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12a A third topology on sequences

Two metrizable topologies on $[0, 1]^\infty$ are mentioned in Sect. 4d. The first one is the compact product topology. The second one is the nonseparable product topology of $([0, 1], d)^\infty$. Now we introduce a third one, the nonseparable topology of uniform convergence, corresponding to a complete metric

\[ (12a1) \quad \rho(x, y) = \sup_k |x(k) - y(k)| \quad \text{for } x, y \in [0, 1]^\infty. \]

In the first topology, the set $x(1, 2, \ldots) = \{ x(n) : n = 1, 2, \ldots \} \subset [0, 1]$ for a typical sequence $x$ is dense in $[0, 1]$, and each point is of multiplicity 1. In the second topology, the set $x(1, 2, \ldots)$ typically contains all rational numbers (therefore, is dense), and each point is of infinite multiplicity. In the third topology, as we’ll see soon, the set $x(1, 2, \ldots)$ typically is nowhere dense, and each point is of multiplicity 1.

Below, $[0, 1]^\infty$ is endowed with the metric (12a1).

12a2 Lemma. \( \forall t \in [0, 1] \; \forall^* x \in [0, 1]^\infty \; t \notin \text{Cl}(x(1, 2, \ldots)). \)

Proof. The function $x \mapsto \text{dist}(t, x(1, 2, \ldots))$ on $[0, 1]^\infty$ is continuous (moreover, Lip(1)), thus, $\{ x : \text{dist}(t, x(1, 2, \ldots)) > 0 \}$ is open. It is dense; indeed, $\forall x \; \forall \varepsilon \; \exists y \; \left( \rho(x, y) \leq \varepsilon \land \text{dist}(t, y(1, 2, \ldots)) \geq \varepsilon \right).$  

\ \n
It follows (via the Baire category theorem) that $\text{Cl}(x(1, 2, \ldots))$ typically misses all rational numbers, and therefore is nowhere dense.

On the other hand...
12a3 Exercise. Prove that \( \forall^* x \in [0, 1]^\infty \ A \cap \text{Cl}(x(1, 2, \ldots)) = \emptyset \)
(a) whenever \( A \) is nowhere dense;
(b) whenever \( A \) is meager.

12a4 Corollary. There exists a null set \( A \subset [0, 1] \) such that \( \forall^* x \in [0, 1]^\infty \ Cl(x(1, 2, \ldots)) \subset A \). (Proof: just take a comeager null set.)

Given a nonempty \( A \subset \{1, 2, \ldots\} \), we consider \( x(A) = \{x(n) : n \in A\} \).

12a5 Lemma. If \( A, B \subset \{1, 2, \ldots\} \) are disjoint then typically \( \text{Cl}(x(A)) \) and \( \text{Cl}(x(B)) \) are disjoint.

Proof. The function \( x \mapsto \text{dist}(x(A), x(B)) \) is continuous (moreover, \( \text{Lip}(2) \)), and \( > 0 \) on a dense open set, since \( \forall x \forall \varepsilon \exists y \left( \rho(x, y) \leq \varepsilon \wedge \text{dist}(y(A), y(B)) \geq \varepsilon \right) \); just take \( y(A) \subset \{0, 2\varepsilon, 4\varepsilon, \ldots\} \) and \( y(B) \subset \{\varepsilon, 3\varepsilon, 5\varepsilon, \ldots\} \). \( \square \)

Multiplicity 1 is thus ensured. Moreover, taking \( A = \{n\} \) and \( B = \{1, 2, \ldots\} \setminus \{n\} \) we see that, typically, each \( x(n) \) is an isolated point of \( x(1, 2, \ldots) \).

On the other hand, \( \forall x \exists A, B \ (A \cap B = \emptyset, \text{dist}(x(A), x(B)) = 0) \) (since \( x(n_k) \to t \) for some \( (n_k)_k \) and \( t \)).

12b Typical set of accumulation points

Consider now the space \( l_\infty(\to \mathbb{R}^n) \) of all bounded sequences \( x = (x(1), x(2), \ldots) \) of points of \( \mathbb{R}^n \), with the metric
\[
\rho(x, y) = \sup_n |x(n) - y(n)|.
\]

This is a nonseparable complete metric (moreover, Banach) space.

For each \( x \in l_\infty(\to \mathbb{R}^n) \) we consider the nonempty compact set of accumulation points
\[
\text{Acc}(x) = \{a : \forall \varepsilon \forall n \exists k \ |x(n + k) - a| \leq \varepsilon \} \in K(\mathbb{R}^n).
\]

12b1 Theorem. For quasi all \( x \in l_\infty(\to \mathbb{R}^n) \) the set \( K = \text{Acc}(x) \) is a nowhere dense perfect null set satisfying\(^1\)
\[
\underline{\dim}_{M}(K) = 0, \quad \overline{\dim}_{M}(K) = n.
\]

No, we do not need to prove this from scratch. Fortunately we can use results of Sect. 10.

