QUANTUM ANALOGUES OF THE BELL INEQUALITIES: THE CASE OF TWO SPATIALLY SEPARATED DOMAINS

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One investigates inequalities for the probabilities and mathematical expectations which follow from the postulates of the local quantum theory. It turns out that the relation between the quantum and the classical correlation matrices is expressed in terms of Grothendieck's known constant. It is also shown that the extremal quantum correlations characterize the Clifford algebra (i.e., canonical anticommutative relations).

The Bell inequalities are inequalities for probabilities that are valid in any local deterministic theory with hidden parameters (briefly and not entirely exactly, these theories will be said to be classical), but need not be true in quantum theory; see, for example, the surveys [1-4], and also [5, pp. 190-193], and [6]. The domain of the probability distributions, admissible in the classical theories, lends itself to a mathematical description [7] and this description is model-independent, i.e., it is not connected with any concrete physical mechanisms. On the other hand, for the probability distributions, admissible in the quantum theory, one considers usually only certain special cases; moreover, in the mentioned paper [7] one has expressed skepticism regarding the possibility of a model-independent approach to quantum probabilities. However, such an approach is possible (it has been communicated by the author in [9]) for a very general situation, allowing many domains in space-time, each in spatially separated domains is considered in more detail in this paper; one proves certain theorems, regarding this case, which have been communicated in [9]. Then we carry out a comparison of the quantum case with the classical one; here, unexpectedly, there arises the Grothendieck constant, known from the geometry of Banach spaces. It turns out that the quantum correlations exceed the classical ones at most \(\sqrt{\chi_2} \) times. In the last section we present some preliminary results for the case of three spatially separated domains.

1. Some Facts about Clifford Algebras

This auxiliary section prepares the technical means, used in the subsequent sections.

By a Clifford algebra \( \mathcal{C}(n) \) we mean a \( \mathcal{C}^* \)-algebra, generated by the Hermitian generators \( X_1, \ldots, X_n \), and by the relations \( X_k^2 = 1 \), \( X_k X_{k+l} = 0 \) for \( k, l = 1, \ldots, n, k+l \). Setting \( X(\tau) = \sum X_k \tau_k \) for \( \tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n \), we achieve an explicit \( \Theta(n) \)-invariance: \( X(\tau) = \tau X(1) \tau^* \) for \( \tau \in \mathbb{R}^n \). It is known that for even \( n \), the \( \mathcal{C}^* \)-algebra \( \mathcal{C}(n) \) is isomorphic to the \( \mathcal{C}^* \)-algebra of matrices of order \( 2^n \), while for odd \( n \), the direct sum of two matrix algebras, each of order \( 2^n \), (see, for example, [10], Subsection 17.3). Consequently, for even \( n \), \( \mathcal{C}(n) \) has, within the accuracy of a unitary equivalence, a unique
irreducible representation (in the space of dimension $2^n$), while for odd $n$, two non-equivalent irreducible representations (in the space of dimension $2^{\frac{n+1}{2}}$).

We consider the tensor product $C(n) \otimes C(n)$ of two Clifford algebras. We are interested in the following Hermitian element of the algebra $C(n) \otimes C(n)$:

$$A = \frac{1}{n} \left( X_1 \otimes X_1 + \ldots + X_n \otimes X_n \right).$$

This can be given another definition, possessing an explicit $\mathcal{O}(n)$-invariance, namely, $A$ is the mean of $X(\mathbf{x}) \otimes X(\mathbf{x})$ over all unit vectors $\mathbf{x}$. We need the spectrum of the operator $\mathbf{A}(\mathbf{x})$ in the irreducible representation $\mathbf{r}$ of the algebra $C(n) \otimes C(n)$. (For $n$ even there is one such representation, while for $n$ odd there are four.)

**Lemma 1.1.** For $n$ even, the spectrum of the operator $\mathbf{A}(\mathbf{x})$ consists of the numbers $4 - \frac{2}{n}$, with multiplicities $C_n^k$, $k = 0, 1, \ldots, n$. For $n$ odd, in two of the four irreducible representations, the spectrum of the operator $\mathbf{A}(\mathbf{x})$ consists of the numbers $4 - \frac{2}{n}$, with multiplicities $C_n^k$, $k = 0, 1, \ldots, n-1$, and in the other two representations, of the numbers $4 - \frac{2}{n}$, with multiplicities $C_n^k$, $k = 0, 1, \ldots, n-2$.

**Proof.** First we consider the joint spectrum of a collection of $n$ commuting operators $A(X_i \otimes X_i), \ldots, A(X_n \otimes X_n)$. Each point of this joint spectrum has the form $\theta = (\theta_1, \ldots, \theta_n)$, $\theta_k = \pm 1$, since $(X_k \otimes X_k)^2 = 1$. First we assume that $n$ is even and we prove that each such point $\theta$ belongs in fact to the spectrum and has multiplicity 1. We make use of the fact that the group $\mathcal{O}(n) \times \mathcal{O}(n)$ acts by automorphisms on the algebra $C(n) \otimes C(n)$.

