8 Relation to the Riemann integral[1]

8a Proper Riemann integral[2]

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ there exist step functions[3] $g, h : \mathbb{R} \to \mathbb{R}$ such that $g \le f \le h$ and $\int h - \int g \le \varepsilon$. Equivalently: the lower integral $\sup_{g \le f} \int g$ and the upper integral $\inf_{h \ge f} \int h$ are equal (and finite). In this case their common value is the Riemann integral $\int f$.

For Riemann integrability it is necessary (and far not sufficient) that f is bounded and has a bounded support.

8a1 Proposition. Every Riemann integrable function is Lebesgue integrable, with the same integral.

Proof. We take step functions $g_n \le f$, $h_n \ge f$ such that $\int g_n \to \int f$ and $\int h_n \to \int f$. WLOG, $g_n \uparrow g$ and $h_n \downarrow h$ (otherwise, use $\max(g_1, \dots, g_n)$ and $\min(h_1, \dots, n_n)$). Taking into account that $g_1 \le g_n \le h_n \le h_1$ and $g_1, h_1 \in L_1$ we get $\int g_n \, \mathrm{d}m \uparrow \int g \, \mathrm{d}m$ and $\int h_n \, \mathrm{d}m \downarrow \int h \, \mathrm{d}m$. Thus, $g \le f \le h$ and $\int g \, \mathrm{d}m = \int h \, \mathrm{d}m$. Therefore $f \in L_1$ and $\int f \, \mathrm{d}m = \lim_n \int g_n \, \mathrm{d}m = \lim_n \int g_n = \int f$. □

All said generalizes readily to functions $\mathbb{R}^d \to \mathbb{R}$.

8b Lebesgue's criterion for Riemann integrability[4]

8b1 Proposition. A bounded function $f : \mathbb{R} \to \mathbb{R}$ with bounded support is Riemann integrable if and only if it is continuous almost everywhere.[5]

[1] See also Jones, Sect. 7A; Capiński & Kopp, Sect. 4.5.

[2] "Bernhard Riemann was not the first to define the concept of a definite integral. However, he was the first to apply a definition of integration to any function, without first specifying what properties the function has." (Jones, p. 161)

[3] By definition, a step function has a finite number of steps.

[4] "It is due to Lebesgue (who lived 1875–1941). However, Riemann actually gave a very similar condition in his 1854 paper." (Jones, p. 163)

[5] Not to be confused with "equal a.e. to a continuous function"; the latter condition is neither necessary nor sufficient (think, why).

Proof. *"Only if":* given a Riemann integrable f, we take step functions $g_n \uparrow g$ and $h_n \downarrow h$ as in the proof of 8a1 and note that $g = h$ a.e. (since $\int (h-g)\,\mathrm{d}m = 0$). For almost every x we have

$$g_n(x) \uparrow f(x)\,, \quad h_n(x) \downarrow f(x)\,, \quad \text{and}$$
$$\text{for every } n, \quad g_n \text{ and } h_n \text{ are continuous at } x\,.$$

By sandwich, it follows that f is continuous at x (think, why).

"If": given that f is a.e. continuous, we define step functions g_n, h_n by

$$g_n(x) = \inf_{t\in I} f(t) \quad \text{for } x \in I\,, \qquad h_n(x) = \sup_{t\in I} f(t) \quad \text{for } x \in I$$

where I runs over binary intervals $\big[2^{-n}k, 2^{-n}(k+1)\big)$, $k \in \mathbb{Z}$. For almost every x, f is continuous at x, which implies $g_n(x) \uparrow f(x)$ and $h_n(x) \downarrow f(x)$ (think, why). Thus, $h_n - g_n \downarrow 0$ a.e.; also, $h_1 - g_1 \in L_1$; therefore $\int h_n - \int g_n = \int (h_n - g_n)\,\mathrm{d}m \to 0$, which shows that f is Riemann integrable. □

All said generalizes readily to functions $\mathbb{R}^d \to \mathbb{R}$.

"This aesthetically pleasing integrability criterion has little practical value" (Bichteler).[1] Well, if you use it when proving simple facts, such as integrability of $\sqrt[3]{f}$ or fg (for integrable f and g), you may find far more elementary, "Lebesgue-free" proofs. But here are harder cases.

8b2 Exercise. Consider functions $f : [0,1] \to \mathbb{R}$ such that the function

$$\operatorname{mid}(-M, f, M) : x \mapsto \begin{cases} -M & \text{when } f(x) \le -M, \\ f(x) & \text{when } -M \le f(x) \le M, \\ M & \text{when } M \le f(x) \end{cases}$$

is integrable for all $M > 0$. Prove that the sum of two such functions is also such function.

8b3 Exercise. Let $f, g : [0,1] \to \mathbb{R}$ be Riemann integrable, $A \subset \mathbb{R}^2$, $\forall x \in [0,1]\ \big(f(x), g(x)\big) \in A$, and $\varphi : A \to \mathbb{R}$ continuous and bounded.[2] Then the function $x \mapsto \varphi\big(f(x), g(x)\big)$ is Riemann integrable.

Prove it.

[1] From book "Integration — a functional approach" by Klaus Bichteler (1998); see Exercise 6.16 on p. 27.

[2] The set A need not be closed, and φ need not be (locally) uniformly continuous.

8b4 Exercise. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be bounded and such that all sections $f(x,\cdot)$ and $f(\cdot,y)$ are Riemann integrable. Then

(a) f need not be Riemann integrable;

(b) f must be Lebesgue integrable.

Prove it.[1]

8c Improper Riemann integral

As was noted in Sect. 1b, a conditionally convergent improper Riemann integral (like $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$) is beyond Lebesgue integration. An absolutely convergent improper Riemann integral of a function $f : \mathbb{R} \to \mathbb{R}$ continuous a.e. is $\int f^+ - \int f^-$. Thus, consider a function $f : \mathbb{R} \to [0,\infty)$ continuous a.e. By 8b1 the Riemann integral $\int 1\!\!1_{[-M,M]} \min\bigl(M, f(x)\bigr)\,\mathrm{d}x$ exists for all $M \in (0,\infty)$. We have $\int 1\!\!1_{[-M,M]} \min\bigl(M, f(x)\bigr)\,\mathrm{d}x \uparrow \int f(x)\,\mathrm{d}x$ (as $M \to \infty$), the improper Riemann integral of f; if (and only if) it is finite, the unsigned function f is improperly Riemann integrable. Now, $f : \mathbb{R} \to \mathbb{R}$ is (absolutely) improperly Riemann integrable, if (and only if) f^-, f^+ are, and in this case $\int f = \int f^+ - \int f^-$.

8c1 Proposition. Every (absolutely) improperly Riemann integrable function is Lebesgue integrable, with the same integral.

Proof. WLOG, $f \ge 0$. We introduce $f_n = 1\!\!1_{[-n,n]} \min(n, f)$ and note that $f_n \uparrow f$ and $\int f_n \uparrow \int f < \infty$. By 8a1, $f_n \in L_1$ and $\int f_n\,\mathrm{d}m = \int f_n$; thus, f is measurable, and $\int f_n\,\mathrm{d}m \uparrow \int f\,\mathrm{d}m$, and so, $\int f\,\mathrm{d}m = \int f$. □

All said generalizes readily to functions $\mathbb{R}^d \to \mathbb{R}$.

[1] Hint: (a) try indicator of an appropriate dense countable set; (b) $f_n(x,y) = f(\frac{k}{n}, y)$ for $\frac{k}{n} \le x < \frac{k+1}{n}$.