5 Moderate deviations in spaces of functions

5a Asymptotically quadratic generating functions

Let \( p, q, \mu, S_n, \Lambda_n, \Lambda_\infty \) and \( A_n \) be as in Sect. 4b, \( \int x^2 \mu(dx) = 1 \) (that is, \( \Lambda''_\mu(0) = 1 \)), and \( p \leq 2 \leq q \) (see 4b4).

5a1 Proposition. For every \( g \in L_q \),

\[
\frac{1}{n\varepsilon^2} \Lambda_n(\varepsilon g) \to \frac{1}{2} \|g\|_2^2 \quad \text{as} \quad \varepsilon \to 0, n \to \infty.
\]

This is a two-dimensional limit; that is,

\[
\forall \delta > 0 \ \exists \varepsilon_0 > 0 \ \exists n_0 \ \forall \varepsilon \leq \varepsilon_0 \ \forall n \geq n_0 \quad \left| \frac{1}{n\varepsilon^2} \Lambda_n(\varepsilon g) - \frac{1}{2} \|g\|_2^2 \right| \leq \delta.
\]

Not the same as \( \lim \varepsilon \lim n \) or \( \lim n \lim \varepsilon \).

First, we improve 4b1, 4b2 for small arguments.

5a2 Lemma. \( \Lambda'_\mu(t) \leq \text{const} \cdot \max(|t|, |t|^{q-1}) \) for all \( t \in \mathbb{R} \).

Proof. For large \( t \) we have \( \Lambda'_\mu(t) = \mathcal{O}(|t|^{q-1}) \) by 4b1; for small \( t \), \( \Lambda'_\mu(t) = \mathcal{O}(|t|) \).

5a3 Lemma. There exists \( C \) such that for all \( g_1, g_2 \in L_q \),

\[
\|\Lambda_\infty(g_1) - \Lambda_\infty(g_2)\| \leq C \|g_1 - g_2\|_q \left( \|g_1\|_q + \|g_1\|_q^{q-1} + \|g_2\|_q + \|g_2\|_q^{q-1} \right).
\]

Proof. Using 5a2 we take \( C \) such that

\[
\forall t_1, t_2 \quad |\Lambda_\mu(t_1) - \Lambda_\mu(t_2)| \leq C|t_1 - t_2| \max(|t_1|, |t_1|^{q-1}, |t_2|, |t_2|^{q-1}) ;
\]

then

\[
\left| \int_0^1 \Lambda_\mu(g_1(x)) \, dx - \int_0^1 \Lambda_\mu(g_2(x)) \, dx \right| \leq \int_0^1 |\Lambda_\mu(g_1(x)) - \Lambda_\mu(g_2(x))| \, dx \leq
\]

\[
\int_0^1 \left( \int |g_1(x) - g_2(x)| \, dx \right) \, dx \leq \int_0^1 \left( \int \max(|g_1(x)|, |g_1(x)|^{q-1}, |g_2(x)|, |g_2(x)|^{q-1}) \, dx \right) \, dx \leq \int_0^1 \left( \int \max(|g_1(x)|, |g_1(x)|^{q-1}, |g_2(x)|, |g_2(x)|^{q-1}) \, dx \right) \, dx
\]

\[
\leq C \int_0^1 \left( \int \max(|g_1(x)|, |g_1(x)|^{q-1}, |g_2(x)|, |g_2(x)|^{q-1}) \, dx \right) \, dx
\]

\[
= C \int_0^1 \left( \int \max(|g_1(x)|, |g_1(x)|^{q-1}, |g_2(x)|, |g_2(x)|^{q-1}) \, dx \right) \, dx
\]

\[
= C \int_0^1 \left( \int \max(|g_1(x)|, |g_1(x)|^{q-1}, |g_2(x)|, |g_2(x)|^{q-1}) \, dx \right) \, dx
\]

\[
= C \int_0^1 \left( \int \max(|g_1(x)|, |g_1(x)|^{q-1}, |g_2(x)|, |g_2(x)|^{q-1}) \, dx \right) \, dx
\]

\[
= C \int_0^1 \left( \int \max(|g_1(x)|, |g_1(x)|^{q-1}, |g_2(x)|, |g_2(x)|^{q-1}) \, dx \right) \, dx
\]

\[
= C \int_0^1 \left( \int \max(|g_1(x)|, |g_1(x)|^{q-1}, |g_2(x)|, |g_2(x)|^{q-1}) \, dx \right) \, dx
\]

\[
= C \int_0^1 \left( \int \max(|g_1(x)|, |g_1(x)|^{q-1}, |g_2(x)|, |g_2(x)|^{q-1}) \, dx \right) \, dx
\]

\[
= C \int_0^1 \left( \int \max(|g_1(x)|, |g_1(x)|^{q-1}, |g_2(x)|, |g_2(x)|^{q-1}) \, dx \right) \, dx
\]
\[ \leq C(|g_1 - g_2|, \max(|g_1|, |g_1|^{-1}, |g_2|, |g_2|^{-1})) \leq \]
\[ \leq C\|g_1 - g_2\|_q \max(|g_1|, |g_1|^{-1}, |g_2|, |g_2|^{-1})\|_p. \]

