6 Small random perturbations of deterministic dynamics

This is basically a discrete-time introduction to Freidlin-Wentzell theory.

6a Adding drift to a random walk

Let S_n and A_n be as in Sect. 5c, 5d; in particular, S_n is a random piecewise-linear function of $C_0[0,1]$,

$$S_n\Big(\frac{k}{n}\Big) = X_1 + \dots + X_k \,;$$

it is driftless: $\mathbb{E}\, X_k = 0$. Let

$$\eta : \mathbb{R} \times [0,1] \to \mathbb{R}$$

be a continuous function satisfying, for a given L,

$$\forall x, y \in \mathbb{R}\ \forall t \in [0,1]\ \ |\eta(x,t) - \eta(y,t)| \le L|x-y| \,.$$

We introduce another random piecewise-linear function R_n of $C_0[0,1]$ via the difference equation[1]

$$\frac{1}{n\varepsilon} R_n\Big(\frac{k+1}{n}\Big) - \frac{1}{n\varepsilon} R_n\Big(\frac{k}{n}\Big) = \frac{1}{n\varepsilon} S_n\Big(\frac{k+1}{n}\Big) - \frac{1}{n\varepsilon} S_n\Big(\frac{k}{n}\Big) + \frac{1}{n}\eta\Big(\frac{1}{n\varepsilon} R_n\Big(\frac{k}{n}\Big), \frac{k}{n}\Big) \,.$$

That is,

$$\begin{aligned} R_n\Big(\frac{k}{n}\Big) &= Y_1 + \dots + Y_k \,, \\ Y_{k+1} &= X_{k+1} + \varepsilon\eta\Big(\frac{Y_1 + \dots + Y_k}{n\varepsilon}, \frac{k}{n}\Big) \,. \end{aligned}$$

[1] A discrete-time counterpart of the stochastic differential equation $\mathrm{d}\frac{1}{n\varepsilon} R_n(t) = \frac{1}{\sqrt{n\varepsilon}} \mathrm{d}\frac{1}{\sqrt{n}} S_n(t) + \eta(\frac{1}{n\varepsilon} R_n(t), t)\, \mathrm{d}t$.

Clearly, the difference equation has one and only one solution. Note the drift: $\mathbb{E}\big(Y_k \,\big|\, Y_1,\dots,Y_{k-1}\big) = \varepsilon\eta\big(\frac{Y_1+\dots+Y_{k-1}}{n\varepsilon}, \frac{k-1}{n}\big)$.

What about large and moderate deviations of R_n? The case $\varepsilon = 1$, $n \to \infty$ leads to large deviations, while the case $\varepsilon \to 0$, $n \to \infty$, $n\varepsilon^2 \to \infty$ leads to moderate deviations. But the relation between $\frac{1}{n\varepsilon}R_n$ and $\frac{1}{n\varepsilon}S_n$ does not depend on ε; thus we postpone ε till Sect. 6b. Generating functions for R_n are hardly useful. Rather, we hope to transfer the rate function of S_n to R_n via the difference equation, or rather, its scaling limit; not quite a differential equation, because of lack of differentiability, rather, an integral equation.

We introduce a mapping $\varphi : C_0[0,1] \to C_0[0,1]$,

$$\varphi(f) = g \quad \Longleftrightarrow \quad \forall t \;\; g(t) = f(t) - \int_0^t \eta\big(f(s), s\big)\,\mathrm{d}s$$

(basically, $f' = g' + \eta(f,\cdot)$, but without differentiability). It is easy to guess that $\frac{1}{n\varepsilon}S_n \approx \varphi\big(\frac{1}{n\varepsilon}R_n\big)$ for large n, and so, the rate function for R_n at f is equal to the rate function for S_n at $g = \varphi(f)$. But wait; is φ one-to-one? Is φ onto? Is φ a homeomorphism?

6a1 Proposition. φ is a homeomorphism of $C_0[0,1]$ onto itself.

