5 Random connected components

5a Connected sets among closed sets

Recall some topological notions and facts.

A metrizable space X is *connected,* if $\operatorname{Clopen}(X) = \{0, X\}$.

A subset $Y \subset X$ is connected if it is itself a connected space. Note that

(5a1) $$\operatorname{Clopen}(Y) \supset \{A \cap Y : A \in \operatorname{Clopen}(X)\} ;$$

in general, these are not equal (indeed, it happens routinely that X is connected while Y is not). For a connected Y,

(5a2) $$\forall A \in \operatorname{Clopen}(X) \;\; \big(Y \subset A \vee Y \subset X \setminus A \big) .$$

For arbitrary Y, choosing a compatible metric ρ on X and denoting as before $\operatorname{dist}(x, A) = \inf\{\rho(x, a) : a \in A\}$, we have

$$\operatorname{Clopen}(Y) = \{ A \subset Y \,:\, \overline{A} \cap (Y \setminus A) = \emptyset \wedge \overline{(Y \setminus A)} \cap A = \emptyset \} = \\ = \{ A \subset Y \,:\, (\forall a \in A \;\operatorname{dist}(a, Y \setminus A) > 0) \wedge (\forall b \in Y \setminus A \;\operatorname{dist}(b, A) > 0) \} .$$

For every $A \in \operatorname{Clopen}(Y)$ there exist open sets $U, V \subset X$ such that $U \cap V = \emptyset$, $U \cap Y = A$ and $V \cap Y = Y \setminus A$; namely, we may take

$$U = \bigcup_{a \in A} B^\circ\big(a, 0.5\rho(a, Y \setminus A)\big) , \quad V = \bigcup_{b \in Y \setminus A} B^\circ\big(b, 0.5\rho(b, A)\big) .$$

(Here $B^\circ(a, r) = \{x : \rho(x, a) < r\}$.) Thus,

(5a3) $$\operatorname{Clopen}(Y) = \\ = \{ Y \cap U \,:\, U, V \text{ are open in } X \;\wedge\; U \cap V = \emptyset \;\wedge\; U \cup V \supset Y \} .$$

5a4 Proposition. For every compact metrizable space X the set

$$\{F \in \mathbf{F}(X) : F \text{ is connected }\}$$

is Borel measurable.

From now on (till 5a9) X is compact, and $(U_n)_n$ is a sequence[1] of open subsets of X such that for all $E, F \in \mathbf{F}(X)$,

$$E \cap F = \emptyset \implies \exists m, n \,(U_m \cap U_n = \emptyset \;\wedge\; E \subset U_m \;\wedge\; F \subset U_n) . \tag{5a5}$$

5a6 Core exercise. Prove existence of such $(U_n)_n$.

Note that $\operatorname{Clopen}(X) \subset \{U_1, U_2, \dots\}$; and by the way, it shows that $\operatorname{Clopen}(X)$ is at most countable (provided that X is compact).[2]

5a7 Core exercise. For every $F \in \mathbf{F}(X)$,

$$\operatorname{Clopen}(F) = \{F \cap U_m : U_m \cap U_n = \emptyset \wedge U_m \cup U_n \supset F\} .$$

Prove it.

5a8 Core exercise. A closed set $F \subset X$ is connected if and only if

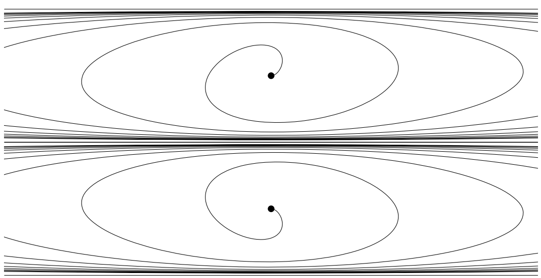
$$\forall m, n \, \big((U_m \cap U_n = \emptyset \wedge U_m \cup U_n \supset F) \implies (U_m \cap F = \emptyset \vee U_n \cap F = \emptyset) \big) .$$

Prove it.

5a9 Core exercise. Prove Prop. 5a4.

Prop. 5a4 fails for Polish (not just compact) spaces. In particular, it fails if X is an infinite-dimensional separable Hilbert space.[3]

Does 5a4 hold for $X = \mathbb{R}^d$? I do not know![4] If you feel enthusiastic to reduce connectedness of a closed set to some property of its compact subsets, take into account the following instructive example of a connected closed subset of $\mathbb{R}^2$:



[1] "Countable superbase", if you like.

[2] For Polish X, $\operatorname{Clopen}(X)$ need not be countable (try a discrete space).