and

\[\| \max(\ldots) \|_p = \| \max(|g_1|^{p/q}, |g_1|, |g_2|^{p/q}, |g_2|) \|_q^{-1} \leq \]
\[\leq \| |g_1|^{p/q} + |g_1| + |g_2|^{p/q} + |g_2| \|_q^{-1} \leq \]
\[= (\| g_1 \|_{p/q} + \| g_1 \|_q + \| g_2 \|_{p/q} + \| g_2 \|_q) \]}
\[\leq (4 \max(\| g_1 \|_{p/q}, \| g_1 \|_q, \| g_2 \|_{p/q}, \| g_2 \|_q)) \]
\[= 4^{q^{-1}} \max(\| g_1 \|_p, \| g_1 \|_q^{-1}, \| g_2 \|_p, \| g_2 \|_q^{-1}) \]
\[\leq 4^{q^{-1}} \max(\| g_1 \|_q, \| g_1 \|_q^{-1}, \| g_2 \|_q, \| g_2 \|_q^{-1}) \]

\[\square\]

5a4 Lemma. For every \( g \in L_q \),

\[ \frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g) \rightarrow \frac{1}{2} \| g \|_2^2 \] as \( \varepsilon \rightarrow 0 \).

Proof. First, the bounded case: \( g \in L_\infty \); we have then

\[ \frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g) = \int_0^1 \frac{1}{\varepsilon^2} \Lambda_\mu(\varepsilon g(x)) \, dx \rightarrow \int_0^1 \frac{1}{2} g^2(x) \, dx, \]

since \( \frac{1}{\varepsilon^2} \Lambda_\mu(\cdot) \rightarrow \frac{1}{2} g^2(\cdot) \) uniformly.

Second, the general case; given \( \delta > 0 \), we take \( g_\delta \in L_\infty \) such that \( \| g_\delta - g \|_q \leq \delta \); by 5a3, \( |\Lambda_\infty(\varepsilon g) - \Lambda_\infty(\varepsilon g_\delta)| \leq \text{const} \cdot \varepsilon^2 \delta \) with a constant that depends on \( g \) but does not depend on \( \varepsilon, \delta \) (as long as \( |\varepsilon| \leq 1, \delta \leq 1 \)). We get

\[ \left| \frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g) - \frac{1}{2} \| g \|_2^2 \right| \leq \]
\[\leq \frac{1}{\varepsilon^2} |\Lambda_\infty(\varepsilon g) - \Lambda_\infty(\varepsilon g_\delta)| + \frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g_\delta) - \frac{1}{2} \| g_\delta \|_2^2 + \frac{1}{2} \| g_\delta \|_2^2 - \frac{1}{2} \| g \|_2^2, \]

\[\leq \text{const} \cdot \delta \]

thus, \( \limsup_{\varepsilon \rightarrow 0} \left| \frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g) - \frac{1}{2} \| g \|_2^2 \right| \leq \text{const} \cdot \delta \) for all \( \delta \).

\[\square\]

Proof of Prop. 5a4: \[ \frac{1}{n} \Lambda_n(\varepsilon g) = \Lambda_\infty(A_n \varepsilon g); \] by 5a3, \[ |\Lambda_\infty(\varepsilon A_n g) - \Lambda_\infty(\varepsilon g)| \leq \text{const} \cdot \varepsilon^2 \| A_n g - g \|_q \]

with a constant that depends on \( g \) but does not depend on \( \varepsilon, n \) (as long as \( |\varepsilon| \leq 1 \)). Thus, \[ \left| \frac{1}{n \varepsilon^2} \Lambda_n(\varepsilon g) - \frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g) \right| \rightarrow 0 \] as \( n \rightarrow \infty \), uniformly on \( |\varepsilon| \leq 1 \). It remains to use 5a4.

\[\square\]
For the (one-dimensional) distribution $\nu_n$ of $(S_n, g)$, similarly to (4b7), we get

\[(5a5) \quad \frac{1}{n \varepsilon^2} \Lambda_{\nu_n}(\varepsilon t) \to \frac{1}{2} \|g\|^2 t^2 \quad \text{as} \quad \varepsilon \to 0, n \to \infty,
\]
since $\Lambda_{\nu_n}(\varepsilon t) = \ln \mathbb{E} \exp(\varepsilon t \langle S_n, g \rangle) = \Lambda_n(\varepsilon t g)$.

5b Gärtner-Ellis, again

**Dimension 1**

Let probability measures $\nu_1, \nu_2, \ldots$ on $\mathbb{R}$ be such that

\[(5b1) \quad \frac{1}{n \varepsilon^2} \Lambda_{\nu_n}(\varepsilon t) \to \frac{1}{2} t^2 \quad \text{as} \quad \varepsilon \to 0, n \to \infty
\]
for all $t \in \mathbb{R}$. (In particular, $\nu_n = \nu^{*n}$ satisfy it, provided that $\int x \nu(dx) = 0$ and $\int x^2 \nu(dx) = 1$, since $\frac{1}{n \varepsilon^2} \Lambda_{\nu_n}(\varepsilon t) = \frac{1}{\varepsilon} \Lambda_{\nu}(\varepsilon t) \to \frac{1}{2} t^2$.)

