalent:

(a) for every $\varepsilon > 0$ there exists a compact set $K \subset \mathcal{X}$ such that

$$\sup_n \big(1 - \mu_n(K) \big) \le \varepsilon \,;$$

(b) for every $\varepsilon > 0$ there exists a compact set $K \subset \mathcal{X}$ such that

$$\limsup_{n\to\infty} \big(1 - \mu_n\big(K_{+\varepsilon}\big) \big) \le \varepsilon \,;$$

(c) for every $\varepsilon > 0$ there exists a finite set $S \subset \mathcal{X}$ such that

$$\sup_n \big(1 - \mu_n\big(S_{+\varepsilon}\big) \big) \le \varepsilon \,.$$

Prove it.

Hint: the implications (c)$\Longrightarrow$(b) and (a)$\Longrightarrow$(b) are trivial; using 9d1 it is not difficult to prove the implication (b)$\Longrightarrow$(c); for proving the implication (c)$\Longrightarrow$(a), do in the spirit of 9d2: take finite sets S_k such that $\sum_k \sup_n \big(1 - \mu_n\big((S_k)_{+1/k}\big)\big) \le \varepsilon$ and consider $K = \cap_k (S_k)_{+1/k}$.

9d4 Definition. A sequence $(\mu_n)_n$ of probability measures on $\mathcal{X}$ is *tight*, if it satisfies the equivalent conditions 9d3(a)–(c).

You may add two more conditions to (b), (c) by choosing independently between lim sup and sup on one hand, and between K and S on the other hand. You may also add one more condition to (a), replacing sup with lim sup. This way you get $2 + 2 \cdot 2 = 6$ equivalent definitions of tightness!

The weak convergence of probability measures on $\mathcal{X}$ is defined by

$$\mu_n \to \mu \quad \Longleftrightarrow \quad \forall f \in C_{\mathrm{b}}(\mathcal{X}) \quad \int f \, \mathrm{d}\mu_n \to \int f \, \mathrm{d}\mu$$

for $\mu, \mu_n \in P(\mathcal{X})$.

9d5 Proposition. Every tight sequence contains a (weakly) convergent subsequence.

Proof. (sketch) If $\mathcal{X}$ is compact then $C_{\mathrm{b}}(\mathcal{X})$ is separable, and the diagonal argument works. In general, we take compact sets $K_i \subset \mathcal{X}$ such that $\mu_n(K_i) \ge 1/i$ for all n and i, and apply the said above to each K_i. Using the diagonal argument again we get a subsequence $(\mu_{n_k})_k$ such that the limit

$$\lim_{k\to\infty} \int_{K_i} f \, \mathrm{d}\mu_{n_k}$$

exists for every $f \in C_{\mathrm{b}}(\mathcal{X})$ and every i. However,

$$\int_{K_i} f \, \mathrm{d}\mu_{n_k} \to \int_{\mathcal{X}} f \, \mathrm{d}\mu_{n_k} \quad \text{as } i \to \infty$$

uniformly in k. $\square$

In fact, a subset of $P(\mathcal{X})$ is tight if and only if its closure is (weakly) compact (Prohorov's theorem), but we do not need it.

Now we turn to *exponential* tightness.

9d6 Exercise. The following three conditions on probability measures $\mu_1, \mu_2, \dots$ on $\mathcal{X}$ are equivalent:

(a) for every $\varepsilon > 0$ there exists a compact set $K \subset \mathcal{X}$ such that

$$\sup_n \bigl(1 - \mu_n(K)\bigr)^{1/p_n} \le \varepsilon \,;$$

(b) for every $\varepsilon > 0$ there exists a compact set $K \subset \mathcal{X}$ such that

$$\limsup_{n\to\infty} \bigl(1 - \mu_n\bigl(K_{+\varepsilon}\bigr)\bigr)^{1/p_n} \le \varepsilon \,;$$

(c) for every $\varepsilon > 0$ there exists a finite set $S \subset \mathcal{X}$ such that

$$\sup_n \bigl(1 - \mu_n\bigl(S_{+\varepsilon}\bigr)\bigr)^{1/p_n} \le \varepsilon \,.$$

Prove it.

