## 1 Basic ideas

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## 1a From Cantor's uncountability theorem to Baire category theorem

By the famous Cantor's uncountability theorem, $\mathbb{R}$ is not countable. Here is one of the proofs. Let $a_{1}, a_{2}, \cdots \in \mathbb{R}$; we need $x \in \mathbb{R}$ such that $\forall n x \neq a_{n}$. To this end we first take $b_{1}<c_{1}$ such that $a_{1} \notin\left[b_{1}, c_{1}\right]$. Then we take $b_{2}<c_{2}$ such that $\left[b_{2}, c_{2}\right] \subset\left[b_{1}, c_{1}\right]$ and $a_{2} \notin\left[b_{2}, c_{2}\right]$. And so on; $\left[b_{1}, c_{1}\right] \supset\left[b_{2}, c_{2}\right] \supset$ $\left[b_{3}, c_{3}\right] \supset \ldots$ Their intersection is not empty, and contains no $a_{n}$.

Can we generalize it to some sets $A_{1}, A_{2}, \cdots \subset \mathbb{R}$ proving that $\cup_{n} A_{n} \neq \mathbb{R}$ ? Yes, provided that these sets satisfy the following.

1a1 Definition. A set $A \subset \mathbb{R}$ is nowhere dense if every nonempty open interval contains some nonempty open subinterval that does not intersect $A$.

1a2 Exercise. A set $A \subset \mathbb{R}$ is nowhere dense if and only if $\operatorname{Int}(\mathrm{Cl}(A))=\emptyset$. Prove it. (Here "Int" stands for interior, and "Cl" for closure.)

1a3 Theorem (Baire). If $A_{1}, A_{2}, \cdots \subset \mathbb{R}$ are nowhere dense then $\operatorname{Int}\left(\cup_{n} A_{n}\right)=$ $\emptyset$.

1a4 Exercise. Prove the theorem.
Equivalently: $\mathbb{R} \backslash \cup_{n} A_{n}$ is dense; that is, $\operatorname{Cl}\left(\mathbb{R} \backslash \cup_{n} A_{n}\right)=\mathbb{R}$.
In particular, $\cup_{n} A_{n} \neq \mathbb{R}$.
Clearly, a singleton is nowhere dense; therefore Cantor's uncountability theorem follows from Baire category theorem.

## 1b From Cantor's uncountability theorem to null sets

Here is another proof of Cantor's uncountability theorem. Let $a_{1}, a_{2}, \cdots \in \mathbb{R}$; we need $x \in \mathbb{R}$ such that $\forall n x \neq a_{n}$. To this end we take $\varepsilon_{1}, \varepsilon_{2}, \cdots>0$ such that $\sum_{n} \varepsilon_{n}<1 / 2$ and consider open intervals $\left(a_{n}-\varepsilon_{n}, a_{n}+\varepsilon_{n}\right)$. A finite number of these intervals cannot cover $[0,1]$ since their total length is less than 1. (Take the Riemann integral of the sum of indicators...) By the Heine-Borel theorem, the infinite sequence of these intervals still does not cover $[0,1]$.

1b1 Definition. A set $A \subset \mathbb{R}$ is a null set if for every $\varepsilon>0$ there exist $\varepsilon_{1}, \varepsilon_{2}, \cdots>0$ and $a_{1}, a_{2}, \cdots \in \mathbb{R}$ such that $A \subset \cup_{n}\left(a_{n}-\varepsilon_{n}, a_{n}+\varepsilon_{n}\right)$ and $2 \sum_{n} \varepsilon_{n} \leq \varepsilon$.

1 b 2 Theorem. If $A_{1}, A_{2}, \cdots \subset \mathbb{R}$ are null sets then $\operatorname{Int}\left(\cup_{n} A_{n}\right)=\emptyset$.
1b3 Exercise. (a) Prove that $\cup_{n} A_{n}$ is also a null set.
(b) Prove the theorem.