**5b2 Example.** It may seem that (4c1) with $\Lambda(t) \sim \frac{1}{2} t^2$ (for $t \to 0$) implies (5b1). But this is an illusion. Here is a counterexample.

Let $\frac{1}{\sqrt{n}} \ll a_n \ll 1$ (that is: $a_n \to 0$ and $\sqrt{n} a_n \to \infty$), and

\[\nu_n = \frac{1}{2} \mu^{*n} + \frac{1}{4} (\delta_{-na_n} + \delta_{na_n});\]

here $\mu = N(0, 1)$ is the standard normal distribution (thus, $\mu^{*n} = N(0, n)$), and $\delta_x$ is the unit atom at $x$. Then

\[\Lambda_{\nu_n}(t) = \ln \left( \frac{1}{2} \exp \frac{nt^2}{2} + \frac{1}{2} \cosh na_n t \right).
\]

On one hand,

\[\frac{1}{n} \Lambda_{\nu_n}(t) \to \frac{1}{2} t^2 \quad \text{as} \quad n \to \infty,
\]
since for $t = 0$ this holds trivially, otherwise $na_n t = o(nt^2)$ for large $n$.

On the other hand, taking $\varepsilon_n$ such that $\frac{1}{\sqrt{n}} \ll \varepsilon_n \ll a_n$ we get

\[\frac{1}{n \varepsilon_n^2} \Lambda_{\nu_n}(\varepsilon_n t) \geq \frac{1}{n \varepsilon_n^2} \ln \left( \frac{1}{4} \exp n a_n \varepsilon_n t \right) = \frac{a_n t}{\varepsilon_n} + O\left( \frac{1}{n \varepsilon_n^2} \right) \to \infty \quad \text{as} \quad n \to \infty.
\]

By the way, these $\nu_n$ violate 5b3 below.

The Legendre transform of $\Lambda(t) = \frac{1}{2} t^2$ is $\Lambda^*(x) = \frac{1}{2} x^2$ (recall 2c6).
5b3 Exercise.

\[ \nu_n\left( n \varepsilon x, \infty \right) \leq \exp\left( -\frac{1}{2}x^2n\varepsilon^2 + o(n\varepsilon^2) \right) \quad \text{for } x \geq 0; \]

\[ \nu_n\left( -\infty, n \varepsilon x \right) \leq \exp\left( -\frac{1}{2}x^2n\varepsilon^2 + o(n\varepsilon^2) \right) \quad \text{for } x \leq 0. \]

Of course, these \( o(\ldots) \) are meant for \( \varepsilon \to 0, n \to \infty \).

Prove it.\(^1\)

It follows that \( \nu_n(n \varepsilon a, n \varepsilon b) \to 1 \) as \( \varepsilon \to 0, n \to \infty, n\varepsilon^2 \to \infty \), whenever \( a < 0 < b \).

For tilted measures \( \nu_{n,\varepsilon t} \), we have \( \Lambda_{\nu_{n,\varepsilon t}}(\varepsilon s) = \Lambda_{\nu_n}(\varepsilon t + \varepsilon s) - \Lambda_{\nu_n}(\varepsilon t) \), thus \( \frac{1}{n\varepsilon^2}\Lambda_{\nu_{n,\varepsilon t}}(\varepsilon s) \to \frac{1}{2}(t + s)^2 - \frac{1}{2}t^2 = ts + \frac{1}{2}s^2 \); the corresponding Legendre transform is \( \Lambda^*_t(x) = \frac{1}{2}(x - t)^2 \) (since generally \( \Lambda^*_t(x) = \Lambda^*(tx + \Lambda(t)) \), as noted after 4c2). Similarly to (4c3),

\[(5b4) \quad \nu_{n,\varepsilon t}(n \varepsilon a, n \varepsilon b) \to 1 \quad \text{as } \varepsilon \to 0, n \to \infty, n\varepsilon^2 \to \infty, \text{whenever } a < t < b.\]

Taking into account that

\[
\frac{d\nu_n}{d\nu_{n,\varepsilon t}}(\varepsilon x) = \exp\left( -\varepsilon t \varepsilon x + \Lambda_{\nu_n}(\varepsilon t) \right) \geq \exp\left( -n\varepsilon^2 \max(ta, tb) + \Lambda_{\nu_n}(\varepsilon t) \right)
\]

for \( x \in (na, nb) \), we get, similarly to (4e4),

\[(5b5) \quad \nu_n(n \varepsilon a, n \varepsilon b) \geq \exp\left( -n\varepsilon^2 \max(ta, tb) + n\varepsilon^2 \cdot \frac{1}{2}t^2 + o(n\varepsilon^2) \right) \]

whenever \( a < t < b \).