[3] This is basically the phenomenon mentioned before 4d10.

[4] I only know that $\{F \in \mathbf{F}(\mathbb{R}^d) : F \text{ is connected}\}$ is coanalytic, since its complement is the image of the Borel set $\{(F_1, F_2) : F_1 \cap F_2 = \emptyset, F_1 \neq \emptyset, F_2 \neq \emptyset\}$ (recall 4d15) under the Borel map $(F_1, F_2) \mapsto F_1 \cup F_2$.

5a10 Remark. As a palliative we may treat a random closed subset of $\mathbb{R}^d$ via the one-point compactification $\mathbb{R}^d \cup \{\infty\}$ (including ∞ into each unbounded closed set). Then all unbounded connected components (if any) are glued together.

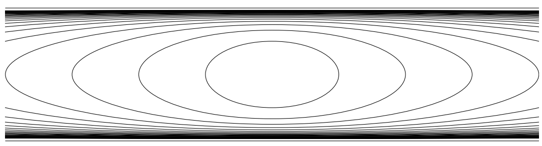
5b Connected components

Consider two equivalence relations on a metrizable space X: points $x, y \in X$ are equivalent, when

$$x, y \in Y \text{ for some connected } Y \subset X\,; \tag{5b1}$$

$$\forall A \in \operatorname{Clopen}(X) \;\; \big(x \in A \iff y \in A \big)\,. \tag{5b2}$$

Clearly, (5b1) implies (5b2) (recall (5a2)). In general they are not equivalent; an example:



Equivalence classes for (5b1) are called *connected components* (of X); for (5b2) — *quasiconnected components.* In general, every quasiconnected component decomposes into connected components. But a compact X is simpler.

5b3 Lemma. For a compact X, (5b1) and (5b2) are equivalent.

Proof. Let Y be an equivalence class for (5b2); we'll prove that Y is connected. By (5a3) it is sufficient to prove $Y \subset U$ or $Y \subset V$ whenever open $U, V \subset X$ satisfy $U \cap V = \emptyset$ and $U \cup V \supset Y$. Compactness gives us $A \in \operatorname{Clopen}(X)$ such that $Y \subset A \subset U \cup V$. Thus, $A \cap U = A \setminus V \in \operatorname{Clopen}(X)$. All points of Y being (5b2)-equivalent, we get $Y \subset A \cap U$ or $Y \subset X \setminus (A \cap U)$; accordingly, $Y \subset U$ or $Y \subset V$. ☐

Note that a compact X can have uncountably many connected components (try the Cantor set).

5b4 Proposition. The following subset of $\mathbf{F}(X) \times \mathbf{F}(X)$ is Borel measurable, provided that X is compact:

$$\{(E, F) : E \text{ is a connected component of } F\}\,.$$

Choosing a compatible metric ρ on X we define for $E, F \in \mathbf{F}(X)$

$$d_{\not\subset}(E, F) = \sup_{x \in E} \operatorname{dist}(x, F) = \sup_{x \in E} \inf_{y \in F} \rho(x, y) = \inf\{r > 0 : E \subset F_{+r}\}\,,$$

$$d_H(E, F) = \max\big(d_{\not\subset}(E, F), d_{\not\subset}(F, E)\big) = \inf\{r > 0 : E \subset F_{+r} \,\wedge\, F \subset E_{+r}\}$$

(as usual, $\inf \emptyset = +\infty$); d_H is a metric on $\mathbf{F}(X) \setminus \{\emptyset\}$, — the well-known Hausdorff metric.[1]

5b5 Lemma. $d_{\not\subset} : \mathbf{F}(X) \times \mathbf{F}(X) \to [0, \infty]$ is a Borel function.

Proof. We take a sequence $(x_n)_n$ dense in X and note that

$$d_{\not\subset}(E, F) = \inf_{r>0} \sup_{n:\operatorname{dist}(x_n,E)\le\varepsilon} \operatorname{dist}(x_n, F)$$

and $\operatorname{dist}(x_n, \cdot)$ is a Borel function by 4d8(a). □

Proof of Prop. 5b4. For $E, F \in \mathbf{F}(X)$, by 5b3, E is a connected component of F if and only if

(a) E is connected, and

(b) $E \subset F$, and E is the intersection of all $A \in \operatorname{Clopen}(F)$ such that $A \supset E$.

Condition (a) leads to a Borel set by Prop. 5a4; we'll prove the same for (b). By compactness, (b) is equivalent to

(b1) for every $\varepsilon > 0$ there exists $A \in \operatorname{Clopen}(F)$ such that $A \supset E$ and $d_{\not\subset}(A, E) \le \varepsilon$.

