Part C: Dependence

Contents

6 Martingales

6a Fusion (via martingales)

Monsters[1] of type A have masses $a_1, a_2, \dots, a_m$; monsters of type B — $b_1, b_2, \dots, b_n$ In the first fight, a_1 eats b_1 with probability $\frac{a_1}{a_1+b_1}$, gets the mass $a_1 + b_1$ and then fights b_2; or b_1 eats a_1 and then fights a_2; and so on.

6a1 Proposition. The monsters of type A win with probability $\frac{A}{A+B}$ where $A = a_1 + \dots + a_m$ and $B = b_1 + \dots + b_n$.

6b Exit time (via martingales)

Let $(S_n)_n$ be the simple random walk, and $T = \inf\{n : |S_n| = 10\}$ (be it finite or infinite).

6b1 Proposition. $T < \infty$ a.s., and $\mathbb{E}\, T = 100$.

6c Branching (via martingales)

Let Z_n be the size of n-th generation (be it the number of animals, neutrons, or men of a given family). Assume that $Z_0 = 1$ always, and each member of the n-th generation produces a random number of offsprings (members of the next generation): either 2 (with probability p) or 0 (with probability $1-p$). That is, conditionally, given $Z_0, \dots, Z_n$, the distribution of $Z_{n+1}/2$ is binomial,

$$\mathbb{P}\big(Z_{n+1} = 2k \,\big|\, Z_0, \dots, Z_n \big) = \binom{Z_n}{k} p^k (1-p)^{Z_n - k} \,.$$

[1] Maybe, banks. . .

This is called the simple branching, or Galton-Watson, process.

For $p \le 0.5$ the process extincts a.s.:

6c1 Proposition. For $p \le 0.5$, $\mathbb{P}\big(\exists n \;\; Z_n = 0\big) = 1$.

For $p > 0.5$ the process either extincts or grows exponentially:

6c2 Proposition. For $p > 0.5$ the limit

$$M_\infty = \lim_{n\to\infty} \frac{Z_n}{(2p)^n}$$

exists and is finite almost surely, and

$$\mathbb{P}\big(M_\infty = 0\big) = \frac{1-p}{p}\,, \quad \mathbb{E}\, M_\infty = 1\,.$$

7 Conditioning

7.1 Theorem (disintegration of measure)**.** For every probability measure μ on $\mathbb{R}^2$ there exist a probability measure ν on $\mathbb{R}$ and a family $(\mu_x)_{x\in\mathbb{R}}$ of probability measures μ_x on $\mathbb{R}$ such that

$$\mu(A) = \int \mu_x(A_x)\, \nu(\mathrm{d}x)$$

for all Borel sets $A \subset \mathbb{R}^2$; here $A_x = \{y : (x,y) \in A\}$.