5 Moderate deviations in spaces of functions

5a Asymptotically quadratic generating functions

Let \( p, q, \mu, S_n, \Lambda_n, \Lambda_\infty \) and \( A_n \) be as in Sect. 4b, \( \int x^2 \mu(dx) = 1 \) (that is, \( \Lambda_\mu''(0) = 1 \)), and \( p \leq 2 \leq q \) (see 4b4).

5a1 Proposition. For every \( g \in L_q \),

\[
\frac{1}{n\varepsilon^2} \Lambda_n(\varepsilon g) \rightarrow \frac{1}{2} \|g\|^2_2 \quad \text{as} \quad \varepsilon \rightarrow 0, \; n \rightarrow \infty .
\]

This is a two-dimensional limit; that is,

\[
\forall \delta > 0 \; \exists \varepsilon_0 > 0 \; \exists n_0 \; \forall \varepsilon \leq \varepsilon_0 \; \forall n \geq n_0 \; \left| \frac{1}{n\varepsilon^2} \Lambda_n(\varepsilon g) - \frac{1}{2} \|g\|^2_2 \right| \leq \delta .
\]

Not the same as \( \lim n \lim \varepsilon \) or \( \lim \varepsilon \lim n \).

First, we improve 4b1, 4b2 for small arguments.

5a2 Lemma. \( \Lambda'_\mu(t) \leq \text{const} \cdot \max(|t|, |t|^{q-1}) \) for all \( t \in \mathbb{R} \).

Proof. For large \( t \) we have \( \Lambda'_\mu(t) = \mathcal{O}(|t|^{q-1}) \) by 4b1; for small \( t \), \( \Lambda'_\mu(t) = \mathcal{O}(|t|) \).

5a3 Lemma. There exists \( C \) such that for all \( g_1, g_2 \in L_q \),

\[
\|\Lambda_\infty(g_1) - \Lambda_\infty(g_2)\| \leq C \|g_1 - g_2\|_q (\|g_1\|_q + \|g_1\|_q^{q-1} + \|g_2\|_q + \|g_2\|_q^{q-1}) .
\]

Proof. Using 5a2 we take \( C \) such that

\[
\forall t_1, t_2 \; |\Lambda_\mu(t_1) - \Lambda_\mu(t_2)| \leq C|t_1 - t_2| \max(|t_1|, |t_1|^{q-1}, |t_2|, |t_2|^{q-1}) ;
\]

then

\[
\left| \int_0^1 \Lambda_\mu(g_1(x)) \, dx - \int_0^1 \Lambda_\mu(g_2(x)) \, dx \right| \leq \int_0^1 |\Lambda_\mu(g_1(x)) - \Lambda_\mu(g_2(x))| \, dx \leq
\]

\[
\left| \int_0^1 \Lambda_\mu(g_1(x)) \, dx - \int_0^1 \Lambda_\mu(g_2(x)) \, dx \right| \leq \int_0^1 |\Lambda_\mu(g_1(x)) - \Lambda_\mu(g_2(x))| \, dx \leq
\]
\[ \leq C(|g_1 - g_2|, \max(|g_1|, |g_1|^{q-1}, |g_2|, |g_2|^{q-1})) \leq C\|g_1 - g_2\|_q \max(|g_1|, |g_1|^{q-1}, |g_2|, |g_2|^{q-1})_p, \]

and
\[
\| \max(\ldots) \|_p = \| \max(|g_1|^{p/q}, |g_1|, |g_2|^{p/q}, |g_2|) \|_q^{q-1} \leq \| |g_1|^{p/q} + |g_1| + |g_2|^{p/q} + |g_2| \|_q \leq (\| |g_1|^{p/q} \|_q + |g_1| + \| |g_2|^{p/q} \|_q + |g_2| \|_q)^{q-1} = \\
= \left( \max(\| |g_1|^{p/q} \|_q, |g_1|, |g_2|^{p/q}, |g_2|\|_q) \right)^{q-1} = \\
= 4^{q-1} \max(\| g_1 \|_p, \| g_1 \|_q, \| g_2 \|_p, \| g_2 \|_q)^{q-1} \leq \\
\leq 4^{q-1} \max(\| g_1 \|_q, \| g_1 \|_q^{q-1}, \| g_2 \|_q, \| g_2 \|_q^{q-1}).
\]

5a4 Lemma. For every \( g \in L_q \),
\[
\frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g) \rightarrow \frac{1}{2} \|g\|_2^2 \quad \text{as} \quad \varepsilon \rightarrow 0.
\]

**Proof.** First, the bounded case: \( g \in L_\infty \); we have then
\[
\frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g) = \int_0^1 \frac{1}{\varepsilon^2} \Lambda_\mu(\varepsilon g(x)) \, dx \rightarrow \int_0^1 \frac{1}{2} g^2(x) \, dx,
\]

since \( \frac{1}{\varepsilon^2} \Lambda_\mu(\varepsilon g(\cdot)) \rightarrow \frac{1}{2} g^2(\cdot) \) uniformly.