Hint: similar to 9d3; (c)$\Longrightarrow$(a): $\bigl(1-\mu_n(K)\bigr)^{1/p_n} \le \bigl(\sum_k \bigl(1-\mu_n((S_k)_{+1/k})\bigr)\bigr)^{1/p_n} \le \sum_k \bigl(1 - \mu_n((S_k)_{+1/k})\bigr)^{1/p_n}$.

9d7 Definition. A sequence $(\mu_n)_n$ of probability measures on $\mathcal{X}$ is *exponentially tight* with rate $(p_n)_n$, if it satisfies the equivalent conditions 9d6(a)–(c).

Once again, you may get 6 equivalent definitions...

9d8 Exercise. Every LD-convergent (with rate $(p_n)_n$) sequence is exponentially tight (with rate $(p_n)_n$).

Prove it.

Hint: 9d6(b), and 4b4(b) via 9c3.

9d9 Exercise. Let $(\mu_n)_n$ be exponentially tight (with rate $(p_n)_n$), and the limit $\|f\|_{\mathrm{lim}} = \lim_{n\to\infty} \|f\|_{L_{p_n}(\mu_n)}$ exists for all $f \in C_{\mathrm{b}}(\mathcal{X})$. Then $\|f\|_{\mathrm{lim}}$ satisfies (9a2), and therefore $(\mu_n)_n$ ls LD-convergent (with rate $(p_n)_n$).

Prove it.

Hint: $\int_{\mathcal{X}} |f|^{p_n} \, \mathrm{d}\mu_n = \int_K |f|^{p_n} \, \mathrm{d}\mu_n + \int_{\mathcal{X}\setminus K} |f|^{p_n} \, \mathrm{d}\mu_n \le \bigl(\max_K |f|\bigr)^{p_n} + \bigl(\varepsilon \max_{\mathcal{X}} |f|\bigr)^{p_n}$.

9d10 Proposition. Let a sequence $(\mu_n)_n$ be exponentially tight with rate $(p_n)_n$. Then there exist $n_1 < n_2 < \dots$ such that the sequence $(\mu_{n_k})_k$ is LD-convergent with rate $(p_{n_k})_k$.

9d11 Exercise. Prove Proposition 9d10.

Hint: similar to (the proof of) Proposition 9d5, but consider $\|f\|_{L_{p_n}(\mu_n)}$ rather than $\int f \, \mathrm{d}\mu_n$. And use 9d9.

9e Inverse contraction principle

Let $\mathcal{X}_1, \mathcal{X}_2$ be Polish spaces, $F : \mathcal{X}_1 \to \mathcal{X}_2$ a continuous map, $(\mu_n)_n$ a sequence of probability measures on $\mathcal{X}_1$, and $(\nu_n)_n$ its image on $\mathcal{X}_2$ (that is, $\nu_n(B) = \mu_n(F^{-1}(B))$ for Borel sets $B \subset \mathcal{X}_2$).

9e1 Theorem. Assume that F is one-to-one and $(\mu_n)_n$ is exponentially tight (with rate $(p_n)_n$), then

(a) if $(\nu_n)_n$ is LD-convergent (with rate $(p_n)_n$), then $(\mu_n)_n$ is LD-convergent (with rate $(p_n)_n$);

(b) if $(\nu_n)_n$ satisfies LDP with rate $(p_n)_n$ and rate function I_2, then $(\mu_n)_n$ satisfies LDP with rate $(p_n)_n$ and rate function I_1 defined by

$$I_1(x) = I_2(F(x)) \quad \text{for } x \in \mathcal{X}_1 \, .$$

Proof. (a) Assume the contrary: $(\mu_n)_n$ is not LD-convergent. Using 9d9 we find $f \in C_\mathrm{b}(\mathcal{X}_1)$ such that $\|f\|_{L_{p_n}(\mu_n)}$ does not converge (as $n \to \infty$). We choose $n_1 < n_2 < \dots$ and $n'_1 < n'_2 < \dots$ such that

$$\lim_k \|f\|_{L_{p_{n_k}}(\mu_{n_k})} \ne \lim_k \|f\|_{L_{p_{n'_k}}(\mu_{n'_k})}$$