We choose $(U_n)_n$ satisfying (5a5). By (5a7), (b1) is equivalent to

(b2) for every $\varepsilon > 0$ there exist m, n such that $U_m \cap U_n = \emptyset$, $U_m \cup U_n \supset F$, $F \cap U_m \supset E$ and $d_{\not\subset}(F \cap U_m, E) \le \varepsilon$.

It remains to check, for given m, n satisfying $U_m \cap U_n = \emptyset$, that each of the following three conditions leads to a Borel set:

(c) $F \subset U_m \cup U_n$,

(d) $E \subset F \setminus U_n$,

(e) $d_{\not\subset}(F \setminus U_n, E) \le \varepsilon$.

We rewrite (c), (d) as

(c1) $F \cap (X \setminus (U_m \cup U_n)) = \emptyset$,

(d1) $d_{\not\subset}(E, F \setminus U_n) = 0$.

For (c1) we use 4d12(b). For (d1) and (e) we use 5b5, taking into account that the map $F \mapsto F \setminus U_n = F \cap (X \setminus U_n)$ is Borel measurable by 4d14. □

We conclude.

[1] In fact, the Hausdorff metric on a compact X turns $\mathbf{F}(X)$ into a compact metric space whose Borel σ-algebra coincides with that of (4d2). The set of connected components of X need not be closed in $(\mathbf{F}(X), d_H)$; an example: $[-1, 0] \cup \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$. For a Polish X, the metric space $(\mathbf{F}(X), d_H)$ is complete but generally nonseparable (try a discrete X). See also Sect. 2 in Beer's article (cited in footnote 5 on page 71).

5b6 Theorem. Let X be a compact metrizable space and S a random closed subset of X. Then the set of all connected components of S[1] is a random Borel subset of the standard[2] Borel space $\mathbf{F}(X)$.

5c Counting the connected components

5c1 Core exercise. Let S be a random measurable subset of a standard Borel space. Then $\{\omega : |S(\omega)| \le n\}$ is measurable for every n. (Here $|S(\omega)|$ is the number of points in $S(\omega)$.)

Prove it.

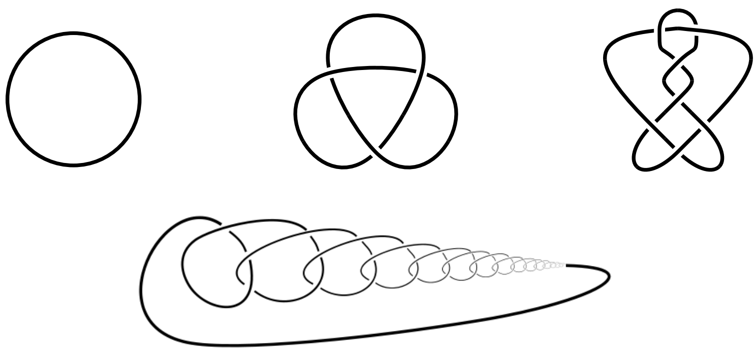
By 5c1 (and Theorem 5b6), given a random closed set in a compact metrizable X, the number of its connected components is a random variable (with values in $\{0, 1, 2, \dots\} \cup \{\infty\}$). In this sense we may count the connected components.[3] Given a Borel set $B \subset \mathbf{F}(X)$, we may count the connected components belonging to B (think, why). For example we may count the connected components contained in a given ball, or intersecting a given ball, etc.

5c2 Core exercise. Let S be a random closed subset of $\mathbb{R}^d$. Then the set of all bounded connected components of S is a random Borel subset of $\mathbf{F}(\mathbb{R}^d)$.

Prove it.

5d Classifying the connected components: random knots

A knot is a subset of $\mathbb{R}^3$ homeomorphic to a circle.



[1] I mean the set of all connected components of $S(\omega)$ as a function of ω.

[2] Recall 4d5.

[3] The set of ω such that $S(\omega)$ is uncountable (and therefore of cardinality continuum, see Kechris, Th. (13.6) or Srivastava, Th. 4.3.5) is also measurable, see Kechris, Th. (29.19).

Knots K_1, K_2 are of the same *type* (in other words, equivalent) if $h(K_1) = K_2$ for some homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$.[1] A knot is *tame* if it is equivalent to a polygonal knot. Or, equivalently, to a smooth knot (continuously differentiable). Otherwise it is a *wild* knot.[2] All knots that lie in a plane are of the same type (the trivial type; the unknot).[3] This is a deep theorem in general, but relatively simple for tame knots.

All tame knot types are a countable set.[4]

5d1 Proposition. Each knot type is a universally measurable[5] subset of $\mathbf{F}(\mathbb{R}^3)$.