Second, the general case; given \( \delta > 0 \), we take \( g_\delta \in L_\infty \) such that \( \| g_\delta - g \|_q \leq \delta \); by 5a3, \( |\Lambda_\infty(\varepsilon g) - \Lambda_\infty(\varepsilon g_\delta)| \leq \text{const} \cdot \varepsilon^2 \delta \) with a constant that depends on \( g \) but does not depend on \( \varepsilon, \delta \) (as long as \( |\varepsilon| \leq 1, \delta \leq 1 \)). We get
\[
\left| \frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g) - \frac{1}{2} \|g\|_2^2 \right| \leq \\
\leq \left| \frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g) - \Lambda_\infty(\varepsilon g_\delta) \right| + \left| \frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g_\delta) - \frac{1}{2} \|g_\delta\|_2^2 \right| + \left| \frac{1}{2} \|g_\delta\|_2^2 - \frac{1}{2} \|g\|_2^2 \right|, \leq \text{const} \cdot \delta
\]

thus, \( \limsup_{\varepsilon \rightarrow 0} \left| \frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g) - \frac{1}{2} \|g\|_2^2 \right| \leq \text{const} \cdot \delta \) for all \( \delta \). \( \square \)

**Proof of Prop. 5a4.** \( \frac{1}{n} \Lambda_n(\varepsilon g) = \Lambda_\infty(A_n \varepsilon g) \); by 5a3, \( |\Lambda_\infty(\varepsilon A_n g) - \Lambda_\infty(\varepsilon g)| \leq \text{const} \cdot \varepsilon^2 \|A_n g - g\|_q \) with a constant that depends on \( g \) but does not depend on \( \varepsilon, n \) (as long as \( |\varepsilon| \leq 1 \)). Thus, \( \left| \frac{1}{n \varepsilon^2} \Lambda_n(\varepsilon g) - \frac{1}{\varepsilon^2} \Lambda_\infty(\varepsilon g) \right| \rightarrow 0 \) as \( n \rightarrow \infty \), uniformly on \( |\varepsilon| \leq 1 \). It remains to use 5a4. \( \square \)
For the (one-dimensional) distribution $\nu_n$ of $\langle S_n, g \rangle$, similarly to (4b7), we get

\[(5a5) \frac{1}{n\epsilon^2} \Lambda_{\nu_n}(\epsilon t) \to \frac{1}{2} \|g\|^2 t^2 \quad \text{as} \quad \epsilon \to 0, n \to \infty, \]

since $\Lambda_{\nu_n}(\epsilon t) = \ln \mathbb{E} \exp(\epsilon t \langle S_n, g \rangle) = \Lambda_n(\epsilon t g)$.

5b G"artner-Ellis, again

**Dimension 1**

Let probability measures $\nu_1, \nu_2, \ldots$ on $\mathbb{R}$ be such that

\[(5b1) \frac{1}{n\epsilon^2} \Lambda_{\nu_n}(\epsilon t) \to \frac{1}{2} t^2 \quad \text{as} \quad \epsilon \to 0, n \to \infty \]

for all $t \in \mathbb{R}$. (In particular, $\nu_n = \nu^*n$ satisfy it, provided that $\int x \nu(dx) = 0$ and $\int x^2 \nu(dx) = 1$, since $\frac{1}{n\epsilon^2} \Lambda_{\nu_n}(\epsilon t) = \frac{1}{\epsilon t} \Lambda_{\nu}(\epsilon t) \to \frac{1}{2} t^2$.)

5b2 Example. It may seem that (4c1) with $\Lambda(t) \sim \frac{1}{2} t^2$ (for $t \to 0$) implies (5b1). But this is an illusion. Here is a counterexample.

Let $\frac{1}{\sqrt{n}} \ll a_n \ll 1$ (that is: $a_n \to 0$ and $\sqrt{na_n} \to \infty$), and

$$
\nu_n = \frac{1}{2} \mu^*n + \frac{1}{4}(\delta_{-na_n} + \delta_{na_n});
$$

here $\mu = N(0,1)$ is the standard normal distribution (thus, $\mu^*n = N(0, n)$), and $\delta_x$ is the unit atom at $x$. Then

$$
\Lambda_{\nu_n}(t) = \ln \left( \frac{1}{2} \exp \frac{nt^2}{2} + \frac{1}{2} \cosh na_n t \right).
$$

On one hand,

$$
\frac{1}{n} \Lambda_{\nu_n}(t) \to \frac{1}{2} t^2 \quad \text{as} \quad n \to \infty,
$$

since for $t = 0$ this holds trivially, otherwise $na_n t = o(nt^2)$ for large $n$.