Similarly to 4c5 (but simpler), if \( x \geq 0 \) and \( \delta > 0 \) then

\[ \nu_n(n \varepsilon x, n \varepsilon (x + \delta)) \geq \exp\left( -\frac{1}{2}x^2n\varepsilon^2 + o(n\varepsilon^2) \right), \]

and similarly to 4c6,

\[ \nu_n(n \varepsilon x, n \varepsilon (x + \delta)) = \exp\left( -\frac{1}{2}x^2n\varepsilon^2 + o(n\varepsilon^2) \right). \]

**DIMENSION \( d \)**

All limits, as well as symbols \( o(\ldots) \), \( O(\ldots) \) are taken for \( \varepsilon \to 0, n \to \infty, n\varepsilon^2 \to \infty \) (unless stated otherwise).

Let probability measures \( \nu_1, \nu_2, \ldots \) on \( \mathbb{R}^d \) be such that

\[(5b6) \quad \frac{1}{n\varepsilon^2}\Lambda_{\nu_n}(\varepsilon t) \to \frac{1}{2}|t|^2 \quad \text{for all } t \in \mathbb{R}^d. \]

\(^1\)Hint: similar to 4c2.
**5b7 Theorem.** (a) For every nonempty closed set $F \subset \mathbb{R}^d$, 
\[
\limsup_{n} \frac{1}{n^2} \ln \nu_n(n \in F) \leq -\min_{x \in F} \frac{1}{2} |x|^2.
\]
(b) For every open set $U \subset \mathbb{R}^d$, 
\[
\liminf_{n} \frac{1}{n^2} \ln \nu_n(n \in U) \geq -\inf_{x \in U} \frac{1}{2} |x|^2.
\]

**5b8 Exercise** (upper bound for a half-space).
\[
\nu_n \left( \{ n \in x : \langle t, x \rangle - \frac{1}{2} |t|^2 \geq c \} \right) \leq \exp \left( -c n \varepsilon^2 + o(n \varepsilon^2) \right)
\]
for all $t \in \mathbb{R}^d$ and $c \geq 0$.
Prove it.

**5b9 Exercise** (half-space not containing the expectation). If $c > 0$, then
\[
\exists \delta > 0 \quad \nu_n \left( \{ n \in x : \langle t, x \rangle \geq c \} \right) = O(e^{-\delta n \varepsilon^2}).
\]
Prove it.

**5b10 Exercise** (lower bound). If $U \subset \mathbb{R}^d$ is open, then 
\[
\ln \nu_n(n \in U) \geq -n \varepsilon^2 \inf_{x \in U} \frac{1}{2} |x|^2 + o(n \varepsilon^2).
\]
Prove it.

**5b11 Exercise.** Prove Theorem 5b7.

The simple rate function $\frac{1}{2} | \cdot |^2$ leads to a simple formula for half-spaces. Every closed half-space $H \subset \mathbb{R}^d$ not containing 0 is
\[
H = \{ x : \langle x, x_H \rangle \geq |x_H|^2 \}
\]
where $x_H$ is the point of $H$ closest to 0. Now, 5b8 with $t = x_H$ and $c = \frac{1}{2} |x_H|^2$ gives 
\[
\nu_n(n \in H) \leq \exp \left( -\frac{1}{2} |x_H|^2 n \varepsilon^2 + o(n \varepsilon^2) \right);
\]
we see very clearly that every $x \neq 0$ belongs to (a) a closed half-space that satisfies the upper bound with rate $\frac{1}{2} |x|^2$, and (b) an open half-space that satisfies the upper bound with rate arbitrarily close to $\frac{1}{2} |x|^2$.

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\textsuperscript{1}Hint: recall the proof of 4c10(a).
5c Exponential tightness

What about a weakly compact set $K \subset L_p$ such that $\mathbb{P}(S_n \notin n\varepsilon K) \leq \exp(-Cn\varepsilon^2 + o(n\varepsilon^2))$ (for a given $C$)? No, this cannot happen. Indeed, on one hand, $K$ must be bounded, that is, $K \subset \{f : \|f\|_p \leq R\}$ for some $R$; on the other hand, $\|S_n\|_1 = |X_1| + \cdots + |X_n|$; $\mathbb{E}\|S_n\|_1 = n\mathbb{E}|X_1|$; $\mathbb{P}(S_n \in n\varepsilon K) \leq \mathbb{P}(\|S_n\|_p \leq n\varepsilon R) \leq \mathbb{P}(\|S_n\|_1 \leq n\varepsilon R)$ is close to 0 (rather than 1) when $n\varepsilon R \ll \mathbb{E}\|S_n\|_1$, that is, $\varepsilon \ll \mathbb{E}|X_1|/R$.

The joint compactification introduced in Sect. 4b and used successfully for large deviations, fails for moderate deviations. We need another joint compactification. The $L_p$-norm feels only absolute values of $X_1, \ldots, X_n$. But we have $\mathbb{E}X_1 = 0$, and cancellation of positive and negative summands should not be ignored.