6a2 Lemma. For all $f_1, f_2 \in C_0[0,1]$,

$$\mathrm{e}^{-L}\|f_1 - f_2\| \le \|\varphi(f_1) - \varphi(f_2)\| \le (1+L)\|f_1 - f_2\|$$

(all norms being in $C_0[0,1]$).

The upper bound is easy. The lower bound will be proved soon.

We introduce a discrete-time counterpart φ_n of φ; $\varphi_n : A_nC_0[0,1] \to A_nC_0[0,1]$ (the n-dimensional subspace $\{f : f = A_nf\}$ of piecewise-linear functions),

$$\varphi_n(f) = g \quad \Longleftrightarrow \quad \forall k \;\; g\Big(\frac{k}{n}\Big) = f\Big(\frac{k}{n}\Big) - \frac{1}{n}\sum_{i=0}^{k-1} \eta\Big(f\Big(\frac{i}{n}\Big), \frac{i}{n}\Big)\,.$$

That is,

$$\varphi_n(f) = g \quad \Longleftrightarrow \quad \forall k \; g\Big(\frac{k+1}{n}\Big) - g\Big(\frac{k}{n}\Big) = f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big) - \frac{1}{n}\eta\Big(f\Big(\frac{k}{n}\Big), \frac{k}{n}\Big)\,.$$

Note that φ_n is a bijection (since the difference equation has one and only one solution), and

$$\varphi_n\Big(\frac{1}{n\varepsilon}R_n\Big) = \frac{1}{n\varepsilon}S_n\,. \tag{6a3}$$

6a4 Exercise. For all $f \in C_0[0,1]$,

$$\|\varphi_n(A_n f) - \varphi(f)\| \to 0\,.$$

Prove it.

In the next exercise $M = \max_{t\in[0,1]} |\eta(0,t)|$; note that $|\eta(x,t)| \le L|x|+M$.

6a5 Exercise. Let $f \in A_n C_0[0,1]$ and $g = \varphi_n(f)$; denote $a_k = \big|g\big(\frac{k}{n}\big)\big|$ and $b_k = \big|f\big(\frac{k}{n}\big) - g\big(\frac{k}{n}\big)\big|$; prove that
(a) $\big|f\big(\frac{k}{n}\big)\big| \le a_k + b_k$;
(b) $b_{k+1} \le b_k + \frac{L}{n}(a_k + b_k) + \frac{M}{n}$;
(c) $b_{k+1} + \|g\| + \frac{M}{L} \le \big(1 + \frac{L}{n}\big)\big(b_k + \|g\| + \frac{M}{L}\big)$;
(d) $b_k \le (\mathrm{e}^L - 1)\big(\|g\| + \frac{M}{L}\big)$;
(e) $\|f\| \le \mathrm{e}^L\|g\| + \frac{\mathrm{e}^L-1}{L}M$.

By 6a4 and 6a5,

$$\|f\| \le \mathrm{e}^L\|\varphi(f)\| + \frac{\mathrm{e}^L - 1}{L}M$$

for all $f \in C_0[0,1]$. That is,

$$L\|f\| + M \le \mathrm{e}^L(L\|\varphi(f)\| + M)\,.$$

6a6 Exercise. Let $f_1, f_2 \in A_n C_0[0,1]$ and $g_1 = \varphi_n(f_1)$, $g_2 = \varphi_n(f_2)$; denote $a_k = \big|g_1\big(\frac{k}{n}\big) - g_2\big(\frac{k}{n}\big)\big|$ and $b_k = \big|f_1\big(\frac{k}{n}\big) - f_2\big(\frac{k}{n}\big) - g_1\big(\frac{k}{n}\big) + g_2\big(\frac{k}{n}\big)\big|$; prove that
(a) $\big|f_1\big(\frac{k}{n}\big) - f_2\big(\frac{k}{n}\big)\big| \le a_k + b_k$;
(b) $b_{k+1} \le b_k + \frac{L}{n}(a_k + b_k)$;
(c) $b_{k+1} + \|g_1 - g_2\| \le \big(1 + \frac{L}{n}\big)(b_k + \|g_1 - g_2\|)$;
(d) $b_k \le (\mathrm{e}^L - 1)\|g_1 - g_2\|$;
(e) $\|f_1 - f_2\| \le \mathrm{e}^L\|g_1 - g_2\|$.

