9 More on differentiation

\section{Finite Taylor expansion}

An infinitely differentiable function $\mathbb{R} \to \mathbb{R}$ need not be analytic. It has a formal Taylor expansion, but maybe of zero radius of convergence, or maybe converging to a different function. An example:

$$f(x) = e^{-1/x} \text{ for } x > 0, \quad f(x) = 0 \text{ for } x \leq 0.$$

\textbf{9a1 Theorem.} \footnote{Exercise 10.2.9 in book: B. Thomson, J. Bruckner, A. Bruckner, “Real analysis”, second edition, 2008.} If an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ is not a polynomial then there exists $x \in \mathbb{R}$ such that $f^{(n)}(x)$ is irrational for all $n$.

Thus, $\exists x \ \forall n \ f^{(n)}(x) \neq 0$.

The set of rational numbers may be replaced with any other countable set.

We’ll prove the theorem via iterated Baire category theorem.

\textbf{9a2 Lemma.} If $f$ is a polynomial on $[a, b]$ and $\forall n \ f(b + \varepsilon_n) = f(b)$ for some $\varepsilon_n \to 0^+$ then $f$ is constant on $[a, b]$.

\footnote{The theorem: Theorem: Let $f(x)$ be $C^\infty$ on $(c, d)$ such that for every point $x$ in the interval there exists an integer $N_x$ for which $f^{(N_x)}(x) = 0$; then $f(x)$ is a polynomial.

is due to two Catalan mathematicians:


The proof can also be found in the book (p. 53):


I will never forget it because in an "Exercise" of the "Opposition" to became "Full Professor" I was posed the following problem:

What are the real functions indefinitely differentiable on an interval such that a derivative vanish at each point?  

Juan Arias de Reyna; see \textcolor{blue}{Question 34059 on Mathoverflow}.}
Proof. We have \( f^{(n)}(b) = 0 \) for \( n = 1, 2, \ldots \) since otherwise \( f(b + \varepsilon) = f(b) + c\varepsilon^k + o(\varepsilon^k) \) for some \( k \geq 1 \) and \( c \neq 0 \).

The same holds for \( f(a - \varepsilon_n) \), of course.

Assume that \( f \) is a counterexample to Theorem 9a1.

Consider a (maybe empty) set \( P_f \) of all maximal nondegenerate intervals \( I \subset \mathbb{R} \) such that \( f \) is a polynomial on \( I \). Note that intervals of \( P_f \) are closed and pairwise disjoint.

9a3 Lemma. The open set

\[ G_f = \bigcup_{I \in P_f} \text{Int} I \]

is dense (in \( \mathbb{R} \)).

Proof. Closed sets \( F_{n,r} = \{ x : f^{(n)}(x) = r \} \) for \( r \in \mathbb{Q} \) and \( n = 0, 1, 2, \ldots \) cover \( \mathbb{R} \). By (5b7), \( \bigcup_{n,r} \text{Int} F_{n,r} \) is dense. Clearly, \( f \) is a polynomial on each interval contained in this dense open set.

It follows that \( P_f \), treated as a totally (in other words, linearly) ordered set, is dense (that is, if \( I_1, I_2 \in P_f \), \( I_1 < I_2 \) then \( \exists I \in P_f \) \( I_1 < I < I_2 \)). It may contain minimal and/or maximal element (unbounded intervals), but the rest of \( P_f \), being an unbounded dense countable totally ordered set, is order isomorphic to \( \mathbb{Q} \cap (0, 1) \) (the proof is similar to the proof of Lemma 2d4; so-called back-and-forth method).

Now we want to contract each interval of \( P_f \) into a point. (We could consider a topological quotient space...)

We take an order isomorphism \( \varphi : P_f \to \mathbb{Q} \) between \( P_f \) and one of \( \mathbb{Q} \cap (0, 1), \mathbb{Q} \cap [0, 1], \mathbb{Q} \cap (0, 1], \mathbb{Q} \cap [0, 1], \mathbb{Q} \), and construct an increasing \( \psi : \mathbb{R} \to [0, 1] \) such that \( \psi(x) = \varphi(I) \) whenever \( x \in I \). Clearly, such \( \psi \) exists and is unique. It is continuous. The image \( \psi(\mathbb{R}) \) is one of \( (0, 1), [0, 1), (0, 1], [0, 1] \). In every case \( \psi(\mathbb{R}) \) is completely metrizable. Note that \( \psi^{-1}(\mathbb{Q}) = \bigcup_{I \in P_f} I \), and \( \psi \) is one-to-one on \( \mathbb{R} \setminus \bigcup_{I \in P_f} I \).