We sacrifice invariance under permutations of the random variables $X_1, \ldots, X_n$ (thus, by the way, complicating generalization to, say, two-dimensional arrays of random variables) and take indefinite integrals of the functions $S_n$ (and others). We move to the space $C_0[0,1]$ of all continuous functions on $[0,1]$ vanishing at 0, and redefine the random function $S_n$ as such piecewise-linear function of $C_0[0,1]$:

$$S_n(x) = \int_0^x (nX_1 \mathbb{I}_{(0,\frac{1}{n})} + \cdots + nX_n \mathbb{I}_{(\frac{x-1}{n},1)}) .$$

Note that indefinite integrals of functions of $L_p$ (or $L_1$) are absolutely continuous; they are dense in the space $C_0[0,1]$, but far not the whole space. In this sense, we really move to a larger space.

We also need Hölder spaces $C_{0,\alpha}$ and Hölder norms $\| \cdot \|_{\alpha}$ for $\alpha \in (0,1)$,

$$\|f\|_{\alpha} = \sup_{0 < r < y < 1} \frac{|f(y) - f(x)|}{(y-x)^{\alpha}} \in [0, \infty]$ for $f \in C_0[0,1],$ 

$$C_{0,\alpha} = \{ f \in C_0[0,1] : \|f\|_\alpha < \infty \} .$$

For $0 < \alpha \leq \beta < 1$ we have $\| \cdot \|_\alpha \leq \| \cdot \|_\beta$ and $C_{0,\alpha} \supset C_{0,\beta}$.

The unit ball $B_\alpha = \{ f : \|f\|_\alpha \leq 1 \}$ is separable, but not compact (in $C_{0,\alpha}$).\footnote{Try $f_n(x) = \min(x^n,1/n)$.} However, $B_\beta$ is compact in $C_0[0,1]$.\footnote{Hint: in this situation, convergence on a dense countable set implies uniform convergence. In fact, moreover, $B_\beta$ is compact in $C_{0,\alpha}$ whenever $0 < \alpha < \beta < 1$; hint: if $f, g \in B_\beta$ satisfy $|f(x) - g(x)| \leq \frac{1}{n}$ for $x = \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$, then $\|f - g\|_\alpha \leq 4/n^{3-\alpha}$.} Note that Hölder functions need not be absolutely continuous.

We also redefine operators $A_n$; now $A_nf$ is the function linear on $[0, \frac{1}{n}]$, $[\frac{1}{n}, \frac{2}{n}]$, $\ldots$, $[\frac{n-1}{n},1]$ and equal to $f$ at $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n},1$. 

\[\begin{align*}
\int_0^x (nX_1 \mathbb{I}_{(0,\frac{1}{n})} + \cdots + nX_n \mathbb{I}_{(\frac{x-1}{n},1)}) .
\end{align*}\]
For a piecewise-linear function \( f = A_n f \) we have
\[
\|f\|_\alpha = \max_{0 \leq k < l \leq n} \frac{1}{(l/k)n^\alpha}\left| f\left(\frac{l}{n}\right) - f\left(\frac{k}{n}\right) \right|;
\]
Indeed, \( \frac{|f(y) - f(x)|}{(y-x)^\alpha} \) cannot be maximal between the nodes \( 0, \frac{1}{n}, \ldots, \frac{n}{n} \) due to concavity of the function \( x \mapsto x^\alpha \). For such \( f \),
\[
\|f\|_\alpha = \max_{0 \leq k < l \leq n} |\langle f', g_{k,l} \rangle| \quad \text{where} \quad g_{k,l} = \frac{n^\alpha}{(l-k)^\alpha} \mathbb{I}_{(\frac{k}{n}, \frac{l}{n})}.
\]
We note that \( \|g_{k,l}\|_q = \left(\frac{l-k}{n}\right)^{1-\alpha} \leq 1 \) for \( \alpha \leq 1/q \). We use \( 5b3 \)
\[
P\left(\|S_n\|_\alpha \geq n\varepsilon x \right) \leq \sum_{k,l} P\left(|\langle S'_n, g_{k,l} \rangle| \geq n\varepsilon x \right) \leq 2\left(n+1\right) \exp\left(-\frac{1}{2}x^2n\varepsilon^2 + o(n\varepsilon^2)\right),
\]
and get
\[
P\left(\|S_n\|_\alpha \geq n\varepsilon x \right) \leq \exp\left(-\frac{1}{2}x^2n\varepsilon^2 + o(n\varepsilon^2) + O(\ln n)\right)
\]
for \( \alpha \leq 1/q \).

From now on, all limits, as well as symbols \( o(\ldots) \), \( O(\ldots) \) are taken for \( \varepsilon \to 0, n \to \infty, \frac{n^2}{\ln n} \to \infty \) (unless stated otherwise). Note the logarithmic gap between moderate deviations and central limit theorem.

Now, for \( \alpha \leq 1/q \) we have
\[
5(\alpha) \quad P\left(\|S_n\|_\alpha \geq n\varepsilon x \right) \leq \exp\left(-\frac{1}{2}x^2n\varepsilon^2 + o(n\varepsilon^2)\right),
\]
which is exponential tightness; \( K_C \) is the ball \( xB_\alpha \) (with \( x \) such that \( x^2/2 = C \)) endowed with the compact topology from \( C_0[0,1] \).