On the other hand, taking $\epsilon_n$ such that $\frac{1}{\sqrt{n}} \ll \epsilon_n \ll a_n$ we get

\[
\frac{1}{n\epsilon_n^2} \Lambda_{\nu_n}(\epsilon_n t) \geq \frac{1}{n\epsilon_n^2} \ln \left( \frac{1}{4} \exp na_n \epsilon_n t \right) = \frac{a_n}{\epsilon_n} t + O\left( \frac{1}{n\epsilon_n^2} \right) \to \infty \quad \text{as} \quad n \to \infty.
\]

By the way, these $\nu_n$ violate 5b3 below.

The Legendre transform of $\Lambda(t) = \frac{1}{2} t^2$ is $\Lambda^*(x) = \frac{1}{2} x^2$ (recall 2c6).
5b3 Exercise.

\[ \nu_n(n \varepsilon x, \infty) \leq \exp\left(-\frac{1}{2}x^2n\varepsilon^2 + o(n\varepsilon^2)\right) \quad \text{for } x \geq 0; \]
\[ \nu_n(-\infty, n \varepsilon x) \leq \exp\left(-\frac{1}{2}x^2n\varepsilon^2 + o(n\varepsilon^2)\right) \quad \text{for } x \leq 0. \]

Of course, these \( o(\ldots) \) are meant for \( \varepsilon \to 0, n \to \infty \).

Prove it.\(^1\)

It follows that \( \nu_n(n \varepsilon a, n \varepsilon b) \to 1 \) as \( \varepsilon \to 0, n \to \infty, n \varepsilon^2 \to \infty \), whenever \( a < 0 < b \).

For tilted measures \( \nu_{n,\varepsilon t} \) we have \( \Lambda_{n,\varepsilon t}(\varepsilon s) = \Lambda_{n}(\varepsilon t + \varepsilon s) - \Lambda_{n}(\varepsilon t) \), thus \( \frac{1}{n \varepsilon^2} \Lambda_{n,\varepsilon t}(\varepsilon s) \to \frac{1}{2}(t + s)^2 - \frac{1}{2}t^2 = ts + \frac{1}{2}s^2 \); the corresponding Legendre transform is \( \Lambda^*_t(x) = \frac{1}{2}(x - t)^2 \) (since generally \( \Lambda^*_t(x) = \Lambda^*(x) - tx + \Lambda(t) \), as noted after 4c2). Similarly to (4c3),

\[ \nu_{n,\varepsilon t}(n \varepsilon a, n \varepsilon b) \to 1 \quad \text{as } \varepsilon \to 0, n \to \infty, n \varepsilon^2 \to \infty, \text{ whenever } a < t < b . \]

Taking into account that

\[ \frac{d\nu_n}{d\nu_{n,\varepsilon t}}(\varepsilon x) = \exp\left(-\varepsilon t\varepsilon x + \Lambda_{n}(\varepsilon t)\right) \geq \exp\left(-n\varepsilon^2 \max(ta, tb) + \Lambda_{n}(\varepsilon t)\right) \]

for \( x \in (na, nb) \), we get, similarly to (4e4),

\[ \nu_n(n \varepsilon a, n \varepsilon b) \geq \exp(-n\varepsilon^2 \max(ta, tb) + n\varepsilon^2 \cdot \frac{1}{2}t^2 + o(n\varepsilon^2)) \]

whenever \( a < t < b \).

Similarly to 4c5 (but simpler), if \( x \geq 0 \) and \( \delta > 0 \) then

\[ \nu_n(n \varepsilon x, n \varepsilon (x + \delta)) \geq \exp\left(-\frac{1}{2}x^2n\varepsilon^2 + o(n\varepsilon^2)\right), \]

and similarly to 4c6,

\[ \nu_n(n \varepsilon x, n \varepsilon (x + \delta)) = \exp\left(-\frac{1}{2}x^2n\varepsilon^2 + o(n\varepsilon^2)\right). \]

**DIMENSION \( d \)**

All limits, as well as symbols \( o(\ldots) \), \( O(\ldots) \) are taken for \( \varepsilon \to 0, n \to \infty, n \varepsilon^2 \to \infty \) (unless stated otherwise).