## 1c Two approaches to small sets and typical objects

1c1 Definition. Given a set $X$, a set $\mathcal{N}$ of subsets of $X$ is called
(a) an ideal $^{1}$ (on $X$ ), if

$$
\begin{aligned}
&(A \subset B \wedge B \in \mathcal{N}) \Longrightarrow A \in \mathcal{N} ; \\
& A, B \in \mathcal{N} \Longrightarrow A \cup B \in \mathcal{N} \\
& \emptyset \in \mathcal{N}
\end{aligned}
$$

(b) a $\sigma$-ideal (on $X$ ), if it is an ideal and

$$
A_{1}, A_{2}, \cdots \in \mathcal{N} \quad \Longrightarrow \quad \cup_{n} A_{n} \in \mathcal{N} .
$$

An ideal (or $\sigma$-ideal) $\mathcal{N}$ on $X$ is proper if $X \notin \mathcal{N}$.
Clearly, null sets are a proper $\sigma$-ideal on $\mathbb{R}$.
The complement of a null set is called a set of full measure.
1c2 Definition. A set $A \subset \mathbb{R}$ is meager ${ }^{2}$ if $A \subset \cup_{n} A_{n}$ for some nowhere dense sets $A_{1}, A_{2}, \cdots \subset \mathbb{R}$.

[^0]Clearly, meager sets are a proper $\sigma$-ideal on $\mathbb{R}$.
The complement of a meager set is called comeager. ${ }^{1}$
When a property holds off a null set (in other words, on a set of full measure), one says that it holds almost everywhere or for almost all elements. Dealing with a probability measure one also says almost sure(ly).

When a property holds off a meager set (in other words, on a comeager set), one says that it holds quasi-everywhere or for quasi all elements. One also says that this property holds generically, for a generic element, or for most of elements. Sometimes the word "typical" is used rather than "generic".

## 1d Compact metrizable spaces; sequence spaces

1d1 Definition. (a) A metric space is a pair $(X, \rho)$ of a set $X$ and a metric $\rho$ on $X$, that is, a function $\rho: X \times X \rightarrow[0, \infty)$ such that $\rho(x, y)=0 \Longleftrightarrow x=y$, $\rho(x, y)=\rho(y, x), \rho(x, z) \leq \rho(x, y)+\rho(y, z)$ for all $x, y, z \in X$.
(b) Let $\rho_{1}, \rho_{2}$ be two metrics on $X ; \rho_{2}$ is stronger than $\rho_{1}$ if $\rho_{2}\left(x_{n}, x\right) \rightarrow 0 \Longrightarrow \rho_{1}\left(x_{n}, x\right) \rightarrow 0$ for all $x, x_{1}, x_{2}, \cdots \in X ;{ }^{2}$ further, $\rho_{1}, \rho_{2}$ are equivalent, if $\rho_{1}\left(x_{n}, x\right) \rightarrow 0 \Longleftrightarrow \rho_{2}\left(x_{n}, x\right) \rightarrow 0$ for all $x, x_{1}, x_{2}, \cdots \in X$.
(c) A metrizable space ${ }^{3}$ is a pair $(X, R)$ where $X$ is a set and $R$ is an equivalence class of metrics on $X$ (metrizable topology; metrics of $R$ are called compatible).
(d) A metrizable space (as well as its metrizable topology) is compact ${ }^{4}$ if every sequence has a convergent subsequence.

Every subset of $\mathbb{R}$ is a metric space with the metric $\rho(x, y)=|x-y|$. This space is compact if and only if the set is closed and bounded.

The Cantor set $C \subset[0,1]$ may be defined as consisting of all numbers of the form

$$
\varphi(x)=\sum_{k=1}^{\infty} \frac{2 x(k)}{3^{k}}
$$

for $x \in\{0,1\}^{\infty}$, that is $x:\{1,2, \ldots\} \rightarrow\{0,1\}$.
1d2 Exercise. (a) $\varphi:\{0,1\}^{\infty} \rightarrow C$ is a bijection;

[^1](b) if $x, x_{1}, x_{2}, \cdots \in\{0,1\}^{\infty}$ then
$$
\varphi\left(x_{n}\right) \underset{n \rightarrow \infty}{\longrightarrow} \varphi(x) \Longleftrightarrow \forall k\left(x_{n}(k) \underset{n \rightarrow \infty}{ } x(k)\right)
$$