(both limits exist, but differ). Using 9d10 we may assume that $(\mu_{n_k})_k$ is LD-convergent with rate $(p_{n_k})_k$, and $(\mu_{n'_k})_k$ is LD-convergent with rate $(p_{n'_k})_k$. The corresponding rate functions I_1, I'_1 on $\mathcal{X}_1$ differ, since $\max_\mathcal{X}\big(|f|\mathrm{e}^{-I_1}\big) \ne \max_\mathcal{X}\big(|f|\mathrm{e}^{-I'_1}\big)$. By Theorem 9b1, I_1 satisfies $I_2(y) = \min\{I_1(x) : F(x) = y\}$, thus, $I_2(F(x)) = I_1(x)$ for all x. Similarly, $I_2(F(x)) = I'_1(x)$ for all x, therefore $I_1 = I_2$; a contradiction.

(b) The relation $I_1(\cdot) = I_2(F(\cdot))$ was verified when proving (a). $\square$

See also [1, Th. 4.2.4, p. 111]; [2, Lemma 3.12, p. 48].

9f Example: endless random walk

We return to the situation of 7a,

$$X_n\Big(\frac{k}{n}\Big) = \frac{s_1 + \dots + s_k}{n} + v\frac{k}{n}\,, \quad \mathbb{P}\big(s_k = -1\big) = \mathbb{P}\big(s_k = +1\big) = \frac{1}{2}\,,$$

$v \in (0,1)$. The events

$$A_n : \quad \exists k \quad X_n\Big(\frac{k}{n}\Big) \le -1$$

cannot be treated via 4b6, since the open set

$$\{w : \inf_t w(t) < -1\}$$

is dense in the corresponding compact space (recall 7a, after (7a1)). Indeed, any change of w after a large t is a small change. This is the product topology (recall (5b6)).

In terms of the process

$$Y_n\Big(\frac{k}{n}\Big) = \frac{s_1 + \dots + s_k}{n}\,,$$

related to X_n by $X_n(t) = Y_n(t) + vt$, we deal with the dense open set

$$\{w : \inf_t (w(t) + vt) < -1\} = \{w : \exists t \; w(t) < -vt - 1\}$$

in the compact space denoted in 7e by $\operatorname{Lip}(1)$.

Given a continuous function $h : [0,\infty) \to (0,\infty)$ such that $1 \ll h(t) \ll t$ for large t (that is, $h(t) \to \infty$ and $h(t)/t \to 0$ as $t \to \infty$), we introduce the set

$$\mathcal{X}_h = \{w \in \operatorname{Lip}(1) : w(\cdot) = o(h(\cdot))\}$$

(that is, $w(t)/h(t) \to 0$ as $t \to \infty$) and equip it with the metric

$$\operatorname{dist}(w, w') = \max_t \frac{|w(t) - w'(t)|}{h(t)}\,.$$

9f1 Exercise. $\mathcal{X}_h$ is a Polish space.

Prove it.

Hint: $\mathcal{X}_h$ is isometric to a closed subet of $C[0,\infty]$ (not just $C[0,\infty)$).

9f2 Exercise. The closure of the open set

$$G = \{w \in \mathcal{X}_h : \min_t (w(t) + vt) < -1\}$$

in $\mathcal{X}_h$ is

$$\overline{G} = \{w \in \mathcal{X}_h : \min_t (w(t) + vt) \le -1\}\,.$$

Prove it.

Hint: first, explain why the minimum is reached.

As before, we endow Lip(1) with the topology of locally iniform convergence.

9f3 Exercise. The embedding $\mathcal{X}_h \to \mathrm{Lip}(1)$ is continuous.