Let probability measures \( \nu_1, \nu_2, \ldots \) on \( \mathbb{R}^d \) be such that

\[ \frac{1}{n \varepsilon^2} \Lambda_{n}(\varepsilon t) \to \frac{1}{2}|t|^2 \quad \text{for all } t \in \mathbb{R}^d . \]

\(^1\)Hint: similar to 4c2.
5b7 Theorem. (a) For every nonempty closed set $F \subset \mathbb{R}^d$,
\[
\limsup \frac{1}{n \varepsilon^2} \ln \nu_n(n \varepsilon F) \leq -\min_{x \in F} \frac{1}{2} |x|^2.
\]
(b) For every open set $U \subset \mathbb{R}^d$,
\[
\liminf \frac{1}{n \varepsilon^2} \ln \nu_n(n \varepsilon U) \geq -\inf_{x \in U} \frac{1}{2} |x|^2.
\]

5b8 Exercise (upper bound for a half-space).
\[
\nu_n \{ \{n \varepsilon x : \langle t, x \rangle - \frac{1}{2}|t|^2 \geq c\} \} \leq \exp(-cn \varepsilon^2 + o(n \varepsilon^2))
\]
for all $t \in \mathbb{R}^d$ and $c \geq 0$.
Prove it.

5b9 Exercise (half-space not containing the expectation). If $c > 0$, then
\[
\exists \delta > 0 \quad \nu_n(\{n \varepsilon x : \langle t, x \rangle \geq c\} = O(e^{-\delta n \varepsilon^2}).
\]
Prove it.

5b10 Exercise (lower bound). If $U \subset \mathbb{R}^d$ is open, then
\[
\ln \nu_n(n \varepsilon U) \geq -n \varepsilon^2 \inf_{x \in U} \frac{1}{2} |x|^2 + o(n \varepsilon^2).
\]
Prove it.

5b11 Exercise. Prove Theorem 5b7\(^1\)

The simple rate function $\frac{1}{2} |\cdot|^2$ leads to a simple formula for half-spaces. Every closed half-space $H \subset \mathbb{R}^d$ not containing 0 is
\[
H = \{ x : \langle x, x_H \rangle \geq |x_H|^2 \}
\]
where $x_H$ is the point of $H$ closest to 0. Now, 5b8 with $t = x_H$ and $c = \frac{1}{2}|x_H|^2$ gives
\[
(5b12) \quad \nu_n(n \varepsilon H) \leq \exp\left(-\frac{1}{2}|x_H|^2 n \varepsilon^2 + o(n \varepsilon^2)\right);
\]
we see very clearly that every $x \neq 0$ belongs to (a) a closed half-space that satisfies the upper bound with rate $\frac{1}{2}|x|^2$, and (b) an open half-space that satisfies the upper bound with rate arbitrarily close to $\frac{1}{2}|x|^2$.

\(^1\)Hint: recall the proof of 4c10(a).
5c Exponential tightness

What about a weakly compact set \( K \subset L_p \) such that \( \mathbb{P}(S_n \notin n\varepsilon K) \leq \exp(-Cn\varepsilon^2 + o(n\varepsilon^2)) \) (for a given \( C \))? No, this cannot happen. Indeed, on one hand, \( K \) must be bounded, that is, \( K \subset \{ f : \|f\|_p \leq R \} \) for some \( R \); on the other hand, \( \|S_n\|_1 = |X_1| + \cdots + |X_n|; \mathbb{E}\|S_n\|_1 = n\mathbb{E}|X_1|; \mathbb{P}(S_n \in n\varepsilon K) \leq \mathbb{P}(\|S_n\|_p \leq n\varepsilon R) \leq \mathbb{P}(\|S_n\|_1 \leq n\varepsilon R) \) is close to 0 (rather than 1) when \( n\varepsilon R \ll \mathbb{E}\|S_n\|_1 \), that is, \( \varepsilon \ll \mathbb{E}|X_1|/R \).

The joint compactification introduced in Sect. 4b and used successfully for large deviations, fails for moderate deviations. We need another joint compactification. The \( L_p \)-norm feels only absolute values of \( X_1, \ldots, X_n \). But we have \( \mathbb{E}X_1 = 0 \), and cancellation of positive and negative summands should not be ignored.

We sacrifice invariance under permutations of the random variables \( X_1, \ldots, X_n \) (thus, by the way, complicating generalization to, say, two-dimensional arrays of random variables) and take indefinite integrals of the functions \( S_n \) vanishing at 0, and redefine the random function \( S_n \) as such piecewise-linear function of \( C_0[0, 1] \):

\[
S_n(x) = \int_0^x \left( nX_1 1_{[0, \frac{1}{n}]} + \cdots + nX_n 1_{[\frac{n-1}{n}, 1]} \right).
\]

Note that indefinite integrals of functions of \( L_p \) (or \( L_1 \)) are absolutely continuous; they are dense in the space \( C_0[0, 1] \), but far not the whole space. In this sense, we really move to a larger space.

We also need Hölder spaces \( C_{0, \alpha} \) and Hölder norms \( \| \cdot \|_\alpha \) for \( \alpha \in (0, 1) \),

\[
\|f\|_\alpha = \sup_{0 < x < y < 1} \frac{|f(y) - f(x)|}{(y - x)^\alpha} \in [0, \infty] \quad \text{for } f \in C_0[0, 1],
\]

\[
C_{0, \alpha} = \{ f \in C_0[0, 1] : \|f\|_\alpha < \infty \}.
\]

For \( 0 < \alpha \leq \beta < 1 \) we have \( \| \cdot \|_\alpha \leq \| \cdot \|_\beta \) and \( C_{0, \alpha} \supset C_{0, \beta} \).