Prove it.
The metric $\rho(x, y)=|\varphi(x)-\varphi(y)|$ is not invariant under permutations of coordinates on $\{0,1\}^{\infty}$, but its equivalence class $R$ is (see $1 \mathrm{~d} 2(\mathrm{~b})$ ). Thus, we have a compact metrizable space $\{0,1\}^{\infty}$, and moreover, the compact metrizable space $\{0,1\}^{S}$ is well-defined for an arbitrary countable set $S$ (irrespective of its enumeration). The space $\{0,1\}^{S}$ may also be thought of as the space of all subsets of $S$.

1d3 Definition. A set $A$ in a metrizable space $X$ is nowhere dense if every nonempty open set contains some nonempty open subset that does not intersect $A$.

Still, $A$ is nowhere dense if and only if $\operatorname{Int}(\mathrm{Cl}(A))=\emptyset$.
1d4 Exercise. (a) Prove that nowhere dense sets are an ideal (on a metrizable space).
(b) On $\mathbb{R}$, prove that they are not a $\sigma$-ideal.

1d5 Exercise. A set $A \subset\{0,1\}^{\infty}$ is nowhere dense if and only if for all $m$ and $t_{1}, \ldots, t_{m} \in\{0,1\}$ there exist $n>m$ and $t_{m+1}, \ldots, t_{n} \in\{0,1\}$ such that all sequences that start with $t_{1}, \ldots, t_{n}$ do not belong to $A$.

Prove it.
1d6 Theorem (Baire). Let $X$ be a compact metrizable space. If $A_{1}, A_{2}, \cdots \subset$ $X$ are nowhere dense then $\operatorname{Int}\left(\cup_{n} A_{n}\right)=\emptyset$.

1d7 Exercise. (a) Prove the theorem.
(b) Find an example of a non-compact metrizable space such that the $\sigma$-ideal of meager sets is not proper.

Thus, the proper $\sigma$-ideal of meager sets is well-defined on every compact metrizable space, in particular, on $\{0,1\}^{\infty}$, and we may speak about generic elements, quasi-everywhere etc. Now, what about null sets? Can we transfer Lebesgue measure from $\mathbb{R}$ to $\{0,1\}^{\infty}$ by $\varphi^{-1}$ ? No, we cannot, since the Cantor set is itself a null set. But on the other hand, endless coin tossing should provide a useful probability measure on $\{0,1\}^{\infty}$; and binary digits can be thought of as endless coin tossing over Lebesgue measure!

We consider the map $\psi:[0,1) \rightarrow\{0,1\}^{\infty}$,

$$
\psi(u)=\left(b_{1}(u), b_{2}(u), \ldots\right),
$$

where $b_{k}(u)$ are the binary digits of $u$, that is,

$$
b_{k}(u) \in\{0,1\}, \quad \sum_{k=1}^{\infty} \frac{b_{k}(u)}{2^{k}}=u, \quad \liminf _{k} b_{k}(u)=0
$$

True, $\psi$ is not a bijection, but do not bother: the countable set $\left\{x: \liminf _{k} x(k)=\right.$ $1\}$ is anyway a null set, and outside it $\psi$ is a bijection,

$$
\psi^{-1}(x)=\sum_{k=1}^{\infty} \frac{x(k)}{2^{k}} .
$$

We transfer Lebesgue measure to $\{0,1\}^{\infty}$ by $\psi$. That is, a set $A \subset\{0,1\}^{\infty}$ is measurable if $\psi^{-1}(A)$ is Lebesgue measurable, and then $\mu(A)$ is equal to the Lebesgue measure of $\psi^{-1}(A)$. This probability measure $\mu$ is sometimes called Lebesgue measure on $\{0,1\}^{\infty}$. ${ }^{1}$ It is invariant under permutations of coordinates on $\{0,1\}^{\infty}$. Thus, we have a probability space $\{0,1\}^{\infty}$, and moreover, the probability space $\{0,1\}^{S}$ is well-defined for an arbitrary countable set $S$ (irrespective of its enumeration). It gives us the proper $\sigma$-ideal of null sets on such space, and we may speak about almost all elements etc.