5d Mogulskii’s theorem, again

We interpret \( \|f'\|_2 \) as \( +\infty \) if \( f \) is not the indefinite integral of a function of \( L_2[0,1] \). As before, all limits, as well as symbols \( o(\ldots) \), \( O(\ldots) \) are taken for \( \varepsilon \to 0, n \to \infty, \frac{n^2}{\ln n} \to \infty \) (unless stated otherwise). Also, \( 1 < p \leq 2 \leq q < \infty ; \frac{1}{p} + \frac{1}{q} = 1 \), and \( \alpha \leq 1/q \).

5d1 Theorem. (a) For every nonempty closed set \( F \subset C_0[0,1] \),
\[
\limsup_{n} \frac{1}{n\varepsilon^2} \ln P\left(\frac{1}{n\varepsilon}S_n \in F \right) \leq -\min_{f \in F} \frac{1}{2} \|f'\|_2^2.
\]
(b) For every open set \( U \subset C_0[0,1] \),
\[
\liminf_{n} \frac{1}{n\varepsilon^2} \ln P\left(\frac{1}{n\varepsilon}S_n \in U \right) \geq -\inf_{f \in U} \frac{1}{2} \|f'\|_2^2.
\]
**5d2 Remark.** Weaker conditions on $F$ and $U$ are sufficient for the theorem (and the proof): for all $R > 0$,

$$F \cap RB_\alpha \text{ is closed},$$

$$U \cap RB_\alpha \text{ is relatively open in } RB_\alpha;$$

here $RB_\alpha = \{ Rf : f \in B_\alpha \} = \{ f : \| f \|_\alpha \leq R \}$.

We choose a dense sequence $x_1, x_2, \ldots \in [0, 1]$ and denote $g_k = \mathbb{I}_{(0,x_k]}$. If $f \in C_0[0, 1]$ is the indefinite integral of a function of $L_2[0, 1]$,

$$f(x) = \int_0^x f'(u) \, du,$$

then clearly $f(x_k) = \langle f', g_k \rangle$. It is convenient to denote $\langle f', g_k \rangle = f(x_k)$ for arbitrary $f \in C_0[0, 1]$ (even though $f'$ is ill-defined). We note that

$$(f_n \to f \text{ in } C_0[0, 1]) \iff \forall k \langle f_n', g_k \rangle \xrightarrow{n \to \infty} \langle f', g_k \rangle$$

for all $f, f_1, f_2, \ldots \in B_\alpha$.

We fix $d$ for a while, and enumerate $x_1, \ldots, x_d$ in ascending order:

$$\{x_1, \ldots, x_d\} = \{y_1, \ldots, y_d\}, \quad 0 < y_1 < \cdots < y_d < 1.$$

Here is an orthonormal basis in the $d$-dimensional space spanned by $g_1, \ldots, g_d$: \[ h_1 = \frac{1}{\sqrt{y_1}} \mathbb{I}_{(0,y_1]}, \quad h_2 = \frac{1}{\sqrt{y_2 - y_1}} \mathbb{I}_{(y_1,y_2]}, \ldots, \quad h_d = \frac{1}{\sqrt{y_d - y_{d-1}}} \mathbb{I}_{(y_{d-1},y_d]} \ldots. \]

Naturally, we let $\langle f', h_i \rangle = \frac{1}{\sqrt{y_i - y_{i-1}}} (f(y_i) - f(y_{i-1}))$ (where $y_0 = 0$). We introduce linear operators $T_d : C_0[0, 1] \to \mathbb{R}^d$ by \[ T_d f = (\langle f', h_1 \rangle, \ldots, \langle f', h_d \rangle); \]

they are continuous.

Similarly to $A_n$, we define operator $\tilde{A}_d : C_0[0, 1] \to C_0[0, 1]$; $\tilde{A}_d f$ is the function linear on $[0, y_1], [y_1, y_2], \ldots, [y_{d-1}, y_d]$, equal to $f$ at $0, y_1, \ldots, y_d$, and constant on $[y_{d-1}, 1]$. Thus, $(\tilde{A}_d f)' = \langle f', h_1 \rangle h_1 + \cdots + \langle f', h_d \rangle h_d$ and $\langle f', (\tilde{A}_d g)' \rangle = \langle (\tilde{A}_d f)', g' \rangle$ (like the orthogonal projection, but $f', g'$ are ill-defined). Note that $\| (\tilde{A}_d f)' \|_2 = \| T_d f \|_2$ and $\langle (\tilde{A}_d f)', (\tilde{A}_d g)' \rangle = \langle T_d f, T_d g \rangle$.