\(^1\)It is also homeomorphic to the Cantor set, as we'll see in 12d.
12b2 Exercise. Let $X,Y$ be metrizable spaces and $f : X \to Y$ be open (it means, the image of every open set is an open set) and continuous. Then the inverse image of a meager set is meager, and the inverse image of a comeager set is comeager.\footnote{Kechris, Sect. 8K, Exer. (8.45).}

Prove it.

According to Remark 1f3, such $f$ may be called genericity preserving (category preserving).

Theorem 12b1 now follows from Theorem 10c1 (and 10c2, 10c5), 12b2 and Prop. 12b3 below.

12b3 Proposition. The map

$$l_\infty(\to \mathbb{R}^n) \ni x \mapsto \text{Acc}(x) \in K(\mathbb{R}^n)$$

is continuous and open.

Proof. First, continuity. If $\rho(x,y) \leq \varepsilon$ and $a \in \text{Acc}(x)$ then $x_{n_k} \to a$ for some $(n_k)_k$, and $y_{n_k} \to b$ for some $(k_i)_i$ and $b$. We have $\rho(a,b) \leq \varepsilon$ and $b \in \text{Acc}(y)$, therefore $\text{Acc}(x) \subset (\text{Acc}(y))_{+\varepsilon}$. Similarly, $\text{Acc}(y) \subset (\text{Acc}(x))_{+\varepsilon}$.

Thus, the map is continuous (and moreover, Lip(1)).

Second, openness. Let $K_1 = \text{Acc}(x)$ and $d_H(K_1, K_2) \leq \varepsilon$; we have to find $y$ close to $x$ such that $K_2 = \text{Acc}(y)$. We choose $z(1), z(2), \ldots \in K_2$ such that $K_2 = \text{Acc}(z)$. We take the first $n_1$ such that $|x(n_1) - z(1)| \leq \varepsilon$ and let $y(n_1) = z(1)$. Then we take the first $n_2 > n_1$ such that $|x(n_2) - z(2)| \leq \varepsilon$ and let $y(n_2) = z(2)$. And so on; $y(n_k) \in K_2$, $|y(n_k) - x(n_k)| \leq \varepsilon$ and $K_2 = \text{Acc}((y(n_k)))_{k}$.

Finally, for every $n \notin \{n_1, n_2, \ldots\}$ we take the first $i$ such that $|z(i) - x(n)| \leq 2\varepsilon$ and let $y(n) = z(i)$, if such $i$ exists; otherwise $y(n) = x(n)$, but this happens only finitely many times, since $\text{dist}(x_n, K_1) \to 0$. We get $\rho(x,y) \leq 2\varepsilon$ and $\text{Acc}(y) = K_2$.

\[\square\]

12b4 Exercise. Let $X,Y$ and $f$ be as in 12b2; assume in addition that $f(X)$ is dense in $Y$. Then for every $A \subset Y$, $f^{-1}(A)$ is nowhere dense if and only if $A$ is nowhere dense.

Prove it.

12b5 Remark. Still, it can happen that $f^{-1}(A)$ is meager but $A$ is not. An example: the projection $\mathbb{R} \times \mathbb{Q} \to \mathbb{R}$.

However, if a meager $f^{-1}(A)$ is of the form $\bigcup_n f^{-1}(A_n)$ with all $f^{-1}(A_n)$ nowhere dense, then $A$ is meager.
12c  Typical measurable function

We turn to the space $L_\infty(\to \mathbb{R}^n)$ of all equivalence classes of Lebesgue measurable functions $f : [0, 1] \to \mathbb{R}^n$, bounded (up to null sets), with the metric

$$
\rho(f, g) = \text{ess sup} |f - g| = \min\{\varepsilon : |f - g| \leq \varepsilon \text{ a.e.}\}.
$$

This is also a nonseparable complete metric (moreover, Banach) space. For each $f \in L_\infty(\to \mathbb{R}^n)$ we consider the nonempty compact set (the support)

$$
\text{Supp}(f) = \{a : \forall \varepsilon \in \mathbb{R}, m(f^{-1}\{a\} + \varepsilon)) > 0\}.
$$

12c1 Exercise. $f(t) \in \text{Supp}(f)$ for almost all $t$.

Prove it.

12c2 Proposition. The map

$$
L_\infty(\to \mathbb{R}^n) \ni f \mapsto \text{Supp}(f) \in K(\mathbb{R}^n)
$$

is continuous and open.

Proof. First, continuity. If $\rho(f, g) \leq \varepsilon$ and $a \in \text{Supp}(f)$ then $m(g^{-1}\{a - \varepsilon - \delta, a + \varepsilon + \delta\}) > 0$ for all $\delta$, therefore $[a - \varepsilon, a + \varepsilon] \cap \text{Supp}(g) \neq \emptyset$; thus, $\text{Supp}(f) \subset (\text{Supp}(g))_{+\varepsilon}$. Similarly, $\text{Supp}(g) \subset (\text{Supp}(f))_{+\varepsilon}$. Thus, the map is continuous (and moreover, Lip(1)).