The unit ball \( B_\alpha = \{ f : \|f\|_\alpha \leq 1 \} \) is separable, but not compact (in \( C_{0, \alpha} \)).\(^1\) However, \( B_\alpha \) is compact in \( C_0[0, 1] \).\(^2\) Note that Hölder functions need not be absolutely continuous.

We also redefine operators \( A_n \); now \( A_nf \) is the function linear on \([0, \frac{1}{n}], \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\] and equal to \( f \) at \( 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1 \).

\(^1\)Try \( f_n(x) = \min(x^n, 1/n) \).

\(^2\)Hint: in this situation, convergence on a dense countable set implies uniform convergence. In fact, moreover, \( B_\beta \) is compact in \( C_{0, \alpha} \) whenever \( 0 < \alpha < \beta < 1 \); hint: if \( f, g \in B_\beta \) satisfy \( |f(x) - g(x)| \leq \frac{1}{n} \) for \( x = \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1 \), then \( \|f - g\|_\alpha \leq 4/n^{3-\alpha} \).
For a piecewise-linear function $f = A_n f$ we have

$$\|f\|_\alpha = \max_{0 \leq k < l \leq n} \frac{1}{(\frac{l}{n} - \frac{k}{n})^\alpha} |f(\frac{l}{n}) - f(\frac{k}{n})|;$$

indeed, $\frac{|f(y) - f(x)|}{(y-x)^\alpha}$ cannot be maximal between the nodes $\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n}{n}$ due to concavity of the function $x \mapsto x^\alpha$. For such $f$,

$$\|f\|_\alpha = \max_{0 \leq k < l \leq n} |\langle f', g_{k,l} \rangle|$$

where $g_{k,l} = \frac{n^\alpha}{(l-k)^\alpha} \mathbb{I}_{(\frac{k}{n}, \frac{l}{n})}$. We note that $\|g_{k,l}\|_q = (\frac{l-k}{n})^{\frac{1}{q} - \alpha} \leq 1$ for $\alpha \leq 1/q$. We use [5b3]

$$\mathbb{P}\left(\|S_n\|_\alpha \geq n\varepsilon x\right) \leq \sum_{k,l} \mathbb{P}\left(|\langle S_n', g_{k,l} \rangle| \geq n\varepsilon x\right) \leq 2\left(n + 1\right) \exp\left(-\frac{1}{2} x^2 n\varepsilon^2 + o(n\varepsilon^2)\right),$$

and get

$$\mathbb{P}\left(\|S_n\|_\alpha \geq n\varepsilon x\right) \leq \exp\left(-\frac{1}{2} x^2 n\varepsilon^2 + o(n\varepsilon^2) + \mathcal{O}(\ln n)\right)$$

for $\alpha \leq 1/q$.

From now on, all limits, as well as symbols $o(\ldots)$, $\mathcal{O}(\ldots)$ are taken for $\varepsilon \to 0$, $n \to \infty$, $\frac{n^2}{\ln n} \to \infty$ (unless stated otherwise). Note the logarithmic gap between moderate deviations and central limit theorem.

Now, for $\alpha \leq 1/q$ we have

(5c1) \hspace{1cm} \mathbb{P}\left(\|S_n\|_\alpha \geq n\varepsilon x\right) \leq \exp\left(-\frac{1}{2} x^2 n\varepsilon^2 + o(n\varepsilon^2)\right),

which is exponential tightness; $K_C$ is the ball $x B_\alpha$ (with $x$ such that $x^2/2 = C$) endowed with the compact topology from $C_0[0,1]$.

5d Mogulskii’s theorem, again

We interpret $\|f'\|_2$ as $+\infty$ if $f$ is not the indefinite integral of a function of $L_2[0,1]$. As before, all limits, as well as symbols $o(\ldots)$, $\mathcal{O}(\ldots)$ are taken for $\varepsilon \to 0$, $n \to \infty$, $\frac{n^2}{\ln n} \to \infty$ (unless stated otherwise). Also, $1 < p \leq 2 \leq q < \infty$: $\frac{1}{p} + \frac{1}{q} = 1$, and $\alpha \leq 1/q$.