Now, Lemma 6a2 follows easily from 6a4 and 6a6. And in addition, for all $f_1, f_2 \in A_n C_0[0,1]$,

$$\mathrm{e}^{-L}\|f_1 - f_2\| \le \|\varphi_n(f_1) - \varphi_n(f_2)\| \le (1+L)\|f_1 - f_2\|\,.$$

6a7 Remark. Alternatively, the lower bound of Lemma 6a2 may be derived from the Gronwall(-Bellmann) lemma:[1] if $u(t) \le \alpha(t) + L\int_0^t u(s)\,\mathrm{d}s$ for all $t \in [0,1]$, then $u(t) \le \alpha(t) + L\int_0^t \alpha(s)\mathrm{e}^{L(t-s)}\,\mathrm{d}s$. To this end, take $u(t) = |f_1(t) - f_2(t) - g_1(t) + g_2(t)|$, $\alpha(t) = L\int_0^t |g_1(s) - g_2(s)|\,\mathrm{d}s$, and obtain $u(t) \le \alpha(t) + L\int_0^t \alpha(s)\mathrm{e}^{L(t-s)}\,\mathrm{d}s = L\int_0^t \mathrm{e}^{L(t-s)}|g_1(s) - g_2(s)|\,\mathrm{d}s \le \|g_1 - g_2\|\big(\mathrm{e}^{Lt} - 1\big)$; $|f_1(t) - f_2(t)| \le |g_1(t) - g_2(t)| + u(t) \le \mathrm{e}^L\|g_1 - g_2\|$.

[1] See "Grönwall's inequality" in Wikipedia.

In order to prove Prop. 6a1 it remains to check that φ is onto.

If $g = \varphi(f)$, then

$$\|f - g\|_{\mathrm{Lip}} \le L\|f\| + M \le \mathrm{e}^L(L\|g\| + M) \tag{6a8}$$

(since $f - g$ is the indefinite integral of $\eta(f, \cdot)$); here

$$\|f\|_{\mathrm{Lip}} = \sup_{0 \le s < t \le 1} \frac{|f(t) - f(s)}{t - s} \quad \text{for } f \in C_0[0,1]\,.$$

The same holds whenever $g = \varphi_n(f)$.

We introduce continuous $\psi_n : C_0[0,1] \to A_n C_0[0,1]$ by

$$\psi_n(g) = \varphi_n^{-1}(A_n g)\,.$$

6a9 Lemma. For every $g \in C_0[0,1]$ the limit

$$f = \lim_{n\to\infty} \psi_n(g)$$

exists, and $\varphi(f) = g$.

Proof. Functions $f_n = \psi_n(g)$ satisfy

$$\|f_n - A_n g\|_{\mathrm{Lip}} \le \mathrm{e}^L(L\|g\| + M)$$

by 6a8, since $\varphi_n(f_n) = A_n g$ and $\|A_n g\| \le \|g\|$. Thus, the sequence $(f_n - A_n g)_n$ is contained in a compact set (a Hölder ball RB_α, $\alpha = 1$); therefore it contains a convergent subsequence $(f_{n_k} - A_{n_k} g)_k$, and then $(f_{n_k})_k$ also converges (since $A_n g \to g$). It is sufficient to prove that the limit $f = \lim_k f_{n_k}$ satisfies $\varphi(f) = g$ (since such f is unique). By 6a4, $\lim_k \varphi_{n_k}(A_{n_k} f) = \lim_n \varphi_n(A_n f) = \varphi(f)$; it remains to prove that $\varphi_{n_k}(A_{n_k} f) \to g$. We have $\|g - A_n g\| \to 0$ and $\|A_n g - \varphi_n(A_n f)\| = \|\varphi_n(f_n) - \varphi_n(A_n f)\| \le (1 + L)\|f_n - A_n f\| \le (1 + L)\|f_n - f\|$, thus, $\|g - \varphi_{n_k}(A_{n_k} f)\| \to 0$. □

6a10 Remark. Alternatively, a solution f of the equation $\varphi(f) = g$ may be obtained by iterations:

$$f_{n+1}(t) = g(t) + \int_0^t \eta\big(f_n(s), s\big)\,\mathrm{d}s\,, \quad f_0(t) = 0\,.$$

They satisfy $|f_{n+1}(t) - f_n(t)| \le (\|g\| + M)(Lt)^n/n!$.