In the space of the representations we have only a projective action of this group but even this is sufficient for our purpose. We take only the subgroup $\{1 \times \mathcal{O}(n)\}$ and in it only the commutative subgroup of transformations of the form $1 \times \mathbf{T}$, where $\mathbf{T} = \text{diag}(\tau_1, \ldots, \tau_n)$ is a diagonal matrix, $\tau_k = \pm 1$. Such a transformation maps $X_K \otimes X_k$ into $\tau_k X_K \otimes X_k$ and, correspondingly, $\theta(\mathbf{t})$, into $\mathbf{T} \mathbf{t} \mathbf{T}^T \theta(\mathbf{t})$. From this it is clear that all the points $\theta$ have the same multiplicity. This multiplicity is equal to 1 since the number $2^n$ of points is equal to the dimension of the space of representations. It remains to note that the spectrum of the operator $\mathbf{A}$ consists of points of the form $\frac{1}{n} \left( \theta_1 + \ldots + \theta_n \right)$.

The case of an odd $n$ is considered in a similar manner, except that instead of the group $\mathcal{O}(n)$ one applies $\mathcal{S}\mathcal{O}(n)$ in connection with which one imposes on $\mathcal{T}$ the condition $\tau_1 \ldots \tau_n = 1$, and the set of $2^n$ points $\theta$ is decomposed into two sets of $2^{n-1}$ points each in accordance with the two values of the product $\theta_1 \ldots \theta_n = \pm 1$, now we take into account that the space of the representation has dimension $2^{n-1}$. The remaining details are left to the reader.

In the sequel we shall be interested only in one point of the spectrum: the unity. We can see that it belongs in fact to the spectrum and its multiplicity is equal to 1.

**Lemma 1.2.** Assume that $\mathbf{A}$ is an irreducible representation of the algebra $C(n) \otimes C(n)$ and let $\psi$ be such that $\mathbf{A}(\mathbf{x}) \psi = \psi$, $|\psi| = 1$, $\mathbf{A}$ being as above. Then

$$\langle \mathbf{A}(X(\mathbf{x}) \otimes X(\mathbf{y})), \psi \rangle = \langle \mathbf{x} \cdot \mathbf{y} \rangle,$$

$$\langle \mathbf{A}(X(\mathbf{x}) \otimes \mathbf{1}) \psi, \psi \rangle = 0,$$

$$\langle \mathbf{A}(\mathbf{1} \otimes X(\mathbf{y})) \psi, \psi \rangle = 0.$$
for all $x, y \in \mathbb{R}^n$.

**Proof.** We know that the mean of $\langle \tilde{\mathcal{R}}(x) \otimes \mathcal{R}(x), \psi, \psi \rangle$ over all unit vectors $x$ is equal to $\langle \tilde{\mathcal{R}}(x), \psi, \psi \rangle = 1$; taking into account that $\langle \tilde{\mathcal{R}}(x) \otimes \mathcal{R}(x), \psi, \psi \rangle \leq 1$ for all such $x$, we obtain that

$$\langle \mathcal{R}(x) \otimes \mathcal{R}(x), \psi, \psi \rangle = 1$$

for all unit vectors $x$. In order to obtain from here the relation

$$\langle \tilde{\mathcal{R}}(x) \otimes \mathcal{R}(y), \psi, \psi \rangle = \langle x, y \rangle$$

one has to show only that

$$\langle \tilde{\mathcal{R}}(x) \otimes \mathcal{R}(y), \psi, \psi \rangle - \langle \tilde{\mathcal{R}}(x) \otimes \mathcal{R}(x), \psi, \psi \rangle = 0.$$

In fact, the asymmetric bilinear form is uniquely determined by its quadratic form. We note that the operator $\tilde{\mathcal{A}}$ is invariant under the automorphism of the algebra $C(H) \otimes C(H)$, which takes $X(x) \otimes 1$ into $1 \otimes X(x)$ and conversely. For $H$ even, to this automorphism there corresponds a unitary operator in the space of the representations; taking into account that the equality $\tilde{\mathcal{R}}(A) \psi = \psi$ determines the vector $\psi$ within the accuracy of a scalar factor, we obtain the required symmetry of the bilinear form. We leave to the reader the case of odd $H$, we only note that for $H = 4m + 1$ the above mentioned automorphism is suitable, while for $H = 4m + 3$ one has to apply the automorphism which takes $X(x) \otimes 1$ into $1 \otimes X(x)$ and conversely. It remains to show that $\langle \tilde{\mathcal{R}}(x) \otimes 1, \psi, \psi \rangle = 0$. But this can be easily obtained from the $SO(H)$-invariance.