Second, openness. Let $K_1 = \text{Supp}(f)$ and $d_H(K_1, K_2) \leq \varepsilon$; we have to find $g$ close to $f$ such that $K_2 = \text{Supp}(g)$. We choose $z(1), z(2), \ldots \in K_2$ such that $K_2 = \text{Cl}(z(1, 2, \ldots))$. We seek $g : [0, 1] \to \{z(1), z(2), \ldots\}$. We consider measurable sets $A_n = f^{-1}([n\varepsilon, n\varepsilon + \varepsilon))$ and for each $n$ such that $m(A_n) > 0$ we take disjoint measurable subsets $A_{n,1}, A_{n,2}, \ldots \subset A_n$ of positive measure.

For every pair $n, k$ satisfying $|z(k) - (n + 0.5)\varepsilon| \leq 2\varepsilon$ we let

$$
g(t) = z(k) \text{ for all } t \in A_{n,k}.
$$

At least one such $n$ exists for every $k$, thus all $z(k)$ belong to $\text{Supp}(g)$. Also, $f(t) \in [n\varepsilon, n\varepsilon + \varepsilon)$, thus $|g(t) - f(t)| \leq 3\varepsilon$.

Finally, at every other point $t$ we let $g(t) = z(i)$ for the first $i$ such that $|f(t) - z(i)| \leq 2\varepsilon$. We get $\rho(f, g) \leq 3\varepsilon$ and $\text{Supp}(g) = K_2$.

Similarly to [12b1] we get:

12c3 Theorem. For quasi all $f \in L_\infty(\to \mathbb{R}^n)$ the set $K = \text{Supp}(f)$ is a nowhere dense perfect null set satisfying

$$
\dim_M(K) = 0, \quad \overline{\dim}_M(K) = n.
$$

It is also homeomorphic to the Cantor set, as we'll see in 12d.
12c4 Exercise. If \( A \subset \mathbb{R}^n \) is meager then \( \forall^* K \in \mathbf{K}(\mathbb{R}^n) \ A \cap K = \emptyset \).
Prove it.

12c5 Corollary. There exists a null set \( A \subset \mathbb{R}^n \) such that \( \forall^* f \in L_\infty(\to \mathbb{R}^n) \ \text{Supp}(f) \subset A \). (Proof: just take a comeager null set.)

12c6 Exercise. If \( A, B \subset [0,1] \) are disjoint measurable sets then typically \( \text{Supp}(f|_A) \) and \( \text{Supp}(f|_B) \) are disjoint.
Prove it.

12c7 Proposition. A typical \( f \in L_\infty(\to \mathbb{R}^n) \) is one-to-one (that is, the equivalence class contains some one-to-one function).

Proof. We correct \( f \) on a null set getting \( f(t) \in \text{Supp}(f|_{[k2^{-n},(k+1)2^{-n})}) \) whenever \( t \in [k2^{-n},(k+1)2^{-n}) \). By 12c6 \( f \) must be one-to-one.

Note that the dimension of [0, 1] is irrelevant! A typical \( f \in L_\infty([0,1]^m \to \mathbb{R}^n) \) is one-to-one also when \( m > n \).

Moreover, Lebesgue measure on [0, 1] was used only via the \( \sigma \)-algebra of measurable sets and the \( \sigma \)-ideal of null sets. All said generalizes readily to a measurable space with a given \( \sigma \)-ideal (under mild conditions). A measure will be more relevant in Sect. 12e.

12d Typical continuous function

A “good” function \( \mathbb{R}^n \to \mathbb{R} \) behaves locally like a (nonconstant) linear function; in particular, for every Lebesgue measurable set \( A \subset \mathbb{R}^n \) of positive measure,

\[
\begin{align*}
  f|_A & \text{ is not one-to-one}, \\
  f(A) & \text{ is not a null set}.
\end{align*}
\]

Let us try to imagine quite the opposite:

\[
\begin{align*}
  f : [0,1]^n \to \mathbb{R} \text{ is continuous,} \\
  \text{and for some set } A \subset [0,1]^n \text{ of full measure,} \\
  (12d1) \quad f|_A & \text{ is one-to-one,} \\
  f(A) & \text{ is a meager set of Hausdorff dimension 0}.
\end{align*}
\]

The latter means that for every \( \varepsilon > 0 \) it is possible to cover \( f(A) \) with countably many balls \( \{x_k\}_{k \in \mathbb{Z}} \) such that \( \sum_k r_k^\varepsilon \leq \varepsilon \).