Prop. 6a1 is now proved. In addition,

$$\psi_n(g) \to \psi(g) \quad \text{for all } g \in C_0[0,1]\,;$$

here $\psi = \varphi^{-1}$ (and $\psi_n = \varphi_n^{-1} \circ A_n$, as before). Also, ψ and all ψ_n satisfy the Lipschitz condition with a single constant (e^L), which implies (think, why) that

$$\psi_n \to \psi \quad \text{uniformly on compact sets.} \tag{6a11}$$

6b Moderate deviations

Similarly to Sect. 5d, we interpret $\|f'\|_2$ as $+\infty$ if f is not the indefinite integral of a function of $L_2[0,1]$. All limits, as well as symbols $o(\dots)$, $\mathcal{O}(\dots)$ are taken for $\varepsilon \to 0, n \to \infty, \frac{n\varepsilon^2}{\ln n} \to \infty$ (unless stated otherwise). Also, $1 < p \le 2 \le q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\alpha \le 1/q$.

6b1 Theorem. (a) For every nonempty closed set $F \subset C_0[0,1]$,

$$\limsup \frac{1}{n\varepsilon^2} \ln \mathbb{P}\Big(\frac{1}{n\varepsilon} R_n \in F \Big) \le - \min_{f \in F} \frac{1}{2} \|\varphi(f)'\|_2^2 \,.$$

(b) For every open set $U \subset C_0[0,1]$,

$$\liminf \frac{1}{n\varepsilon^2} \ln \mathbb{P}\Big(\frac{1}{n\varepsilon} R_n \in U \Big) \ge - \inf_{f \in U} \frac{1}{2} \|\varphi(f)'\|_2^2 \,.$$

***Proof** of Theorem 6b1(b).* For arbitrary $\delta > 0$ we introduce the open set $U_{-\delta} = \{ f \in U : \operatorname{dist}(f, C_0[0,1] \setminus U) > \delta \}$, note that $\cup_{\delta>0} U_{-\delta} = U$, therefore $\inf_{f \in U} I(f) = \inf_{\delta>0} \inf_{f \in U_{-\delta}} I(f)$, where $I(f) = \frac{1}{2} \|\varphi(f)'\|_2^2$; it is sufficient to prove, for all $\delta > 0$, that

$$\liminf \frac{1}{n\varepsilon^2} \ln \mathbb{P}\Big(\frac{1}{n\varepsilon} R_n \in U \Big) \ge - \inf_{f \in U_{-\delta}} I(f) \,,$$

that is (recall (6a3)),

$$\liminf \frac{1}{n\varepsilon^2} \ln \mathbb{P}\Big(\psi_n\Big(\frac{1}{n\varepsilon} S_n \Big) \in U \Big) \ge - \inf_{f \in U_{-\delta}} I(f) \,.$$

Using the exponential tightness (for S_n) we take $R > 0$ such that $\limsup \frac{1}{n\varepsilon^2} \ln \mathbb{P}\big(\|S_n\|_\alpha > Rn\varepsilon \big) < - \inf_{f \in U_{-\delta}} I(f)$. By (6a11), $\psi_n \to \psi$ uniformly on RB_α; thus, for all n large enough, for all $g \in RB_\alpha$, $\|\psi_n(g) - \psi(g)\| \le \delta$, and therefore $\psi(g) \in U_{-\delta} \implies \psi_n(g) \in U$. We have