We define \( E_{n,r} \subset \psi(\mathbb{R}) \) for \( r \in \mathbb{Q} \) and \( n = 0, 1, 2, \ldots \) as follows:

\[ E_{n,r} = \{ x : \psi^{-1}(x) \subset F_{n,r} \} \]

9a4 Lemma. Each \( E_{n,r} \) is closed in \( \psi(\mathbb{R}) \).
Proof. Given $x_1 > x_2 > \ldots, x_k \in E_{n,r}$, $x_k \downarrow x$ in $\psi(\mathbb{R})$, we take $t_k \in \psi^{-1}(x_k) \subset F_{n,r}$ and note that $t_1 > t_2 > \ldots, t_k \downarrow t \in \psi^{-1}(x)$, $f^{(n)}(t_k) = r$ for all $k$, thus $f^{(n)}(t) = r$, that is, $t \in F_{n,r}$.

If $x$ is irrational then $x \in E_{n,r}$ since $\psi^{-1}(x) = \{t\}$.

If $x$ is rational then $\psi^{-1}(x) = [s,t]$, and $f^{(n)}(\cdot) = r$ on $[s,t]$ by Lemma 9a2 (applied to $f^{(n)}$).

The case $x_k \uparrow x$ is similar. \hfill \Box

9a5 Exercise. Each $E_{n,r}$ is nowhere dense in $\psi(\mathbb{R})$.

Prove it.

Now Theorem 9a1 follows from the Baire category theorem (applied the second time).

9a6 Corollary. If an infinitely differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ has only finitely many non-zero partial derivatives at every point then $f$ is a polynomial.

Proof. Let $d = 2$ (the general case is similar).

By Theorem 9a1 for every $x \in \mathbb{R}$ the function $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is a polynomial; similarly, each $f(\cdot, y)$ is a polynomial. Introducing the set $A_n$ of all $x \in \mathbb{R}$ such that $f(x,\cdot)$ is a polynomial of degree $\leq n$ we have $A_n \uparrow \mathbb{R}$, therefore $A_n$ is infinite (moreover, uncountable) for $n$ large enough. The same holds for $f(\cdot, y)$ and $B_n$.

For $x \in A_n$ the coefficients $a_0(x), \ldots, a_n(x)$ of the polynomial $f(x,\cdot)$ are linear functions of $f(x,y_0), \ldots, f(x,y_n)$ provided that $y_0, \ldots, y_n \in B_n$ are pairwise different. Therefore these coefficients are polynomials (in $x$), of degree $\leq n$.

We get a polynomial $P : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,y) = P(x,y)$ for $x \in A_n$, $y \in \mathbb{R}$. For every $y \in \mathbb{R}$ two polynomials $f(\cdot, y)$ and $P(\cdot, y)$ coincide on the infinite set $A_n$, therefore they coincide on the whole $\mathbb{R}$. \hfill \Box

A very similar (and a bit simpler) argument gives an interesting purely topological result.

9a7 Theorem. 1 If $[0,1]$ is the disjoint union of countably many closed sets then one of the sets is the whole $[0,1]$ (and others are empty).

Proof. (sketch). Assume the contrary: $[0,1] = \cup_n F_n$, $F_n \neq \emptyset$ are closed. (Finitely many sets cannot do because of connectedness.) Then $\cup_n \text{Int } F_n$ is dense in $[0,1]$.

Consider a (maybe empty) set $P$ of all maximal nondegenerate intervals $I \subset [0, 1]$ such that $\exists n \ I \subset F_n$. Note that intervals of $P_f$ are closed and pairwise disjoint. The open set $G = \bigcup_{I \in P} \text{Int} I$ is dense in $[0, 1]$, since it contains $\bigcup_n \text{Int} F_n$.

It follows that $P$, treated as a totally ordered set, is dense. Thus, the set $C = [0, 1] \setminus G$ is perfect, with no interior (and in fact, homeomorphic to the Cantor set).

As before, each $F_n \cap C$ is nowhere dense in $C$. (Hint: if an endpoint of an interval $I \in P$ belongs to $F_n \cap C$ then $I \subset F_n$.)

It remains to apply the Baire category theorem (in the second time).

9a8 Corollary. If the cube $[0, 1]^d$ is the disjoint union of countably many closed sets then one of the sets is the whole $[0, 1]^d$ (and others are empty).

Proof. Let $d = 2$ (the general case is similar).

Assume the contrary: $[0, 1]^2 = \bigcup_n F_n$, $F_n$ are closed.

By Theorem 9a7, each $\{x\} \times [0, 1]$ is contained in a single $F_n$. The same holds for each $[0, 1] \times \{y\}$. Thus, it is a single $n$. \qed

I wonder, is it true for an arbitrary continuum (that is, a compact connected metrizable space)?

9b Continuous and nowhere differentiable

9b1 Theorem. There exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that for every $x \in (0, 1)$, $f$ is not differentiable at $x$.