\(^1\)A set of Hausdorff dimension 0 need not be meager. Moreover, it can be comeager! An example: Liouville numbers. (See Oxtoby Sect. 2 or A. Bruckner, J. Bruckner, B. Thomson “Real analysis” (second edition, 2008), Problem 10:8.3.) On the other hand, \( \dim_M(B) < n \) implies that \( B \) is meager (and moreover, nowhere dense), just because \( \dim_M(B) = \dim_M(Cl(B)) \).
What do you think about existence of such \( f \)?

A measurable (rather that continuous) function with similar properties can be constructed using well-known tricks with digits; say (for \( n = 2 \))

\[
f(x, y) = (0.γ_1γ_2 \ldots)_3 \quad \text{whenever } x = (0.β_1β_2 \ldots)_2, \ y = (0.β'_1β'_2 \ldots)_2,
\]

\[
γ_1 = 2β_1, \ γ_2 = 2β'_1, \ γ_3 = 2β_2, \ γ_4 = 2β'_2, \ γ_5 = 2β_3, \ldots
\]

This \( f \) is Riemann integrable (recall 5e) but has a dense set of discontinuity points. It is hard to believe that such a function can be continuous. But...

**12d2 Theorem.** 2 Every continuous function \([0, 1]^n \to \mathbb{R}\) is the sum of two functions satisfying \([12d1]\).

Have you any idea, why? Wait a little...

Given a metrizable space \( X \), we consider the space \( C_b(X \to \mathbb{R}^n) \) of all bounded continuous functions \( f : X \to \mathbb{R}^n \) with the metric

\[
ρ(f, g) = \sup_{x \in X} |f(x) - g(x)|.
\]

**12d3 Proposition.** Let \( X \) be a metrizable space and \( Y \subset X \) a closed set. Then the map

\[
C_b(X \to \mathbb{R}^n) \ni f \mapsto f|_Y \in C_b(Y \to \mathbb{R}^n)
\]

is continuous and open.

**Proof.** Continuity is evident. Openness follows easily from the Tietze-[Urysohn-Brouwer-Lebesgue] extension theorem: for every \( g \in C_b(Y \to \mathbb{R}) \) there exists \( f \in C_b(X \to \mathbb{R}) \) such that \( f|_Y = g \) and \( \sup_X |f| = \sup_Y |g| \).

It follows by \([12b2]\) that \( f|_Y \) is typical if \( f \) is typical. Thus, being interested in “very disconnected” subsets, we turn to “very disconnected” spaces.

The set \( \text{Clopen}(X) \) of all clopen (that is, open-and-closed) sets in \( X \) is an algebra of sets. If \( \text{Clopen}(X) = \{\emptyset, X\} \), \( X \) is called connected. If \( \text{Clopen}(X) \) is a basis (of the topology), \( X \) is called zero-dimensional.\(^3\) Also, \( X \) is called perfect, if it has no isolated points.

**12d4 Lemma.** A typical set of \( \mathbf{K}(\mathbb{R}^n) \) is zero-dimensional.

\(^1\)Hausdorff dimension less than 1 (rather than 0).

\(^2\)See also Bruckner, Bruckner, Thomson Exer. 10:7.9.

\(^3\)If \( X \) is zero-dimensional then clearly \( x \) is totally disconnected, that is, contains no connected subset of more than one point. The converse holds (for compact \( X \); and fails for some subsets of \( \mathbb{R}^2 \)), but we do not need it.
Proof. Given $\varepsilon > 0$, consider all $K$ such that every coordinate of every point of $K$ belongs to $\mathbb{R} \setminus \varepsilon\mathbb{Z}$. They are a dense open set in $K(\mathbb{R}^n)$, and every point has a clopen $\varepsilon\sqrt{n}$-small neighborhood. Quasi all $K$ satisfy this condition for all $\varepsilon = 1/k$, $k = 1, 2, \ldots$ \hfill \Box \\

If $X$ is a (nonempty) perfect zero-dimensional compact (metrizable) space then clearly 
* every nonempty clopen subset of $X$ is such space;
* for every $n$ there exists a partition of $X$ into $n$ clopen sets;\footnote{A partition is a covering by nonempty, pairwise disjoint sets.}
* for every $\varepsilon$ there exists a finite partition of $X$ into $\varepsilon$-small (that is, of diameter $\leq \varepsilon$) clopen sets;
* for every $\varepsilon$, for every $n$ large enough, there exists a partition of $X$ into $n\varepsilon$-small clopen sets.

12d5 Lemma. All perfect zero-dimensional compact spaces are mutually homeomorphic (and therefore homeomorphic to the Cantor set).