$$\begin{aligned} \mathbb{P}\Big(\psi_n\Big(\frac{1}{n\varepsilon} S_n \Big) \in U \Big) &\ge \mathbb{P}\Big(\frac{1}{n\varepsilon} S_n \in RB_\alpha \text{ and } \psi\Big(\frac{1}{n\varepsilon} S_n \Big) \in U_{-\delta} \Big) \ge \\ &\ge \mathbb{P}\Big(\frac{1}{n\varepsilon} S_n \in \varphi(U_{-\delta}) \Big) - \mathbb{P}\big(\|S_n\|_\alpha > Rn\varepsilon \big) \,; \end{aligned}$$

by Theorem 5d1(b),

$$\begin{aligned} \liminf \frac{1}{n\varepsilon^2} \ln \mathbb{P}\Big(\frac{1}{n\varepsilon} S_n \in \varphi(U_{-\delta}) \Big) &\ge - \inf_{g \in \varphi(U_{-\delta})} \frac{1}{2} \|g'\|_2^2 = \\ &= - \inf_{f \in U_{-\delta}} \frac{1}{2} \|\varphi(f)'\|_2^2 = - \inf_{f \in U_{-\delta}} I(f) \,, \end{aligned}$$

while $\mathbb{P}\big(\|S_n\|_\alpha > Rn\varepsilon \big)$ is exponentially smaller. $\square$

***Proof** of Theorem 6b1(a).* For arbitrary $\delta > 0$ we introduce the closed set $F_{+\delta} = \{f : \operatorname{dist}(f, F) \le \delta\} \subset C_0[0,1]$, and note that $\cap_{\delta>0} F_{+\delta} = F$. The function $I : f \mapsto \frac{1}{2}\|\varphi(f)'\|_2^2$ is lower semicontinuous (recall 5d3), and moreover, sets $\{f : I(f) \le c\} = \psi\big(\{g : \|g'\|_2^2 \le c\}\big)$ are compact (recall Sect. 4b; the ball B_2 is compact in $\|\cdot\|_{\text{int}}$).

Claim: $\min_{f\in F} I(f) = \sup_{\delta>0} \min_{f\in F_{+\delta}} I(f)$. *Proof:* if $c < \min_{f\in F} I(f)$, then $F \cap \{f : I(f) \le c\} = \emptyset$, that is, $\cap_{\delta>0}\big(F_{+\delta} \cap \{f : I(f) \le c\}\big) = \emptyset$; by compactness, for some $\delta > 0$ we have $F_{+\delta} \cap \{f : I(f) \le c\} = \emptyset$, that is, $\min_{f\in F_{+\delta}} I(f) > c$.

It is sufficient to prove, for all $\delta > 0$, that

$$\limsup \frac{1}{n\varepsilon^2} \ln \mathbb{P}\Big(\frac{1}{n\varepsilon} R_n \in F\Big) \le -\min_{f\in F_{+\delta}} I(f)\,,$$

that is (recall (6a3)),

$$\limsup \frac{1}{n\varepsilon^2} \ln \mathbb{P}\Big(\psi_n\Big(\frac{1}{n\varepsilon} S_n\Big) \in F\Big) \le -\min_{f\in F_{+\delta}} I(f)\,.$$

Using the exponential tightness (for S_n) we take $R > 0$ such that $\limsup \frac{1}{n\varepsilon^2} \ln \mathbb{P}\big(\|S_n\|_\alpha > Rn\varepsilon\big) < -\min_{f\in F_{+\delta}} I(f)$. By (6a11), $\psi_n \to \psi$ uniformly on RB_α; thus, for all n large enough, for all $g \in RB_\alpha$, $\|\psi_n(g) - \psi(g)\| \le \delta$, and therefore $\psi_n(g) \in F \implies \psi(g) \in F_{+\delta}$. We have

$$\mathbb{P}\Big(\frac{1}{n\varepsilon} S_n \in RB_\alpha \text{ and } \psi_n\Big(\frac{1}{n\varepsilon} S_n\Big) \in F\Big) \le \mathbb{P}\Big(\psi\Big(\frac{1}{n\varepsilon} S_n\Big) \in F_{+\delta}\Big) \le$$
$$\le \mathbb{P}\Big(\frac{1}{n\varepsilon} S_n \in \varphi(F_{+\delta})\Big)\,;$$