## 1e "Almost all" versus "quasi all": first examples

1e1 Example. The famous strong law of large numbers (SLLN) states that

$$
\begin{equation*}
\lim _{n} \frac{1}{n} \sum_{k=1}^{n} x(k)=\frac{1}{2} \quad \text { for almost all } x \in\{0,1\}^{\infty} . \tag{a}
\end{equation*}
$$

In contrast,

$$
\begin{equation*}
\liminf _{n} \frac{1}{n} \sum_{k=1}^{n} x(k)=0, \limsup _{n} \frac{1}{n} \sum_{k=1}^{n} x(k)=1 \quad \text { for quasi all } x \in\{0,1\}^{\infty}, \tag{b}
\end{equation*}
$$

as we will see soon.
1e2 Example. Consider sets

$$
A_{n}=\{x: x(1)=x(n+1), x(2)=x(n+2), \ldots, x(n)=x(2 n)\} \subset\{0,1\}^{\infty} .
$$

Clearly, $\mu\left(A_{n}\right)=2^{-n}$, thus $\sum_{n} \mu\left(A_{n}\right)<\infty$; by the first Borel-Cantelli lemma,

$$
\begin{equation*}
\mu\left(\limsup _{n} A_{n}\right)=0 . \tag{a1}
\end{equation*}
$$

[^2]In other words, almost every $x$ belongs to $A_{n}$ only for finitely many $n$. Equivalently, ${ }^{1}$

$$
\begin{equation*}
\sum_{n} \mathbf{1}_{A_{n}}(x)<\infty \quad \text { for almost all } x \in\{0,1\}^{\infty} \tag{a2}
\end{equation*}
$$

( $\mathbf{1}_{A}$ being the indicator of $A$ ). In contrast,

$$
\begin{equation*}
\sum_{n} \mathbf{1}_{A_{n}}(x)=\infty \quad \text { for quasi all } x \in\{0,1\}^{\infty} \tag{b}
\end{equation*}
$$

as we will see soon. That is, quasi every $x$ belongs to $A_{n}$ for infinitely many $n$. (Of course, the infinite set of $n$ depends on $x$.)

1e3 Exercise. Denote by $B_{n}$ the complement of $A_{n}$, and by $C_{n}$ the set $B_{n} \cap B_{n+1} \cap \ldots$ Prove that
(a) $C_{n}$ is closed;
(b) $C_{n}$ is nowhere dense.

Thus, $C=\cup_{n} C_{n}$ is meager, and its complement $\cap_{n}\left(A_{n} \cup A_{n+1} \cup \ldots\right)=$ $\limsup A_{n} A_{n}$ is comeager, which proves 1 e 2 (b).

1e4 Exercise. Now consider sets $A_{n}=\left\{x: x(n)=x(n+1)=\cdots=x\left(n^{2}\right)=\right.$ $0\}$. Prove that
(a) the set $\limsup _{n} A_{n}$ is comeager;
(b) $\lim \inf _{n} \frac{1}{n} \sum_{k=1}^{n} x(k)=0$ for all $x \in \limsup { }_{n} A_{n}$.

A half of $1 \mathrm{e1}(\mathrm{~b})$ is thus proved; the other half is similar.

## 1f Digits of a typical number

We return to the map $\psi:[0,1) \rightarrow\{0,1\}^{\infty}, \psi(u)=\left(b_{1}(u), b_{2}(u), \ldots\right)$ where $b_{k}(u)$ are the binary digits of $u$. Of course, $\psi$ is discontinuous; and nevertheless...

1f1 Exercise. Prove that
(a) If $A \subset\{0,1\}^{\infty}$ is nowhere dense then $\psi^{-1}(A) \subset[0,1)$ is nowhere dense.
(b) If $A \subset\{0,1\}^{\infty}$ is meager then $\psi^{-1}(A) \subset[0,1)$ is meager.
(c) If $A \subset\{0,1\}^{\infty}$ is comeager then $\psi^{-1}(A) \subset[0,1)$ is comeager.