Proof. Given such spaces $X, Y$, we take partitions $X = \bigcup_{k_1=1}^{n_1} X_{k_1}$, $Y = \bigcup_{k_1=1}^{n_1} Y_{k_1}$ into 1-small clopen sets. Then, partitions $X_{k_1} = \bigcup_{k_2=1}^{n_2} X_{k_1,k_2}$, $Y_{k_1} = \bigcup_{k_2=1}^{n_2} Y_{k_1,k_2}$ into 1/2-small clopen sets. And so on. Finally, we consider $G_1 = \bigcup_{k_1=1}^{n_1} X_{k_1} \times Y_{k_1} \subset X \times Y$, $G_2 = \bigcup_{k_1=1}^{n_1} \bigcup_{k_2=1}^{n_2} X_{k_1,k_2} \times Y_{k_1,k_2} \subset X \times Y$ and so on, and note that $G = \cap_n G_n$ is the graph of a homeomorphism $X \to Y$. \hfill \Box \\

12d6 Corollary. A typical set of $K(\mathbb{R}^n)$ is homeomorphic to the Cantor set.

Amazingly, the Cantor set in $\mathbb{R}^n$ can be knotted! See “Antoine’s necklace” in Wikipedia.\footnote{Image from Wikipedia.} I wonder, is this typical?

If $X$ is a (nonempty) compact (metrizable) space then clearly 
* every nonempty closed subset of $X$ is such space;
* for every $\varepsilon$ there exists a finite covering of $X$ by $\varepsilon$-small closed sets;
* for every $\varepsilon$, for every $n$ large enough, there exists a covering of $X$ by $n$ $\varepsilon$-small closed sets. (Not necessarily different...)
12d7 Lemma. Every compact space is a continuous image of the Cantor set.

Proof. Let $C$ be the Cantor set and $X$ a compact space. We take a partition $C = \biguplus_{k_1=1}^{n_1} C_{k_1}$ of $C$ into 1-small clopen sets and a covering $X = \bigcup_{k_2=1}^{n_2} X_{k_2}$ of $X$ by 1-small closed sets. Then, $C_{k_1} = \biguplus_{k_2=1}^{n_2} C_{k_1,k_2}$ and $X_{k_1} = \bigcup_{k_2=1}^{n_2} X_{k_1,k_2}$, with 1/2-small sets. And so on. We define $G_1, G_2, \ldots$ and $G$ as before and note that $G$ is the graph of a continuous map $C \to X$. □

12d8 Proposition. The map

$$C(C \to \mathbb{R}^n) \ni f \mapsto f(C) \in K(\mathbb{R}^n)$$

is continuous and open.

Here $C$ is the Cantor set, and $C(C \to \mathbb{R}^n)$ is the space of all continuous maps $C \to \mathbb{R}^n$ with the metric $\rho(f, g) = \max_{x \in C} |f(x) - g(x)|$.

Proof. Continuity (and even Lip(1)) is evident; openness will be proved.

Let $K_1 = f(C)$ and $d_H(K_1, K_2) \leq \varepsilon$; we need $g$ close to $f$ such that $K_2 = g(C)$. We take a finite partition $C = C_1 \uplus \cdots \uplus C_m$ of $C$ into clopen sets $C_k$ such that $\text{diam}(f(C_k)) \leq \varepsilon$. Sets

$$X_k = (f(C_k))_{+\varepsilon} \cap K_2$$

are a covering of $K_2$ by closed sets. We take $g_k \in C(C_k \to \mathbb{R}^n)$ such that $g_k(C_k) = X_k$ and combine them into $g \in C(C \to \mathbb{R}^n)$, then $g(C) = K_2$ and $\rho(f, g) \leq 2\varepsilon$. □

Similarly to [12b] we get:

12d9 Corollary. For quasi all $f \in C(C \to \mathbb{R}^n)$ the set $K = f(C)$ is a nowhere dense null set homeomorphic to the Cantor set, satisfying $\dim_M(K) = 0$, $\overline{\dim}_M(K) = n$.

12d10 Exercise. If $A, B$ are disjoint clopen subsets of the Cantor set then typically $f(A)$ and $f(B)$ are disjoint.

Prove it.

It follows that a typical $f$ is one-to-one. Therefore (by compactness) it is a homeomorphism between $C$ and $f(C)$. Thus, we improve [12d9].

12d11 Theorem. For quasi all $f \in C(C \to \mathbb{R}^n)$, $f$ is a homeomorphism of $C$ onto a nowhere dense null set $K = f(C)$ satisfying

$$\dim_M(K) = 0, \quad \overline{\dim}_M(K) = n.$$
Now (at last) we are in position to attack Theorem 12d2.

12d12 Theorem. 1 There exists a set $A \subset [0,1]^{n}$ of full measure such that for quasi all $f \in C([0,1]^{n} \to \mathbb{R})$,

$f|_{A}$ is one-to-one, 
$f(A)$ is a meager set of Hausdorff dimension 0.