We consider the complete metric space $C[0, 1]$ of all continuous $f : [0, 1] \to \mathbb{R}$ (separable, in fact). We define continuous functions $\varphi_n : C[0, 1] \to \mathbb{R}$ by

$$\varphi_n(f) = \min_{k=1,\ldots,n} \left| f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right|.$$ 

Clearly, $\varphi_n \to 0$ pointwise. What about the rate of convergence? We take arbitrary $\varepsilon_n \to 0$ and examine $\frac{1}{\varepsilon_n} \varphi_n$.

9b2 Exercise. $\limsup_{n \to \infty, g \to f} \frac{1}{\varepsilon_n} \varphi_n(g) = \infty$ for all $f \in C[0, 1]$.

Prove it.

By Prop. 5b9,

$$\limsup_{n \to \infty} \frac{1}{\varepsilon_n} \varphi_n(f) = \infty$$
for quasi all $f \in C[0,1]$.

On the other hand, if $f$ is differentiable at $x_0 \in (0,1)$ then $f(x) - f(x_0) = O(|x - x_0|)$, that is,

$$\exists C \forall x \in [0,1] \ |f(x) - f(x_0)| \leq C|x - x_0|.$$  

Taking $k$ such that $\frac{k-1}{n}, \frac{k}{n} \in [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]$ we get $|f(\frac{k}{n}) - f(\frac{k-1}{n})| \leq \frac{2C}{n}$. Thus,

$$\forall n \ \varphi_n(f) \leq \frac{2C}{n}.$$  

By (9b3), such $f$ are a meager set, which proves Theorem 9b1.

9b4 Exercise. There exists a continuous function $f : [0,1] \to \mathbb{R}$ such that for every $x \in (0,1)$

$$\limsup_{y \to x^-} |f(y) - f(x)| \log \log \log \frac{1}{|y - x|} = \infty,$$

$$\limsup_{y \to x^+} |f(y) - f(x)| \log \log \log \frac{1}{|y - x|} = \infty.$$  

Prove it.

However, $|f(y) - f(x)|$ cannot be replaced with $f(y) - f(x)$. If $C > f(1) - f(0)$ then there exists $x \in (0,1)$ such that

$$\limsup_{y \to x^+} \frac{f(y) - f(x)}{y - x} \leq C$$

and moreover, $\sup_{y \in [x,1]} \frac{f(y) - f(x)}{y - x} \leq C$. Proof (sketch): choose $b \in (f(1) - C, f(0))$ and take the greatest $x$ such that $f(x) \geq Cx + b$.

9c Differentiable and nowhere monotone

9c1 Theorem. There exists a differentiable function $f : [0,1] \to \mathbb{R}$ such that for every $(a,b) \subset [0,1]$, $f$ is not monotone on $(a,b)$.

9c2 Lemma. There exists a strictly increasing differentiable function $f : [0,1] \to \mathbb{R}$ such that $f'(\cdot) = 0$ on a dense set.

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1C.E. Weil (1976) “On nowhere monotone functions”, Proc. AMS 56, 388–389. (Yes, two pages!) See also Sect. 10.7.2 in “Real analysis”.

Proof. We’ll construct a continuous strictly increasing surjective \( g : [0, 1] \to [0, 1] \) such that the inverse function \( f = g^{-1} : [0, 1] \to [0, 1] \) has the needed properties. It is sufficient to ensure that (finite or infinite) derivative \( g'(\cdot) \in (0, \infty] \) exists everywhere (and never vanishes), and is infinite on a dense set.

A function 
\[
\alpha(x) = x^{1/3}
\]
is strictly increasing (on \( \mathbb{R} \)), with \( \alpha'(0) = +\infty \) and \( \alpha'(x) \in (0, \infty) \) for \( x \neq 0 \). We introduce 
\[
A = \max_{h \neq 0} \frac{\alpha(1 + h) - \alpha(1)}{h\alpha'(1)} \in (0, \infty)
\]
(this continuous function vanishes on \( \pm \infty \); in fact, \( A = 4 \)) and note that 
\[
(9c3) \quad \frac{\alpha(x + h) - \alpha(x)}{h\alpha'(x)} \leq A
\]
for all \( h \neq 0 \) and \( x \) (since for \( x \neq 0 \) it equals \( x^{1/3}(\alpha(1 + \frac{h}{x}) - \alpha(1)) = \frac{\alpha(1 + \frac{h}{x}) - \alpha(1)}{\frac{2}{3} \alpha'(1)} \)).