Now we have three “incarnations” of the $d$-dimensional Euclidean vector space:
\* \( \mathbb{R}^d \);
\* subspace of \( L_2[0,1] \) spanned by \( g_1, \ldots, g_d \) or, equivalently, by \( h_1, \ldots, h_d \), with the norm \( \| \cdot \|_2 \) (step functions);
\* subspace \( \{ f : \tilde{A}_d f = f \} \) of \( C_0[0,1] \), with the norm \( f \mapsto \| f' \|_2 \) (polygonal functions).

They are intertwined by a commutative diagram of linear isometries:

\[
\begin{array}{ccc}
\text{polygonal} & \xleftarrow{\text{projection of}} & \text{step} \\
\xleftarrow{\mathbb{R}^d} & & \xrightarrow{T_d f} \\
& & \xrightarrow{\text{step}} \xleftarrow{\int} \\
& & f \xrightarrow{T_d f} f'
\end{array}
\]

We turn to \( d \to \infty \). Clearly,

\[
f_n \to f \text{ in } C_0[0,1] \iff \forall d \ T_d f_n \xrightarrow{n \to \infty} T_d f
\]

for all \( f, f_1, f_2, \ldots \in B_\alpha \).

If \( d_1 \leq d_2 \), then \( \tilde{A}_{d_1} \tilde{A}_{d_2} = \tilde{A}_{d_1} \tilde{A}_{d_2} \tilde{A}_{d_1} \), and \( (\tilde{A}_{d_1} f)' \) is the orthogonal projection of \( (\tilde{A}_{d_2} f)' \). Thus, \( \|(\tilde{A}_{d} f)'\|_2 \) is increasing (in \( d \)).

**5d3 Lemma.** \( \|(\tilde{A}_{d} f)'\|_2 \uparrow \| f' \|_2 \) (be it finite or infinite) as \( d \to \infty \).

**Proof.** On one hand, if \( f' \in L_2 \), then \( (\tilde{A}_{d} f)' \) is the orthogonal projection of \( f' \) to the subspace spanned by \( g_1, \ldots, g_d \); the union of these subspaces is dense in \( L_2 \), thus, \( \|(\tilde{A}_{d} f)'\|_2 \uparrow \| f' \|_2 \).

On the other hand, assume that \( \lim_d \|(\tilde{A}_{d} f)'\|_2 < \infty \); we have to prove that \( f' \in L_2 \). The series

\[
(\tilde{A}_1 f)' + (\tilde{A}_2 f - \tilde{A}_1 f)' + (\tilde{A}_3 f - \tilde{A}_2 f)' + \ldots
\]

consists of orthogonal summands, and its partial sums are bounded. It follows easily that these partial sums are a Cauchy sequence. Thus, the series converges:

\[
(\tilde{A}_d f)' \to \varphi \text{ for some } \varphi \in L_2.
\]

We note that \( \langle (\tilde{A}_d f)', g_d \rangle = \langle f', g_d \rangle \) when \( k \geq d \); thus, it equals \( \langle \varphi, g_d \rangle \); that is, \( \int_0^x \varphi(u) \, du = f(x_d) \) for all \( d \); this shows that \( \varphi = f' \).

Denote by \( \nu_{d,n} \) the distribution of \( T_d S_n \). By 5a1,

\[
\frac{1}{n \varepsilon^2} \Lambda_{\nu_{d,n}}(\varepsilon t_1, \ldots, \varepsilon t_d) \to \frac{1}{2}(t_1^2 + \cdots + t_d^2) \text{ as } n \to \infty
\]

for all \( (t_1, \ldots, t_d) \in \mathbb{R}^d \), since \( \Lambda_{\nu_{d,n}}(t_1, \ldots, t_d) = \ln \mathbb{E} \exp(\varepsilon t_1 \langle S_n, h_1 \rangle + \cdots + \varepsilon t_d \langle S_n, h_d \rangle) = \ln \mathbb{E} \exp(S_n \varepsilon t_1 h_1 + \cdots + \varepsilon t_d h_d) = \Lambda_n(\varepsilon t_1 h_1 + \cdots + \varepsilon t_d h_d). \)
Thus, Theorem 5b7 (as well as 5b8–5b12) applies to \( \nu_{d,n} \) for given \( d \). That theorem is formulated for \( \mathbb{R}^d \), but may be transferred readily to the “step” or “polygonal” space. In all cases, the rate function is \( \frac{1}{2} \| \cdot \|^2 \).

5d4 Exercise. Let \( g \in C_0[0,1] \) satisfy \( g = \tilde{A}_d g \) (for a given \( d \)), and \( H = \{ f \in C_0[0,1] : \langle f', g' \rangle \geq \| g' \|_2^2 \} \) (even though \( f' \) is ill-defined...). Then

(a) \( H = \{ f \in C_0[0,1] : \langle T_d f, T_d g \rangle \geq |T_d g|^2 \} \);
(b) \( \mathbb{P}\{ S_n \in n \varepsilon H \} \leq \exp\left( -\frac{1}{2} \| g' \|_2^2 n \varepsilon^2 + o(n \varepsilon^2) \right) \).

Prove it.