A subset of $\mathbb{R}$ is zero-dimensional if and only if its complement is dense (think, why). Thus, a closed subset of $\mathbb{R}$ is zero-dimensional if and only if it is nowhere dense. By 1d4(a), the union of two zero-dimensional closed subsets of $\mathbb{R}$ is zero-dimensional. 2

12d13 Lemma. There exist perfect zero-dimensional sets $K_{n} \subset [0,1]$ such that $K_{1} \subset K_{2} \subset \ldots \subset [0,1]$.

Proof. Monotonicity can be achieved by taking $K_{1} \subset K_{1} \cup K_{2} \subset \ldots$ (since a finite union of perfect zero-dimensional subsets of $[0,1]$ is perfect and zero-dimensional). It remains to find, for a given $\varepsilon$, a perfect zero-dimensional $K \subset [0,1]$ satisfying $m(K) \geq 1 - \varepsilon$.

We take a dense sequence of pairwise disjoint closed intervals $[x_{k}, x_{k} + \delta_{k}] \subset [0,1]$ such that $\sum_{k} \delta_{k} \leq \varepsilon$, let $K = [0,1] \setminus \cup_{k} (x_{k}, x_{k} + \delta_{k})$ and note that $K$ is perfect and zero-dimensional.

The same for $[0,1]^{n}$ follows immediately: take $K_{n} \subset K_{n} \subset \ldots \subset [0,1]^{n}$.

12d14 Lemma. If $\dim_{H}(A) = 0$ then $A$ is of Hausdorff dimension 0.

Proof. It is possible to cover $A$ with $N_{\delta}(A)$ balls of radius $\delta$. We have $\liminf_{\delta \to 0^+} \frac{\log N_{\delta}(A)}{\log 1/\delta} = 0$. Given $\varepsilon$, we take $\delta$ such that $\log N_{\delta}(A) \leq \frac{1}{2} \varepsilon \log 1/\delta \leq \varepsilon \log 1/\delta - \log 1/\varepsilon$, then $\delta^{\varepsilon} N_{\varepsilon}(A) \leq \varepsilon$.

12d15 Lemma. Sets of Hausdorff dimension 0 are a $\sigma$-ideal.

Proof. Let $A = A_{1} \cup A_{2} \cup \ldots$, each $A_{k}$ being of Hausdorff dimension 0. Given $\varepsilon$, for each $k$ we cover $A_{k}$ with balls $\{x_{k,i}\}_{i} \times r_{k,i}$ such that $\sum_{i} r_{k,i}^{\varepsilon} \leq 2^{-i} \varepsilon$; then $\sum_{k,i} r_{k,i}^{\varepsilon} \leq \varepsilon$.

Proof of Theorem 12d12. We take perfect zero-dimensional $K_{1} \subset K_{2} \subset \ldots \subset [0,1]^{n}$ such that $m(K_{i}) \uparrow 1$ and let $A = \cup_{i} K_{i}$. By 12d5 each $K_{i}$ is homeomorphic to the Cantor set. Thus, Theorem 12d11 applies to quasi all $f \in C(K_{i} \to \mathbb{R})$. By 12d3 (and 12b2) the same holds for quasi all $f \in$ 1

See also Bruckner, Bruckner, Thomson, Exercise 10:7.6.

2In more general spaces this fact holds but is harder to prove.
C([0,1]^n → R) restricted to K_i. That is, for each i, f|_{K_i} is a homeomorphism of K_i onto a nowhere dense null set f(K_i) satisfying $\dim_M(f(K_i)) = 0$ (and $\dim_M(f(K_i)) = 1$). It follows that $f|_A$ is one-to-one and f(A) is meager. By 12d14, each f(K_i) is of Hausdorff dimension 0. By 12d15, f(A) is of Hausdorff dimension 0.

12d16 Remark. Our choice of A ensures, in addition, that for every meager B ⊂ R^n
$$\forall^* f \in C([0,1]^n \to R) \ f(A) \cap B = \emptyset.$$ Thus, there exists a null set B ⊂ R^n such that
$$\forall^* f \in C([0,1]^n \to R) \ f(A) \subset B.$$ (Similar to 12c4, 12c5.)

Proof of Theorem 12d2. By Theorem 12d12 quasi all $f \in C([0,1]^n \to R)$ satisfy 12d14. Given $g \in C([0,1]^n \to R)$, a map $f \mapsto g - f$ is a homeomorphism of $C([0,1]^n \to R)$. Thus, also $g - f$ satisfies 12d14 for quasi all f. □

12e Another topology on measurable functions

We turn to the space $L_1(\to R^n)$ of all equivalence classes of Lebesgue integrable functions $f : [0,1] \to R^n$ with the metric
$$\rho(f,g) = \int |f - g| \ dm.$$ This is a Polish (in fact, Banach) space.

12e1 Lemma. $\forall x \in R^n \ \forall^* f \in L_1(\to R^n) \ m\{t : f(t) = x\} = 0$.