Similarly to Sect. 5a we choose some \( a_n, c_n \in (0, 1) \) such that \( a_n \) are pairwise distinct, dense, and \( \sum c_n < \infty \). The series 
\[
\beta(x) = \sum_{n=1}^{\infty} c_n \alpha(x - a_n)
\]
converges uniformly on \( [0, 1] \) (since \( |\alpha(\cdot)| \leq 1 \) and \( \sum c_n < \infty \)). The series \( \sum_{n=1}^{\infty} c_n \alpha'(x - a_n) \) converges (to a finite sum) for some \( x \) and diverges (to \( +\infty \)) for other \( x \) (in particular, for \( x \in \{a_1, a_2, \ldots\} \)). We consider \( \beta_n(x) = \sum_{k=1}^{n} c_k \alpha(x - a_k) \) and \( \gamma_n(x) = \beta(x) - \beta_n(x) = \sum_{k=n+1}^{\infty} c_k \alpha(x - a_k) \). By \( (9c3) \),
\[
0 \leq \frac{\gamma_n(x + h) - \gamma_n(x)}{h} \leq A \sum_{k=n+1}^{\infty} c_k \alpha'(x - a_k)
\]
for all \( h \neq 0 \) and \( x \). Thus (similarly to Sect. 5a)
\[
\liminf_{h \to 0} \frac{\beta(x + h) - \beta(x)}{h} \leq \limsup_{h \to 0} \frac{\beta(x + h) - \beta(x)}{h} \leq \beta'(x) + A \sum_{k=n+1}^{\infty} c_k \alpha'(x - a_k)
\]
therefore
\[
\beta'(x) = \sum_{n=1}^{\infty} c_n \alpha'(x - a_n) \in (0, \infty]
\]
for all $x$.

It remains to take $g(x) = \frac{\beta(x) - \beta(0)}{\beta(1) - \beta(0)}$. □

Do not think that $\beta'(\cdot) = \infty$ only on the countable set $\{a_1, a_2, \ldots\}$. Amazingly, $f'(x) = 0$ for quasi all $x \in [0, 1]$ (and therefore $\beta'(x) = \infty$ for quasi all $x \in [0, 1]$). Here is why. By 5b2 and 5c5, $f'$ is of Baire class 1, thus, $\{x : f'(x) \neq 0\}$ is an $F_\sigma$ set, and $\{x : f'(x) = 0\}$ is a $G_\delta$ set;¹ being dense it must be comeager (as noted before 5c2).

We introduce the space $D$ of all bounded derivatives on $(0, 1)$; that is, of $F'$ for all differentiable $F : (0, 1) \to \mathbb{R}$ such that $F'$ is bounded. We endow $D$ with the metric

$$\rho(f, g) = \sup_{x \in (0, 1)} |f(x) - g(x)|.$$

9c4 Exercise. (a) $D$ is a complete metric space. (b) $D$ is not separable. Prove it.

We consider a subspace $D_0$ of all $f \in D$ such that $f(x) = 0$ for quasi all $x$. As noted above, this happens if and only if $f(\cdot) = 0$ on a dense set. By 9c2 $D_0$ is not $\{0\}$; moreover, for every $x \in (0, 1)$ there exists $f \in D_0$ such that $f(x) \neq 0$ (try $f(ax + b)$).

9c5 Exercise. (a) $D_0$ is a vector space; that is, a linear combination of two functions of $D_0$ is a function of $D_0$. (b) $D_0$ is a closed subset of $D$. Prove it.

Given $(a, b) \subset (0, 1)$, the set

$$E_{a,b} = \{ f \in D_0 : \forall x \in (a, b) \ f(x) \geq 0 \}$$

is closed (evidently). Given $f \in E_{a,b}$, we take $x \in (a, b)$ such that $f(x) = 0$ and $g \in D_0$ such that $g(x) > 0$. Then $f - \varepsilon g \in D_0$ and $f - \varepsilon g \notin E_{a,b}$ for all $\varepsilon > 0$; thus, $f$ is not an interior point of $E_{a,b}$. We see that $E_{a,b}$ is nowhere dense. Similarly, $-E_{a,b} = \{ f \in D_0 : \forall x \in (a, b) \ f(x) \leq 0 \}$ is nowhere dense. It follows that quasi all functions of $D_0$ change the sign on every interval. Theorem 9c1 is thus proved.

¹A straightforward representation

$$f'(x) = 0 \iff \forall \varepsilon \exists \delta \forall h \ (|h| < \delta \implies |f(x + h) - f(x)| \leq \varepsilon|h|)$$

gives only $F_\sigma$. Taking into account that $f$ is differentiable we have another representation

$$f'(x) = 0 \iff \forall \varepsilon \exists h \ (|h| < \varepsilon \land |f(x + h) - f(x)| < \varepsilon|h|)$$

that gives $G_\delta$. 
Hints to exercises

9a5: otherwise, some interval of $P_f$ is not maximal.
9b2: $g(\frac{k}{n}) = f(\frac{k}{n}) \pm \sqrt{\varepsilon_n}$.
9b4: similar to 9b1.
9c4: (a) $D$ is closed in the space of all bounded functions; (b) try shifts of a discontinuous derivative.