5d1 Theorem. (a) For every nonempty closed set $F \subset C_0[0,1]$,

$$\limsup \frac{1}{n^2} \ln \mathbb{P}\left(\frac{1}{n\varepsilon} S_n \in F\right) \leq -\min_{f \in F} \frac{1}{2} \|f'\|_2^2.$$

(b) For every open set $U \subset C_0[0,1]$,

$$\liminf \frac{1}{n^2} \ln \mathbb{P}\left(\frac{1}{n\varepsilon} S_n \in U\right) \geq -\inf_{f \in U} \frac{1}{2} \|f'\|_2^2.$$
5d2 Remark. Weaker conditions on $F$ and $U$ are sufficient for the theorem (and the proof): for all $R > 0$,

$$F \cap RB_\alpha \text{ is closed,}$$

$$U \cap RB_\alpha \text{ is relatively open in } RB_\alpha;$$

here $RB_\alpha = \{RF : f \in B_\alpha\} = \{f : \|f\|_\alpha \leq R\}$.

We choose a dense sequence $x_1, x_2, \cdots \in [0, 1]$ and denote $g_k = \mathbb{1}_{(0,x_k)}$. If $f \in C_0[0,1]$ is the indefinite integral of a function of $L_2[0,1]$,

$$f(x) = \int_0^x f'(u) \, du,$$

then clearly $f(x_k) = \langle f', g_k \rangle$. It is convenient to denote $\langle f', g_k \rangle = f(x_k)$ for arbitrary $f \in C_0[0,1]$ (even though $f'$ is ill-defined). We note that

$$(f_n \to f \text{ in } C_0[0,1]) \iff \forall k \langle f'_n, g_k \rangle \to \langle f', g_k \rangle$$

for all $f, f_1, f_2, \cdots \in B_\alpha$.

We fix $d$ for a while, and enumerate $x_1, \ldots, x_d$ in ascending order:

$$\{x_1, \ldots, x_d\} = \{y_1, \ldots, y_d\}, \quad 0 < y_1 < \cdots < y_d < 1.$$

Here is an orthonormal basis in the $d$-dimensional space spanned by $g_1, \ldots, g_d$:

$$h_1 = \frac{1}{\sqrt{y_1}} \mathbb{1}_{(0,y_1)}, \quad h_2 = \frac{1}{\sqrt{y_2 - y_1}} \mathbb{1}_{(y_1,y_2)}, \ldots, \quad h_d = \frac{1}{\sqrt{y_d - y_{d-1}}} \mathbb{1}_{(y_{d-1},y_d)}.$$ 

Naturally, we let $\langle f', h_i \rangle = \frac{1}{\sqrt{y_i - y_{i-1}}} (f(y_i) - f(y_{i-1}))$ (where $y_0 = 0$). We introduce linear operators $T_d : C_0[0,1] \to \mathbb{R}^d$ by

$$T_d f = (\langle f', h_1 \rangle, \ldots, \langle f', h_d \rangle);$$

they are continuous.

Similarly to $A_n$, we define operator $\tilde{A}_d : C_0[0,1] \to C_0[0,1]$; $\tilde{A}_d f$ is the function linear on $[0, y_1], [y_1, y_2], \ldots, [y_{d-1}, y_d]$, equal to $f$ at $0, y_1, \ldots, y_d$, and constant on $[y_{d-1}, 1]$. Thus, $(\tilde{A}_d f)' = \langle f', h_1 \rangle h_1 + \cdots + \langle f', h_d \rangle h_d$ and $(\tilde{A}_d f)' = \langle (\tilde{A}_d f)', g' \rangle$ (like the orthogonal projection, but $f'$, $g'$ are ill-defined). Note that $\|(\tilde{A}_d f)\|_2 = \|T_d f\|_2$ and $(\langle \tilde{A}_d f)', (\tilde{A}_d g) \rangle = (T_d f, T_d g)$.

Now we have three “incarnations” of the $d$-dimensional Euclidean vector space:
* $\mathbb{R}^d$;
* subspace of $L_2[0, 1]$ spanned by $g_1, \ldots, g_d$ or, equivalently, by $h_1, \ldots, h_d$; 
    with the norm $\| \cdot \|_2$ (step functions);
* subspace $\{ f : \tilde{A}_d f = f \}$ of $C_0[0, 1]$, with the norm $f \mapsto \| f' \|_2$ (polygonal functions).

They are intertwined by a commutative diagram of linear isometries:

\[
\begin{array}{ccc}
\text{polygonal} & \xleftarrow{\varepsilon t} \text{step} & f \\
\mathbb{R}^d & \quad & f' \\
& \text{projection of } \tilde{A}_d & \text{to the subspace spanned by } x_1, \ldots, x_d \\
& \xleftarrow{\tilde{A}_d} \text{subspace of } \{ \langle \cdot, h \rangle : h \in \mathcal{H} \} & \text{with the norm } \parallel \cdot \parallel_{\mathcal{H}} \\
& \xleftarrow{\text{with the norm } \parallel \cdot \parallel_{C_0[0, 1]}} \text{subspace of } \{ \langle \cdot, h \rangle : h \in \mathcal{H} \} & \text{with the norm } \parallel \cdot \parallel_{C_0[0, 1]} \\
& \xleftarrow{\tilde{A}_d^* \text{ or } \tilde{A}_d^*} \text{subspace of } \{ \langle \cdot, h \rangle : h \in \mathcal{H} \} & \text{with the norm } \parallel \cdot \parallel_{C_0[0, 1]} \end{array}
\]

We turn to $d \to \infty$. Clearly,

\[f_n \to f \text{ in } C_0[0, 1] \iff \forall d \quad T_d f_n \underset{n \to \infty}{\longrightarrow} T_d f \]

for all $f, f_1, f_2, \ldots \in B_\alpha$.