Proof. For every $\varepsilon > 0$ the set $\{f : m\{t : f(t) = x\} < \varepsilon\}$ is open and dense in $L_1(\to R^n)$. □

12e2 Exercise. If $A \subset R^n$ is meager then $\forall^* f \in L_1(\to R^n) \ m(f^{-1}(A)) = 0$. Prove it.

12e3 Corollary. There exists a null set $A \subset R^n$ such that for quasi all $f \in L_1(\to R^n)$, $f(\cdot) \in A$ almost everywhere. (Proof: just take a comeager null set.) Similarly to 12c, we may define the support (closed rather than compact), but this time it is the whole $R^n$. 
12e4 Lemma. For every nonempty open $G \subset \mathbb{R}$, $\forall^* f \in L_1(\rightarrow \mathbb{R}^n)$ $m(f^{-1}(G)) > 0$.

Proof. Take continuous $\varphi : \mathbb{R}^n \to [0, \infty)$ that vanishes outside $G$ but not everywhere. Then $f \mapsto \int \varphi(f(\cdot)) \, dm$ is a continuous function on $L_1(\rightarrow \mathbb{R}^n)$, positive on a dense set.

The same holds for $f|_A$ for an arbitrary measurable $A \subset [0, 1]$ of positive measure (but not for all $A$ simultaneously, of course). Do you think it leads to infinite multiplicity? No, it does not. The result is similar to 12c7 but the proof is harder.

12e5 Proposition. A typical $f \in L_1(\rightarrow \mathbb{R}^n)$ is one-to-one (that is, the equivalence class contains some one-to-one function).

12e6 Lemma. If $A, B \subset [0, 1]$ are disjoint measurable sets then for a typical $f \in L_1(\rightarrow \mathbb{R}^n)$,

$$\forall s \in A \forall t \in B \ f(s) \neq f(t)$$

for some choice of a function within the given equivalence class.

Proof. Given $\varepsilon > 0$, we introduce a set $G_\varepsilon$ of all $f$ such that there exist measurable $A_1 \subset A$, $B_1 \subset B$ satisfying

$$m(A \setminus A_1) < \varepsilon, \quad m(B \setminus B_1) < \varepsilon, \quad \text{ess inf}_{s \in A_1, t \in B_1} |f(s) - f(t)| > 0.$$ 

It is sufficient to prove that a typical $f$ belongs to all $G_\varepsilon$. We note that $G_\varepsilon$ is a dense set (even for $\varepsilon = 0$) by the argument of the proof of 12a5. It remains to prove that $G_\varepsilon$ is open (for $\varepsilon > 0$, of course).

Given $f \in G_\varepsilon$ and $A_1, B_1$, we take $\delta > 0$ such that $m(A \setminus A_1) \leq \varepsilon - \delta$, $m(B \setminus B_1) \leq \varepsilon - \delta$ and $\text{ess inf}_{s \in A_1, t \in B_1} |f(s) - f(t)| \geq \delta$. For arbitrary $g \in L_1(\rightarrow \mathbb{R}^n)$ we have

$$m\{t : |f(t) - g(t)| \geq \delta/3\} \leq \frac{3}{\delta} \|f - g\|.$$ 

If $\|f - g\| < \delta^2/3$ then the set $Z = \{t : |f(t) - g(t)| \geq \delta/3\}$ satisfies $m(Z) < \delta$.

Taking $A_2 = (A \setminus A_1) \setminus Z$, $B_2 = (B \setminus B_1) \setminus Z$ we get $m(A \setminus A_2) \leq m(A \setminus A_1) + \delta < \varepsilon$, $m(B \setminus B_2) < \varepsilon$, and $\text{ess inf}_{s \in A_2, t \in B_2} |f(s) - f(t)| \geq \delta - 2\delta/3 > 0$. □

Proof of Prop. 12e5. We correct $f$ on a null set getting $f([0, 1/2)) \cap f([1/2, 1)) = \emptyset$. Then we correct $f|_{[0,1/2)}$ (without increasing its image) getting $f([0, 1/4)) \cap f([1/4, 1/2)) = \emptyset$. And so on. □
Instead of the support, now we examine the distribution of $f$; this is a probability measure $\mu_f$ on $\mathbb{R}^n$ defined by

$$\mu_f(B) = m(f^{-1}(B)) \quad \text{for Borel sets } B \subset \mathbb{R}^n.$$ 

In general, a probability measure on $\mathbb{R}^n$ decomposes into purely atomic part (concentrated on a finite or countable set of atoms), absolutely continuous part (that has a density w.r.t. Lebesgue measure) and singular part (concentrated on an $m$-null set but atom-free).

- By 12e5, $\mu_f$ is typically atom-free.
- By 12e3, $\mu_f$ is typically singular.

Integrability of $f$ implies $\int_{\mathbb{R}^n} |x| \mu_f(dx) < \infty$.