Thus, all tame knots are also a universally measurable set.

By 5c2 and 5d1, given a random closed set in $\mathbb{R}^3$, we may count its connected components that are (a) tame knots; (b) tame knots of a given type; (c) tame knots of a given type that are contained in a given ball; etc.

You may think about a random smooth map $f : \mathbb{R}^3 \to \mathbb{R}^2$ (or $\mathbb{R}^3 \to \mathbb{C}$) such that almost surely, f is regular (that is, $\operatorname{rank} \mathrm{d}f(\cdot) = 2$) at all x satisfying $f(x) = 0$. Then all bounded connected components of the random closed set $f^{-1}(0)$ are tame knots (almost surely).

The group $\operatorname{Homeo}(\mathbb{R}^3)$ of all homeomorphisms $\mathbb{R}^3 \to \mathbb{R}^3$ acts on the standard Borel space $\mathbf{F}(\mathbb{R}^3)$.

Here is a generalization of Prop. 5d1.

5d2 Proposition. For every $F \in \mathbf{F}(\mathbb{R}^3)$ its orbit $\{h(F) : h \in \operatorname{Homeo}(\mathbb{R}^3)\}$ is a universally measurable subset of $\mathbf{F}(\mathbb{R}^3)$.

5d3 Core exercise. The set $C(\mathbb{R}^3 \to \mathbb{R}^3)$ of all continuous maps $\mathbb{R}^3 \to \mathbb{R}^3$ endowed with the σ-algebra generated by evaluations $f \mapsto f(x)$ is a standard Borel space.

Prove it.

5d4 Core exercise. $f(x)$ is jointly measurable in $f \in C(\mathbb{R}^3 \to \mathbb{R}^3)$ and $x \in \mathbb{R}^3$.

Prove it.

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[1] See for instance Sect. 1.1 in: R.H. Crowell, R.H. Fox, "Introduction to knot theory", Springer 1963.

[2] Crowell and Fox, Sect. 1.2.

[3] Crowell and Fox, Sect. 1.2.

[4] Crowell and Fox, Chapter I, Exercise 5. In fact, they are a semigroup isomorphic to $\{1, 2, 3, \dots\}$ with multiplication. By the way, some wild knots are infinite products of tame knots.

[5] In fact, Borel measurable, but this is harder to prove.

5d5 Core exercise. The composition $g \circ f \in C(\mathbb{R}^3 \to \mathbb{R}^3)$ is jointly measurable in $f, g \in C(\mathbb{R}^3 \to \mathbb{R}^3)$.

Prove it.

We embed the set $\operatorname{Homeo}(\mathbb{R}^3)$ into $C(\mathbb{R}^3 \to \mathbb{R}^3) \times C(\mathbb{R}^3 \to \mathbb{R}^3)$ as follows:

$$h \mapsto (h, h^{-1}) \,.$$

By 5d5 the image $\{ (f, g) : f \circ g = \mathrm{id} \wedge g \circ f = \mathrm{id} \}$ is a Borel set, therefore a standard Borel space. We endow $\operatorname{Homeo}(\mathbb{R}^3)$ with the corresponding σ-algebra and observe that

$$\operatorname{Homeo}(\mathbb{R}^3) \text{ is a standard Borel space.}^1 \qquad \text{(5d6)}$$

5d7 Extra exercise. $\operatorname{Homeo}(\mathbb{R}^3)$ is both a Borel subset and a measurable subspace of $C(\mathbb{R}^3 \to \mathbb{R}^3)$.

Prove it.[2]

(By the way, it follows that the map $f \mapsto f^{-1}$ is Borel measurable on $\operatorname{Homeo}(\mathbb{R}^3)$ treated as a subset of $C(\mathbb{R}^3 \to \mathbb{R}^3)$.)

5d8 Core exercise. Let $F \subset \mathbb{R}^3$ be a closed set, then the map

$$C(\mathbb{R}^3 \to \mathbb{R}^3) \ni f \mapsto f(F) \in \mathbf{F}(\mathbb{R}^3)$$

is Borel measurable.

Prove it.

Thus, the orbit $\{h(F) : h \in \operatorname{Homeo}(\mathbb{R}^3)\}$ is the image of the standard Borel space $\operatorname{Homeo}(\mathbb{R}^3)$ under the Borel map $h \mapsto h(F)$.

Here is a topology-free counterpart of Def. 3e1.

5d9 Definition. A subset of a countably separated measurable space is *analytic* if it is the image of some standard Borel space under some measurable map.

Does 5d9 conflict with 3e1? No, it does not.