Our space \( C_0[0,1] \) is not a finite-dimensional Euclidean space, nor a Hilbert space, and still, every \( f \neq 0 \) belongs to an open half-space that satisfies the upper bound with rate arbitrarily close to \( \Lambda^*_e(f) \). Indeed, if \( c < \Lambda^*_e(f) \) (being the latter finite or infinite), then \( \frac{1}{2}\| (A_d f)' \|_2^2 > c \) for \( d \) large enough; we take such \( d \), and introduce \( g = (1 - \delta) \tilde{A}_d (f) \) with \( \delta > 0 \) small enough, then \( \frac{1}{2}\| g' \|_2^2 \geq c \) and \( g = \tilde{A}_d g \); the half-space \( H = \{ f_1 \in C_0[0,1] : \langle f_1', g' \rangle > \| g' \|_2^2 \} \) is open in \( C_0[0,1] \) (think, why), \( f \in H \) (think, why), and \( \mathbb{P}\{ S_n \in n \varepsilon H \} \leq \exp\left( -cn \varepsilon^2 + o(n \varepsilon^2) \right) \) by 5d4(b).

5d5 Exercise. Prove Theorem 5d1(a).

5d6 Exercise. Let \( U \subset C_0[0,1] \) be open, and \( f_0 \in U \cap B_\alpha \). Then there exist \( d \) and \( \delta > 0 \) such that

\[
\forall f \in B_\alpha \left( \| T_d f - T_d f_0 \| \leq \delta \implies f \in U \right).
\]

Prove it.\(^1\)

5d7 Exercise. \( \| f \|_{1/2} \leq \| f' \|_2 \) for all \( f \in C_0[0,1] \) (be the norms finite or infinite). (Here \( \| \cdot \|_{1/2} \) is the Hölder norm for \( \alpha = 1/2 \), while \( \| \cdot \|_2 \) is the \( L_2 \) norm.)

Prove it.

Also, \( \alpha \leq \frac{1}{q} \) and \( p \leq 2 \leq q \), thus, \( \| f \|_\alpha \leq \| f \|_{1/2} \leq \| f' \|_2 \).

\textbf{Proof of Theorem 5d1(b).}\(^2\) Let \( f_0 \in U \); we’ll prove that \( \liminf \frac{1}{n \varepsilon^2} \ln \mathbb{P}\{ S_n \in n \varepsilon U \} \geq -\frac{1}{2} \| f_0' \|_2^2 \), assuming \( \| f_0' \|_2 < \infty \) (otherwise the claim is void). We take \( R > \| f_0' \|_2 \), then \( f_0 \in RB_\alpha \) by 5d7, and \( \limsup \frac{1}{n \varepsilon^2} \ln \mathbb{P}\{ \| S_n \|_\alpha \geq Rn \varepsilon \} < -\frac{1}{2} \| f_0' \|_2^2 \) by 5c1. Exercise 5d6 gives \( d \) and \( \delta > 0 \) such that \( \forall f \in RB_\alpha \left( \| T_d f - T_d f_0 \| \leq \delta \implies f \in U \right) \). It is sufficient to prove that

\[
\liminf \frac{1}{n \varepsilon^2} \ln \mathbb{P}\left( \| T_d \frac{S_n}{n \varepsilon} - T_d f_0 \| < \delta \right) \geq -\inf_{x: \| x - T_d f_0 \| < \delta} \frac{1}{2} |x|^2,
\]

\(^1\)Hint: similar to 4e7.

\(^2\)Quite similar to the proof of Theorem 4e1(b).
since \( \inf_{x:|x-Td_{f_0}|<\delta} \frac{1}{2}|x|^2 \leq \frac{1}{2}|Td_{f_0}|^2 = \frac{1}{2}\|\hat{A}_{d_0}f_0\|^2 \leq \frac{1}{2}\|f_0\|^2 \). Theorem 5b7(b) gives the needed inequality, since \( \nu_{d,n}(\{n\in\mathbb{N} : |x-Td_{f_0}| < \delta\}) = P(12d_{n}\xi-Td_{f_0} < \delta). \)

**5d8 Exercise.** A fair coin is tossed \( n \) times, giving \( (\beta_1, \ldots, \beta_n) \in \{0, 1\}^n \). Given a continuous \( \varphi: [0, 1] \to (0, \infty) \), consider

\[
p_n = P\left( \forall k=1, \ldots, n \quad \frac{2(\beta_1 + \cdots + \beta_k) - k}{n^{2/3}} \leq \varphi\left( \frac{k}{n} \right) \right).
\]

(a) Prove that

\[
p_n = 1 - \exp\left(-an^{1/3}(1+o(1))\right)
\]

for some \( a > 0 \);

(b) find \( a \) when \( \varphi(x) = 1 + vx \) for a given \( v > 0 \);

(c) find \( a \) when \( \varphi(x) = \max(1 + vx, y) \) for given \( v > 0 \) and \( y > 1 \);

(d) find \( a \) when \( \varphi(x) = 1 + cx^2 \) for a given \( c > 0 \);

(e) find \( a \) when \( \varphi(x) = 1 + c\sqrt{x} \) for a given \( c > 0 \).

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