The set $\mathcal{P}_1(\mathbb{R}^n)$ of all (Borel) probability measures on $\mathbb{R}^n$ satisfying $\int_{\mathbb{R}^n} |x| \mu(dx) < \infty$ is endowed with the so-called transportation metric

$$\rho(\mu_1, \mu_2) = \inf_{f_1, f_2: \mu_1 = \mu_{f_1}, \mu_2 = \mu_{f_2}} \rho(f_1, f_2).$$

Note that a sequence of purely atomic measures can converge to an absolutely continuous measure; and a sequence of absolutely continuous measures can converge to a purely atomic measure. In fact, each of the three sets of measures (purely atomic, singular, and absolutely continuous) is dense in $\mathcal{P}_1(\mathbb{R}^n)$.

**12e7 Proposition.** The map

$$L_1(\to \mathbb{R}^n) \ni f \mapsto \mu_f \in \mathcal{P}_1(\mathbb{R}^n)$$

is continuous and open.

**Proof.** Continuity (and even Lip(1)) is evident; openness will be proved.

Let $\mu_1 = \mu_{f_1}$ and $\rho(\mu_1, \mu_2) \leq \varepsilon$; we need $f_2$ close to $f_1$ such that $\mu_2 = \mu_{f_2}$. We take $g_1, g_2 \in L_1(\to \mathbb{R}^n)$ such that

$$\rho(g_1, g_2) \leq 2\varepsilon.$$

We introduce

$$A_k = f_1^{-1}([k \varepsilon, k \varepsilon + \varepsilon]), \quad B_k = g_1^{-1}([k \varepsilon, k \varepsilon + \varepsilon])$$

for $k \in \mathbb{Z}$ and note that $m(A_k) = m(B_k)$ (since $\mu_{f_1} = \mu_{g_1}$). For each $k$ such that $m(A_k) > 0$ we take a measure preserving map $\varphi_k : A_k \to B_k$ (try increasing $\varphi_k$ such that $\forall x \ m(A_k \cap (\infty, x]) = m(B_k \cap (\infty, \varphi_k(x)))$). We define $f_2$ by

$$f_2(t) = g_2(\varphi_k(t)) \quad \text{for } t \in A_k$$
and note that $\mu_{f_2} = \mu_{g_2} = \mu_2$ since for every Borel set $B \subset \mathbb{R}$,
\[
m(f_2^{-1}(B)) = \sum_k m(f_2^{-1}(B) \cap A_k) = \sum_k m\{s \in A_k : g_2(\varphi_k(s)) \in B\} = \sum_k m\{t \in B_k : g_2(t) \in B\} = m(g_2^{-1}(B)).
\]

It remains to prove that $f_2$ is close to $f_1$. We have
\[
\rho(f_1, f_2) = \int |f_1 - f_2| \, dm = \sum_k \int_{A_k} |f_1 - f_2| \, dm \leq \sum_k \int_{A_k} (|f_1 - k\varepsilon| + |k\varepsilon - f_2|) \, dm \leq \varepsilon + \sum_k \int_{A_k} (|f_2 - k\varepsilon| \, dm = \\
= \varepsilon + \sum_k \int_{B_k} (|g_2 - k\varepsilon| \, dm \leq \varepsilon + \sum_k \int_{B_k} (|g_2 - g_1| + |g_1 - k\varepsilon|) \, dm \leq 2\varepsilon + \rho(g_1, g_2) \leq 4\varepsilon.
\]

\[\square\]

12e8 Exercise. A typical measure is atom-free.
Prove it.

12e9 Exercise. A typical measure is singular.
Prove it.

Minkowski (or “box”) dimension of a measure is defined by
\[
\dim_M \mu = \liminf_{\mu(B) \to 1} \dim_M B, \quad \overline{\dim}_M \mu = \liminf_{\mu(B) \to 1} \overline{\dim}_M B
\]
where $B$ runs over all Borel sets.

It appears that\footnote{J. Myjak, R. Rudnicki (2002) “On the box dimension of typical measures”, Monatsh. Math. \textbf{136}, 1143–150.} for quasi all $\mu \in \mathcal{P}_1(\mathbb{R}^n),$
\[
\dim_M \mu = 0, \quad \overline{\dim}_M \mu = n.
\]

By 12e7 (and 12b2) for quasi all $f \in L_1(\to \mathbb{R}^n),$
\[
\dim_M \mu f = 0, \quad \overline{\dim}_M \mu f = n.
\]
Hints to exercises

12a3: (a) try dist(A, x(1, 2, . . .)); (b) use (a).
12c1: 5d5 can help.
12c4: similar to 12a3
12c6: similar to 12a5
12d10: similar to 12a5
12e2: recall 12a3
12e8: no, 12e5 is of no help (I think so). Rather, prove that all µ satisfying ∀x µ({x}) < ε are an open set.
12e9: use 12e3 and 12b5 if you like. Or do not.

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