by Theorem 5d1(a),

$$\limsup \frac{1}{n\varepsilon^2} \ln \mathbb{P}\Big(\frac{1}{n\varepsilon} S_n \in \varphi(F_{+\delta})\Big) \le -\min_{g\in\varphi(F_{+\delta})} \frac{1}{2}\|g'\|_2^2 =$$
$$= -\min_{f\in F_{+\delta}} \frac{1}{2}\|\varphi(f)'\|_2^2 = -\min_{f\in F_{+\delta}} I(f)\,,$$

while $\mathbb{P}\big(\|S_n\|_\alpha > Rn\varepsilon\big)$ is exponentially smaller. □

6b2 Exercise. Assume that the amount R of a fissile material in an apparatus fluctuates around its equilibrium value R_{eq} according to such a discrete-time stochastic model:

$$R_k - R_{\text{eq}} = \Big(1 - \frac{\alpha}{n}\Big)(R_{k-1} - R_{\text{eq}}) \pm 1 \quad \text{for } k = 1, \dots, n\,; \quad R_0 = \beta R_{\text{eq}}\,;$$

here -1 and $+1$ are equiprobable, independent of the past; and $\sqrt{n} \ll R_{\rm eq} \ll n$. We need to find (in a rough approximation) the probability of reaching a critical level $\gamma R_{\rm eq}$ (during the time n). And α, β, γ are given positive parameters.

Reformulate this problem in terms of Theorem 6b1 (but do not solve it yet...)

6c Against a stream, at the speed of the stream

We consider a negative stationary drift,

$$\eta(x,t) = -v(x)\,, \quad v : \mathbb{R} \to (0,\infty) \ \text{ continuous}\,,$$

and examine the distribution of the random variable

$$M_n = \max_{k=0,\dots,n} \frac{1}{n\varepsilon} R_n\Big(\frac{k}{n}\Big) = M\Big(\frac{1}{n\varepsilon} R_n\Big)\,, \quad M(f) = \max_{0\le t\le 1} f(t)\,.$$

For every $c \in (0,\infty)$ the closed set $F_c = \{f : M(f) \ge c\}$ and the open set $G_c = \{f : M(f) > c\}$ satisfy

$$\min_{f\in F_c} I(f) = \inf_{f\in G_c} I(f) \qquad \text{(denote it } I_c\text{)}$$

(here $I(f) = \frac12 \|\varphi(f)'\|_2^2 = \frac12 \int_0^1 \big(f'(t) - \eta(f(t),t)\big)^2 \,\mathrm{d}t = \frac12 \int_0^1 \big(f'(t) + v(f(t))\big)^2 \,\mathrm{d}t$), which follows easily from continuity of $\lambda \mapsto I(\lambda f)$. By Theorem 6b1,

$$\mathbb{P}\big(M_n > c\big) = \exp\big(-n\varepsilon^2 I_c + o(n\varepsilon^2)\big)\,;$$

we want to know I_c.

For every absolutely continuous $f \in C_0[0,1]$ such that $f' \in L_2[0,1]$ we have

$$\frac12\big(f'(t) + v(f(t))\big)^2 - \underbrace{\frac12\big(f'(t) - v(f(t))\big)^2}_{\ge 0} = 2f'(t)v(f(t)) = 2\frac{\mathrm{d}}{\mathrm{d}t} V(f(t))$$

where $V(x) = \int_0^x v(u)\,\mathrm{d}u$. Thus, $2V(M(f)) = \max_{0\le t\le 1} 2V(f(t)) \le I(f)$;

$$I_c \ge 2\int_0^c v(x)\,\mathrm{d}x\,.$$

In order to reach equality we consider the solution f of the differential equation

$$f'(t) = v(f(t))\,;$$

in terms of the function $W(x) = \int_0^x \frac{\mathrm{d}u}{v(u)}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} W(f(t)) = 1\,; \quad W(f(t)) = t\,; \quad f(t) = W^{-1}(t)\,.$$

Clearly, $M(f) = f(1) = W^{-1}(1)$ and $I(f) = 2V(M(f)) = 2V(W^{-1}(1))$. Thus,

$$I_c = 2 \int_0^c v(x)\,\mathrm{d}x \quad \text{for } c = W^{-1}(1)\,.$$

A wonder: the least unlikely way to reach the level c against the stream is, to move at the speed of the stream (but in the opposite direction)!