5d10 Lemma. For every subset A of a separable metrizable space, the following two conditions are equivalent:

(a) A is the image of some Polish space under some continuous map;

(b) A is the image of some standard Borel space under some Borel map.

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[1] It is in fact a Polish group (and therefore, by 4d7, a standard Borel space).

[2] In fact, this follows from standardness by a general theorem; but you are asked to prove it explicitly.

Proof. (a)$\Longrightarrow$(b): the Polish space is a standard Borel space by 4d7, and the continuous map is Borel measurable.

(b)$\Longrightarrow$(a): we may assume that the standard Borel space is a Borel subset of the Cantor set, or even the whole Cantor set (extend the map by a constant...); it remains to use Prop. 3d1. □

And here is a counterpart of Theorem 3f22.

5d11 Theorem. Analytic sets in countably separated measurable spaces are universally measurable.

Proof. Let $(X, \mathcal{A})$ be a standard Borel space, $(Y, \mathcal{B})$ a countably separated measurable space, $f : X \to Y$ a measurable map, and μ a probability measure on $(Y, \mathcal{B})$; we have to prove that $f(X)$ is μ-measurable.

Without loss of generality we may assume that $(Y, \mathcal{B})$ is a Borel space. Proof: first, by 1d35, $(Y, \mathcal{B}_1)$ is a Borel space for some $\mathcal{B}_1 \subset \mathcal{B}$; second, f is measurable from $(X, \mathcal{A})$ to $(Y, \mathcal{B}_1)$; third, every $(\mu|_{\mathcal{B}_1})$-measurable set is μ-measurable.

By 1d40 we may assume that $Y \subset \mathbb{R}$. Moreover, we may assume that $Y = \mathbb{R}$. Proof: we define a probability measure ν on $\mathbb{R}$ by $\nu(B) = \mu(B \cap Y)$ for $B \in \mathcal{B}(\mathbb{R})$ and observe that every ν-measurable subset of Y is μ-measurable.

It remains to use Lemma 5d10 and Theorem 3f22. □

Propositions 5d2 and 5d1 follow.

Proposition 5d2 (and its proof) generalizes readily to arbitrary locally compact separable metrizable spaces (in place of $\mathbb{R}^3$).

So, we may count tame knots. What about wild knots?

5d12 Proposition. All knots are a universally measurable subset of $\mathbf{F}(\mathbb{R}^3)$.

Thus, wild knots are also a universally measurable set.

The circle is a compact metrizable space; knots are its homeomorphic images in $\mathbb{R}^3$.

5d13 Core exercise. Let K be a compact metrizable space, and (similarly to 5d3) $C(K \to \mathbb{R}^3)$ the standard Borel space of continuous maps. Then the set of all homeomorphisms $f : K \to f(K) \subset \mathbb{R}^3$ is a Borel subset of $C(K \to \mathbb{R}^3)$.

Prove it.

Similarly to 5d8, the map

$$C(K \to \mathbb{R}^3) \ni f \mapsto f(K) \in \mathbf{F}(\mathbb{R}^3)$$

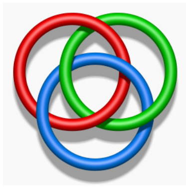
is Borel measurable. Its image

$$\{F : F \text{ is homeomorphic to } K\} \subset \mathbf{F}(\mathbb{R}^3)$$

is analytic, thus, universally measurable. Prop. 5d12 follows.

In fact, the set of all knots is Borel measurable.[1]

You may think also about random links, and a lot of other random geometric objects.



[1]The same holds for subsets of a given σ-compact separable metrizable space X, that are homeomorphic to a given compact metrizable space K; see: C. Ryll-Nardzewski, "On a Freedman's problem", Fund. Math. **57** (1965), 273–274. The corresponding Borel complexity is not bounded in K even if $X = [0,1] \times [0,1]$; see Fact 3.12 in: A. Marcone, "Complexity of sets and binary relations in continuum theory: a survey" (2005). For $K = \mathbb{S}^1$ this Borel set is $F_{\sigma\delta}$ and not $G_{\delta\sigma}$; see Lemma 6.2 and Theorem 8.5 in: R. Camerlo, U.B. Darji, A. Marcone, "Classification problems in continuum theory", Trans. AMS **357**:11, 4301–4328 (2005).

Hints to exercises

5a6: consider a countable base and all finite unions of its sets.

5c1: use 3f24, taking into account that the Borel space is countably separated.

5c2: recall 5a10.

5d3: recall 2c10(c).

5d4: similar to (4b4).

5d5: use 5d4.

5d8: use a dense sequence.

Index