If $d_1 \leq d_2$, then $\tilde{A}_{d_1} \tilde{A}_{d_2} = \tilde{A}_{d_2} = \tilde{A}_{d_2} \tilde{A}_{d_1}$, and $(\tilde{A}_{d_2} f)'$ is the orthogonal projection of $(\tilde{A}_{d_2} f)'$. Thus, $\|(\tilde{A}_{d_2} f)'\|_2$ is increasing (in $d$).

**5d3 Lemma.** $\|(\tilde{A}_d f)'\|_2 \leq \|f'\|_2$ (be it finite or infinite) as $d \to \infty$.

**Proof.** On one hand, if $f' \in L_2$, then $(\tilde{A}_d f)'$ is the orthogonal projection of $f'$ to the subspace spanned by $g_1, \ldots, g_d$; the union of these subspaces is dense in $L_2$, thus, $\|(\tilde{A}_d f)'\|_2 \leq \|f'\|_2$.

On the other hand, assume that $\lim_d \|(\tilde{A}_d f)'\|_2 < \infty$; we have to prove that $f' \in L_2$. The series

\[(\tilde{A}_1 f)' + (\tilde{A}_2 f - \tilde{A}_1 f)' + (\tilde{A}_3 f - \tilde{A}_2 f)' + \ldots\]

consists of orthogonal summands, and its partial sums are bounded. It follows easily that these partial sums are a Cauchy sequence. Thus, the series converges:

\[(\tilde{A}_d f)' \to \varphi \quad \text{for some } \varphi \in L_2.\]

We note that $\langle (\tilde{A}_d f)', g_d \rangle = \langle f', g_d \rangle$ when $k \geq d$; thus, it equals $\langle \varphi, g_d \rangle$; that is, $\int_0^{x_d} \varphi(u) \, du = f(x_d)$ for all $d$; this shows that $\varphi = f'$.

Denote by $\nu_{d,n}$ the distribution of $T_d S_n$. By **5a1**

\[
\frac{1}{n \varepsilon^2} \Lambda_{\nu_{d,n}}(\varepsilon t_1, \ldots, \varepsilon t_d) \to \frac{1}{2} (t_1^2 + \cdots + t_d^2) \quad \text{as } n \to \infty
\]

for all $(t_1, \ldots, t_d) \in \mathbb{R}^d$, since $\Lambda_{\nu_{d,n}}(t_1, \ldots, t_d) = \ln \mathbb{E} \exp(\varepsilon t_1 \langle S_n, h_1 \rangle + \cdots + \varepsilon t_d \langle S_n, h_d \rangle) = \ln \mathbb{E} \exp(S_n \varepsilon t_1 h_1 + \cdots + \varepsilon t_d h_d) = \Lambda_n(\varepsilon t_1 h_1 + \cdots + \varepsilon t_d h_d)$.
Thus, Theorem 5b7 (as well as 5b8 (5b12)) applies to \( \nu_{d,n} \) for given \( d \). That theorem is formulated for \( \mathbb{R}^d \), but may be transferred readily to the “step” or “polygonal” space. In all cases, the rate function is \( \frac{1}{2} \| \cdot \|^2 \).

5d4 Exercise. Let \( g \in C_0[0,1] \) satisfy \( g = \tilde{A}_dg \) (for a given \( d \)), and \( H = \{ f \in C_0[0,1] : \mathcal{L}_H f, \mathcal{L}_H g \geq \| g' \|^2_2 \} \) (even though \( f' \) is ill-defined...). Then

(a) \( H = \{ f \in C_0[0,1] : \mathcal{L}_H f, \mathcal{L}_H g \geq | \mathcal{T}_d g |^2 \} \);
(b) \( \mathbb{P} \{ S_n \in n \epsilon H \} \leq \exp \left( -\frac{1}{2} \| g' \|^2_2 n \epsilon^2 + o(n \epsilon^2) \right) \).

Prove it.