[^3]1f2 Exercise. Let $A \subset\{0,1\}^{\infty}$. Prove or disprove:
(a) If $\psi^{-1}(A)$ is nowhere dense then $A$ is nowhere dense.
(b) If $\psi^{-1}(A)$ is meager then $A$ is meager.

1f3 Remark. A map satisfying the equivalent conditions 1f1(b,c) (but not necessarily (a)) may be called genericity preserving. ${ }^{1}$ Informally, such map transforms a generic element of the first space into a generic element of the second space.

Combining 1f1 with 1 e 1 (b) and 1 e 2 (b) we see that quasi all $u \in[0,1$ ) satisfy

$$
\liminf _{n} \frac{1}{n} \sum_{k=1}^{n} b_{k}(u)=0, \quad \limsup \frac{1}{n} \sum_{k=1}^{n} b_{k}(u)=1
$$

and the relation

$$
b_{1}(u)=b_{n+1}(u), \ldots, b_{n}(u)=b_{2 n}(u)
$$

holds for infinitely many $n$.
All said about $\{0,1\}^{\infty}$ and binary digits generalizes readily to $\{0,1, \ldots, 9\}^{\infty}$ and decimal digits, as well as any other basis. Given comeager sets $A_{p} \subset$ $\{0, \ldots, p-1\}^{\infty}$, we observe for a generic number $u \in[0,1)$ the following property: for every basis $p=2,3, \ldots$ the corresponding digits of $u$ are a sequence that belongs to $A_{p}$.

## Hints to exercises

1a4: $\left[b_{1}, c_{1}\right] \supset\left[b_{2}, c_{2}\right] \supset \ldots$
1d2. if $x(1)=y(1), \ldots, x(n)=y(n)$ then $|\varphi(x)-\varphi(y)| \leq \frac{2}{3^{n+1}}+\frac{2}{3^{n+2}}+\ldots$;
otherwise $|\varphi(x)-\varphi(y)| \geq \frac{2}{3^{n}}-\frac{2}{3^{n+1}}-\frac{2}{3^{n+2}}-\ldots$
1d4; (a) $\left[b_{1}, c_{1}\right] \supset\left[b_{2}, c_{2}\right] \supset\left[b_{3}, c_{3}\right]$; (b) the union can be dense.
1 d 7 ( a$)$ similar to 1 a 4 with balls rather than intervals; (b) try a dense countable set.
1e3: (b) use 1 d 5 .
1e4, (b) try $n \in\{1,4,9,16, \ldots\}$
1f1: (a) by 1 d 5 every binary interval $\left[\frac{k}{2^{n}}, \frac{k+1}{2^{n}}\right.$ ) contains a binary subinterval such that... (b), (c) follow from (a).
1f2: consider $\{0,1\}^{\infty} \backslash \psi([0,1))$.

[^4]
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[^0]:    ${ }^{1}$ This notion of set theory is different from (but related to) ideals in ring theory, order theory etc.
    ${ }^{2}$ Or of the first category.

[^1]:    ${ }^{1}$ Or residual.
    ${ }^{2}$ However, a Cauchy sequence in $\left(X, \rho_{2}\right)$ need not be Cauchy in $\left(X, \rho_{1}\right)$.
    ${ }^{3}$ Equivalently, and usually, a metrizable space is defined as a special case of a topological space; but here we do not need the notion of general (not just metrizable) topological space.
    ${ }^{4}$ Equivalently (for metrizable spaces), and usually, a compact space is defined by the Heine-Borel property: every open cover has a finite subcover.

[^2]:    ${ }^{1}$ It is in fact the Haar measure on the topological group $\left(\mathbb{Z}_{2}\right)^{\infty}$.

[^3]:    ${ }^{1}$ The sum of the indicators is integrable, therefore, finite almost everywhere. (This is the proof of the first Borel-Cantelli lemma.)

[^4]:    ${ }^{1}$ According to Melleray and Tsankov, a continuous map with this property is called category-preserving; see arXiv:1201.4447, Def. 2.7.