Prove it.

The distribution μ_n of the process Y_n is a probability measure on Lip(1). We want to choose h such that $\mu_n(\mathcal{X}_h) = 1$ and moreover, $(\mu_n)_n$ is exponentially tight in $\mathcal{X}_h$ (not just in Lip(1)).

9f4 Exercise. For every n and $c > 0$,

$$\mathbb{P}\Big(\frac{s_1 + \dots + s_n}{\sqrt{n}} \ge c \Big) \le \exp(-c^2/2)\,.$$

Prove it.

Hint:

$$\mathbb{P}\big(s_1 + \dots + s_n \ge c\sqrt{n} \big) \le \frac{\mathbb{E} \exp\big(\lambda(s_1 + \dots + s_n)\big)}{\exp(\lambda c \sqrt{n})} \le \exp\Big(\frac{n\lambda^2}{2} - \lambda c \sqrt{n} \Big)$$

for $\lambda > 0$; choose the optimal λ.

9f5 Exercise. Prove that[1]

$$\limsup_{n\to\infty} \frac{s_1 + \dots + s_n}{\sqrt{2n \ln n}} \le 1 \quad \text{a.s.}$$

Hint: $\sum_n \mathbb{P}\big(s_1 + \dots + s_n \ge c\sqrt{2n\ln n} \big) < \infty$ for $c > 1$.

[1]In fact, by the law of the iterated logarithm,

$$\limsup_{n\to\infty} \frac{s_1 + \dots + s_n}{\sqrt{2n \ln\ln n}} = 1 \quad \text{a.s.,}$$

but we do not need it.

We get $\mu_n(\mathcal{X}_h) = 1$ provided that $\sqrt{t \ln t} = o(h(t))$, that is, $\frac{h(t)}{\sqrt{t \ln t}} \to \infty$ as $t \to \infty$.

9f6 Exercise. For every n and $c > 0$,

$$\mathbb{P}\big(\exists t \;\; Y_n(t) \ge c\sqrt{(t+1)\ln(t+2)}\big) \le \frac{2^{-nc^2/2}}{1 - 2^{-c^2/2}} \,.$$

Prove it.

Hint:

$$\sum_{k=0}^\infty \exp\Big(-\frac12 c^2 (n+k) \ln\Big(\frac{k}{n} + 2\Big)\Big) \le 2^{-nc^2/2} \sum_{k=0}^\infty 2^{-kc^2/2} \,.$$

9f7 Exercise. If $\frac{h(t)}{\sqrt{t \ln t}} \to \infty$ as $t \to \infty$ then $(\mu_n)_n$ is exponentially tight in $\mathcal{X}_h$.

Prove it.

Hint: $\{ w \in \operatorname{Lip}(1) : \forall t \;\; |w(t)| \le c\sqrt{(t+1)\ln(t+2)} \}$ is a compact set in $\mathcal{X}_h$.

Combining 9f7, 9f3 and Theorem 9e1 we conclude that $(\mu_n)_n$ ls LD-convergent in $\mathcal{X}_h$ provided that $\frac{h(t)}{\sqrt{t \ln t}} \to \infty$ as $t \to \infty$. It satisfies LDP in $\mathcal{X}_h$ with the rate function

$$J(w) = \int_0^\infty J_0\big(w'(t)\big)\,\mathrm{d}t \,.$$

Finally, combining 9f2, 4b6 and 7a we get

$$\big(\mathbb{P}\big(A_n\big)\big)^{1/n} \to \exp\Big(-\min_{t>0} t J_0\Big(\frac1t + v\Big)\Big) \quad \text{as } n \to \infty \,.$$

References

[1] A. Dembo, O. Zeitouni, *Large deviations techniques and applications,* Jones and Bartlett publ., 1993.

[2] J. Feng, T.G. Kurtz, *Large deviations for stochastic processes,* 2005, http://www.math.wisc.edu/∼kurtz/feng/ldp.htm