8 Relation to the Riemann integral

8a Proper Riemann integral

Recall that a function \( f : \mathbb{R} \to \mathbb{R} \) is Riemann integrable if and only if for every \( \varepsilon > 0 \) there exist step functions \( g, h : \mathbb{R} \to \mathbb{R} \) such that \( g \leq f \leq h \) and \( \int h - \int g \leq \varepsilon \). Equivalently: the lower integral \( \sup_{g \leq f} \int g \) and the upper integral \( \inf_{h \geq f} \int h \) are equal (and finite). In this case their common value is the Riemann integral \( \int f \).

For Riemann integrability it is necessary (and far not sufficient) that \( f \) is bounded and has a bounded support.

8a1 Proposition. Every Riemann integrable function is Lebesgue integrable, with the same integral.

**Proof.** We take step functions \( g_n \leq f, h_n \geq f \) such that \( \int g_n \to \int f \) and \( \int h_n \to \int f \). WLOG, \( g_n \uparrow g \) and \( h_n \downarrow h \) (otherwise, use \( \max(g_1, \ldots, g_n) \) and \( \min(h_1, \ldots, h_n) \)). Taking into account that \( g_1 \leq g_n \leq h_n \leq h_1 \) and \( g_1, h_1 \in L_1 \) we get \( \int g_n \, dm \uparrow \int g \, dm \) and \( \int h_n \, dm \downarrow \int h \, dm \). Thus, \( g \leq f \leq h \) and \( \int g \, dm = \int h \, dm \). Therefore \( f \in L_1 \) and \( \int f \, dm = \lim_n \int g_n \, dm = \lim_n \int f \).

All said generalizes readily to functions \( \mathbb{R}^d \to \mathbb{R} \).

8b Lebesgue’s criterion for Riemann integrability

8b1 Proposition. A bounded function \( f : \mathbb{R} \to \mathbb{R} \) with bounded support is Riemann integrable if and only if it is continuous almost everywhere.

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1See also Jones, Sect. 7A; Capiński & Kopp, Sect. 4.5.
2“Bernhard Riemann was not the first to define the concept of a definite integral. However, he was the first to apply a definition of integration to any function, without first specifying what properties the function has.” (Jones, p. 161)
3By definition, a step function has a finite number of steps.
4“It is due to Lebesgue (who lived 1875–1941). However, Riemann actually gave a very similar condition in his 1854 paper.” (Jones, p. 163)
5Not to be confused with “equal a.e. to a continuous function”; the latter condition is neither necessary nor sufficient (think, why).
Proof. “Only if”: given a Riemann integrable $f$, we take step functions $g_n \uparrow g$ and $h_n \downarrow h$ as in the proof of 8a1 and note that $g = h$ a.e. (since $\int (h - g) \, dm = 0$). For almost every $x$ we have

$$g_n(x) \uparrow f(x), \quad h_n(x) \downarrow f(x),$$

and for every $n$, $g_n$ and $h_n$ are continuous at $x$.

By sandwich, it follows that $f$ is continuous at $x$ (think, why).

“If”: given that $f$ is a.e. continuous, we define step functions $g_n, h_n$ by

$$g_n(x) = \inf_{t \in I} f(t) \text{ for } x \in I, \quad h_n(x) = \sup_{t \in I} f(t) \text{ for } x \in I$$

where $I$ runs over binary intervals $[2^{-n}k, 2^{-n}(k+1))$, $k \in \mathbb{Z}$. For almost every $x$, $f$ is continuous at $x$, which implies $g_n(x) \uparrow f(x)$ and $h_n(x) \downarrow f(x)$ (think, why). Thus, $h_n - g_n \downarrow 0$ a.e.; also, $h_1 - g_1 \in L_1$; therefore $\int h_n - \int g_n = \int (h_n - g_n) \, dm \to 0$, which shows that $f$ is Riemann integrable.

All said generalizes readily to functions $\mathbb{R}^d \to \mathbb{R}$.

“This aesthetically pleasing integrability criterion has little practical value” (Bichteler).\(^1\) Well, if you use it when proving simple facts, such as integrability of $\sqrt{f}$ or $fg$ (for integrable $f$ and $g$), you may find far more elementary, “Lebesgue-free” proofs. But here are harder cases.

8b2 Exercise. Consider functions $f : [0, 1] \to \mathbb{R}$ such that the function

$$\text{mid}(-M, f, M) : x \mapsto \begin{cases} 
-M & \text{when } f(x) \leq -M, \\
M & \text{when } M \leq f(x) \leq M, \\
f(x) & \text{when } -M \leq f(x) \leq M, 
\end{cases}$$

is integrable for all $M > 0$. Prove that the sum of two such functions is also such function.

8b3 Exercise. Let $f, g : [0, 1] \to \mathbb{R}$ be Riemann integrable, $A \subset \mathbb{R}^2$, $\forall x \in [0, 1] \, (f(x), g(x)) \in A$, and $\varphi : A \to \mathbb{R}$ continuous and bounded.\(^2\) Then the function $x \mapsto \varphi(f(x), g(x))$ is Riemann integrable.

Prove it.

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\(^1\)From book “Integration — a functional approach” by Klaus Bichteler (1998); see Exercise 6.16 on p. 27.

\(^2\)The set $A$ need not be closed, and $\varphi$ need not be (locally) uniformly continuous.
8b4 Exercise. Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be bounded and such that all sections $f(x, \cdot)$ and $f(\cdot, y)$ are Riemann integrable. Then
(a) $f$ need not be Riemann integrable;
(b) $f$ must be Lebesgue integrable.
Prove it.\footnote{Hint: (a) try indicator of an appropriate dense countable set; (b) $f_n(x, y) = f\left(\frac{k}{n}, y\right)$ for $\frac{k}{n} \leq x < \frac{k+1}{n}$.}

8c Improper Riemann integral

As was noted in Sect. 1b, a conditionally convergent improper Riemann integral (like $\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}$) is beyond Lebesgue integration. An absolutely convergent improper Riemann integral of a function $f : \mathbb{R} \to \mathbb{R}$ continuous a.e. is $\int f^+ - \int f^-$. Thus, consider a function $f : \mathbb{R} \to [0, \infty)$ continuous a.e. By $8b1$, the Riemann integral $\int \mathbb{1}_{[-M,M]} \min(M, f(x)) \, dx$ exists for all $M \in (0, \infty)$. We have $\int \mathbb{1}_{[-M,M]} \min(M, f(x)) \, dx \uparrow \int f(x) \, dx$ (as $M \to \infty$), the improper Riemann integral of $f$; if (and only if) it is finite, the unsigned function $f$ is improperly Riemann integrable. Now, $f : \mathbb{R} \to \mathbb{R}$ is (absolutely) improperly Riemann integrable, if (and only if) $f^-, f^+$ are, and in this case $\int f = \int f^+ - \int f^-$.\footnote{Proposition. Every (absolutely) improperly Riemann integrable function is Lebesgue integrable, with the same integral.}

Proof. WLOG, $f \geq 0$. We introduce $f_n = \mathbb{1}_{[-n,n]} \min(n, f)$ and note that $f_n \uparrow f$ and $\int f_n \uparrow \int f < \infty$. By $8a1$, $f_n \in L_1$ and $\int f_n \, dm = \int f_n$; thus, $f$ is measurable, and $\int f_n \, dm \uparrow \int f \, dm$, and so, $\int f \, dm = \int f$. \hfill $\square$

All said generalizes readily to functions $\mathbb{R}^d \to \mathbb{R}$.\footnote{Proposition. Every (absolutely) improperly Riemann integrable function is Lebesgue integrable, with the same integral.}