For $c \in \big(0, W^{-1}(1)\big)$ we use a function f_c such that

$$f_c'(t) = \begin{cases} v(f(t)) & \text{for } t < W(c), \\ -v(f(t)) & \text{for } t > W(c). \end{cases}$$

Clearly, $f_c(t) = W^{-1}(t)$ for $t \le W(c)$; $M(f_c) = f_c(W(c)) = c$; and $I(f_c) = \frac{1}{2} \int_0^1 \big(f_c'(t) + v(f_c(t))\big)^2\,\mathrm{d}t = \frac{1}{2} \int_0^{W(c)} (\dots)^2\,\mathrm{d}t = 2V(f_c(W(c))) = 2V(c)$, thus, $I_c = 2V(c)$ for $c \le W^{-1}(1)$; that is,

$$I_c = 2 \int_0^c v(x)\,\mathrm{d}x \quad \text{whenever} \quad \int_0^c \frac{\mathrm{d}x}{v(x)} \le 1\,.$$

6c1 Exercise. Apply this technique in the situation of Exer. 6b2. Which values of α, β, γ are within reach?

However, for $c > W^{-1}(1)$ we need another approach. We introduce closed sets

$$F_{t,c} = \{f : f(t) = c\} \quad \text{for } 0 < t \le 1, c > 0$$

and note that $F_c = \cup_{0<t\le1} F_{t,c}$ (think, why); it follows (think, why) that

$$\min_{f \in F_c} I(f) = \min_{0<t\le1} \min_{f \in F_{t,c}} I(f)\,.$$

We'll find $\min_{f \in F_{t,c}} I(f)$; denote it $I_{t,c}$. Clearly, $I_{t,c} < \infty$.

6d Against a stream, at the acceleration of the stream

We return to $\eta(x,t)$ (not just $-v(x)$), but now we assume that $\eta \in C^1([0,1] \times \mathbb{R})$.

6d1 Lemma. Let $c > 0$. If f is a minimizer of $I(\cdot)$ on $F_{1,c}$, then $f \in C^2[0,1]$ and

$$f''(t) = a\big(f(t), t\big) \quad \text{for } 0 < t < 1\,,$$

where

$$a(x,t) = \eta_{,1}(x,t)\eta(x,t) + \eta_{,2}(x,t)\,.$$

Here $\eta_{,1}(x,t) = \frac{\partial}{\partial x}\eta(x,t)$ and $\eta_{,2}(x,t) = \frac{\partial}{\partial t}\eta(x,t)$. The function $a(\cdot,\cdot)$ is called *convective acceleration;* here is why. If g satisfies $g'(t) = \eta(g(t),t)$ on some open interval of t, then $g''(t) = a(g(t),t)$ for these t (since $\frac{\mathrm{d}}{\mathrm{d}t}\eta(g(t),t) = \eta_{,1}(g(t),t)g'(t) + \eta_{,2}(g(t),t)$).

A wonder: against the stream, we need not move at the speed of the stream, but we must move at the acceleration of the stream!

Our random process is Markovian (and remains Markovian if conditioned on $f(1) = c$), it forgets the past (remembers only the present, a single number $f(t)$); no inertia, no mass; and nevertheless, the acceleration is relevant. Wonders never cease!