Our space \( C_0[0,1] \) is not a finite-dimensional Euclidean space, nor a Hilbert space, and still, every \( f \neq 0 \) belongs to an open half-space that satisfies the upper bound with rate arbitrarily close to \( \Lambda_*^\infty(f) \). Indeed, if \( c < \Lambda_*^\infty(f) \) (being the latter finite or infinite), then \( \frac{1}{2} \| (\Lambda_d f)^* \|^2_2 > c \) for \( d \) large enough; we take such \( d \), and introduce \( g = (1 - \delta) \tilde{A}_d f \) with \( \delta > 0 \) small enough, then \( \frac{1}{2} \| g' \|^2_2 \geq c \) and \( g = \tilde{A}_d g \); the half-space \( H = \{ f_1 \in C_0[0,1] : (f_1, g') > \| g' \|^2_2 \} \) is open in \( C_0[0,1] \) (think, why), \( f \in H \) (think, why), and \( \mathbb{P} \{ S_n \in n \epsilon H \} \leq \exp \left( -cn \epsilon^2 + o(n \epsilon^2) \right) \) by 5d4 (b).

5d5 Exercise. Prove Theorem 5d11(a).

5d6 Exercise. Let \( U \subset C_0[0,1] \) be open, and \( f_0 \in U \cap B_{\alpha} \). Then there exist \( d \) and \( \delta > 0 \) such that

\[
\forall f \in B_{\alpha} \left( | T_d f - T_d f_0 | \leq \delta \implies f \in U \right).
\]

Prove it.\(^1\)

5d7 Exercise. \( \| f \|_{1/2} \leq \| f' \|_2 \) for all \( f \in C_0[0,1] \) (be the norms finite or infinite). (Here \( \| \cdot \|_{1/2} \) is the Hölder norm for \( \alpha = 1/2 \), while \( \| \cdot \|_2 \) is the \( L_2 \) norm.)

Prove it.

Also, \( \alpha \leq \frac{1}{q} \) and \( p \leq 2 \leq q \), thus, \( \| f \|_{\alpha} \leq \| f \|_{1/2} \leq \| f' \|_2 \).

**Proof of Theorem 5d11(b).**\(^2\) Let \( f_0 \in U \); we’ll prove that \( \liminf \frac{1}{n \epsilon^2} \ln \mathbb{P} \left( S_n \in n \epsilon U \right) \geq -\frac{1}{2} \| f'_0 \|_2^2 \), assuming \( \| f'_0 \|_2 < \infty \) (otherwise the claim is void). We take \( R > \| f'_0 \|_2 \), then \( f_0 \in RB_{\alpha} \) by 5d7, and \( \limsup \frac{1}{n \epsilon^2} \ln \mathbb{P} \left( \| S_n \|_{\alpha} \geq R n \epsilon \right) < -\frac{1}{2} \| f'_0 \|_2^2 \) by 5c1. Exercise 5d6 gives \( d \) and \( \delta > 0 \) such that \( \forall f \in RB_{\alpha} \left( | T_d f - T_d f_0 | \leq \delta \implies f \in U \right) \). It is sufficient to prove that

\[
\liminf \frac{1}{n \epsilon^2} \ln \mathbb{P} \left( \left| T_d \frac{S_n}{n \epsilon} - T_d f_0 \right| < \delta \right) \geq -\inf_{x : \| x - T_d f_0 \| < \delta} \frac{1}{2} | x |^2.
\]

\(^1\)Hint: similar to 4e7.

\(^2\)Quite similar to the proof of Theorem 4e1(b).
since \(\inf_{x:|x-Td_0|<\delta}\frac{1}{2}|x|^2 \leq \frac{1}{2}|Td_0|^2 = \frac{1}{2}\|\hat{A}_d f_0\|^2 \leq \frac{1}{2}\|f_0\|^2\). Theorem 5b gives the needed inequality, since \(\nu_{d,n}(\{n\in\mathbb{N}: |x-Td_0|<\delta\}) = P(\|T_d a_n - T_d f_0\|<\delta)\).

5d8 Exercise. A fair coin is tossed \(n\) times, giving \((\beta_1, \ldots, \beta_n) \in \{0, 1\}^n\). Given a continuous \(\varphi: [0, 1] \to (0, \infty)\), consider

\[
p_n = P(\forall k = 1, \ldots, n \frac{2(\beta_1 + \cdots + \beta_k) - k}{n^{2/3}} \leq \varphi(\frac{k}{n})).
\]

(a) Prove that

\[
p_n = 1 - \exp\left(-an^{1/3}(1 + o(1))\right)
\]

for some \(a > 0\);

(b) find \(a\) when \(\varphi(x) = 1 + vx\) for a given \(v > 0\);

(c) find \(a\) when \(\varphi(x) = \max(1 + vx, y)\) for given \(v > 0\) and \(y > 1\);

(d) find \(a\) when \(\varphi(x) = 1 + cx^2\) for a given \(c > 0\);

(e) find \(a\) when \(\varphi(x) = 1 + c\sqrt{x}\) for a given \(c > 0\).

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