***Proof** of Lemma 6d1.* We know that $I(f) < \infty$, thus, f is absolutely continuous, and $f' \in L_2[0,1]$.

For every $g \in C^1[0,1]$ such that $g(0) = g(1) = 0$ we have $I(f+\lambda g) \ge I(f)$ for all $\lambda \in \mathbb{R}$, and for $\lambda \to 0$,

$$\begin{aligned} I(f+\lambda g) &= \frac{1}{2}\int_0^1 \big(f'(t) + \lambda g'(t) - \eta(f(t)+\lambda g(t),t)\big)^2\,\mathrm{d}t = \\ &= \frac{1}{2}\int_0^1 \big(f'(t) - \eta(f(t),t) + \lambda(g'(t) - \eta_{,1}(f(t),t))g(t) + o(\lambda)\big)^2\,\mathrm{d}t\,; \end{aligned}$$

$$\frac{I(f+\lambda g) - I(f)}{\lambda} \to \int_0^1 \big(f'(t) - \eta(f(t),t)\big)\big(g'(t) - \eta_{,1}(f(t),t))g(t)\big)\,\mathrm{d}t\,;$$

this integral must vanish for all g.

We apply one of the "fundamental lemmas of calculus of variations": if $\varphi, \psi \in L_1[0,1]$ satisfy $\int_0^1(\varphi g' + \psi g) = 0$ for all $g \in C^1[0,1]$ such that $g(0) = g(1) = 0$, then φ is absolutely continuous and $\varphi' = \psi$ (almost everywhere).

In our case, $\varphi(t) = f'(t) - \eta(f(t),t)$ and $\psi(t) = -\eta_{,1}(f(t),t)\varphi(t)$. Continuity of φ implies continuity of ψ (since $\eta_{,1}(f,\cdot)$ is continuous) and f' (since $\eta(f,\cdot)$ is continuous); thus, $\varphi \in C^1[0,1]$ (since $\varphi' = \psi$), and $f \in C^1[0,1]$. Hence $f' \in C^1[0,1]$ (since $\eta(f,\cdot) \in C^1[0,1]$), that is, $f \in C^2[0,1]$.

Finally,

$$f''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi(t) + \eta(f(t),t)\big) = \psi(t) + \eta_{,1}(f(t),t)f'(t) + \eta_{,2}(f(t),t) =$$

$$= \eta_{,1}(f(t),t)\underbrace{\big(f'(t)-\varphi(t)\big)}_{=\eta(f(t),t)} + \eta_{,2}(f(t),t) = a(f(t),t)\,.$$

□

6d2 Remark. Generalization to $F_{t,c}$ is straightforward: if f is a minimizer of $I(\cdot)$ on $F_{t,c}$, then $f|_{[0,t]} \in C^2[0,t]$, $f''(s) = a(f(s),s)$ for $0 < s < t$, and $f'(s) = \eta(f(s),s)$ for $t < s < 1$.

Now, let $\eta(x,t) = -v(x)$ again (and $v(x) > 0$ as before, but in addition, $v \in C^1$). The convective acceleration being $a(x,t) = a(x) = v(x)v'(x) = -U'(x)$ where $U(x) = -\frac{1}{2}v^2(x)$, we get

$$f''(x) = a(f(t)) = -U'(f(t))\,,$$

nothing but the motion of a particle (of mass 1) in the potential U. (Did you know?) The energy conservation applies:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\big(f'(t)\big)^2 + U\big(f(t)\big)\Big) = f'(t)f''(t) + U'\big(f(t)\big)f'(t) = 0\,.$$

Good luck: the second-order differential equation is reduced to the first-order one,

$$f'(t) = \sqrt{v^2(f(t)) + b}\,;$$

the constant b (twice the "energy") should conform to the boundary conditions $f(0) = 0$, $f(t) = c$. The value $b = 0$ leads to the case $f'(t) = v(f(t))$ of Sect. 6c; positive b lead to faster increase of f. For $b > 0$, the change of variable $x = f(t)$ gives

$$I_{t,c} = \frac{1}{2}\int_0^c \big(\sqrt{v^2(x)+b} + v(x)\big)^2 \frac{\mathrm{d}x}{\sqrt{v^2(x)+b}}\,,$$

while b and c are related via the condition $f(t) = c$, that is,

$$\int_0^c \frac{\mathrm{d}x}{\sqrt{v^2(x)+b}} = t\,.$$

Now you may reconsider Exer. 6c1 for other values of α, β, γ.

Index