**Lemma 1.9.** Assume that the representation $\tilde{\mathcal{R}}$ of the algebra $C(H) \otimes C(H)$ is such that there exists a cyclic (in another terminology: totalizing) vector $\psi$ with the property $\tilde{\mathcal{R}}(A) \psi = \psi$. Then the representation $\tilde{\mathcal{R}}$ is either irreducible, or it is a sum of two nonequivalent irreducible representations (of course, the second case is possible only for odd $H$).

**Proof.** The existence of irreducible subrepresentations does not cause any doubt in view of the finite-dimensionality of the algebra. One has to prove only that the representation $\tilde{\mathcal{R}}$ cannot contain two equivalent irreducible subrepresentations. We assume that the opposite: $\tilde{\mathcal{R}}$ is equivalent to the sum $\tilde{\mathcal{R}}_1 \oplus \tilde{\mathcal{R}}_1 \oplus \tilde{\mathcal{R}}_2$ and $\tilde{\mathcal{R}}_1$ is irreducible; then correspondingly $\psi = \psi_1 \otimes \lambda \psi_1 \otimes \psi_2$, in fact the property $\tilde{\mathcal{R}}_1(A) \psi = \psi_1$ determines the vector $\psi_1$ within the accuracy of a scalar factor. But the space of vectors of the form $\psi_1 \otimes \lambda \psi_1 \otimes \psi_2$ (with fixed $\lambda$ and arbitrary $\psi_1, \psi_2$) is invariant relative to $\tilde{\mathcal{R}}_1 \oplus \tilde{\mathcal{R}}_1 \oplus \tilde{\mathcal{R}}_2$, this contradicts the cyclicity of the vector $\psi$.

2. Quantum Realizability of Correlation Matrices

An $m \times n$ matrix $C$ with real elements is said to be a quantum realized correlation matrix if it satisfies the first (or the second) conditions of the subsequent theorem. For the set of all quantum realized correlation $m \times n$ matrices we introduce the notation $\text{Cov}(m, n)$. The fourth (or the fifth) condition of the subsequent theorem yields a simpler geometric description of such matrices.

**Theorem 2.1.** For any $m \times n$ matrix $C = \{c_{k\ell}\}$ with real elements, the following five statements are equivalent:
(1) There exists a $C^*$-algebra $\mathcal{A}$ with identity, Hermitian elements $A_1, ..., A_m, B_1, ..., B_n \in \mathcal{A}$, and a state $\varphi$ on $\mathcal{A}$ such that for any $k, \ell, z, \bar{z}$, we have:

$$\varphi(A_k B_\ell B_{\bar{z}} \bar{A}_z) = c_k \ell.$$

$$-1 \leq A_k \leq 1; \quad -1 \leq B_\ell \leq 1;$$

$$\varphi(A_k B_\ell) = c_k \ell.$$

(2) There exist Hermitian operators $A_1, ..., A_m, B_1, ..., B_n$ and a density matrix $\psi$ in the Hilbert space $H$ of finite or countable dimension, such that for any $k, \ell$, one has $A_k B_\ell = B_\ell A_k$; the spectrum of each of the operators $A_k, B_\ell$ lies on $[-1, 1]$;

$$\text{Tr}(A_k B_\ell \psi) = c_k \ell.$$

(3) The same as (2) and, in addition, $A_k^2 = 1$, $B_\ell^2 = 1$, $\text{Tr}(A_k \psi) = 0$, $\text{Tr}(B_\ell \psi) = 0$ for all $k, \ell$; and $H = H_1 \otimes H_2$, $A_k = A_k^{(1)} \otimes A_k^{(2)}$, $B_\ell = B_\ell^{(1)} \otimes B_\ell^{(2)}$, where $A_k^{(1)}, B_\ell^{(2)}$ are some operators in $H_1, H_2$, respectively; $I^{(1)}, I^{(2)}$ are the identity operators; and all the anticommutators $A_k^{(1)} A_k^{(2)} + A_k^{(2)} A_k^{(1)}$, $B_\ell^{(1)} B_\ell^{(2)} + B_\ell^{(2)} B_\ell^{(1)}$, are scalar operators (i.e., multiples of the identity operators) and the spaces $H_1, H_2$ are finite-dimensional, moreover

$$2 \log_2 \dim H_1 \leq \begin{cases} m & \text{for even } n, \\ m+1 & \text{for odd } n, \\ n & \text{for even } n, \\ n+1 & \text{for odd } n. \end{cases}$$

$$2 \log_2 \dim H_2 \leq \begin{cases} n & \text{for even } n, \\ n+1 & \text{for odd } n. \end{cases}$$

(4) In the Euclidean space of dimension $m+n$, there exist unit vectors $x_1, ..., x_m, y_1, ..., y_n$, such that $\langle x_k, y_\ell \rangle = c_k \ell$ for all $k, \ell$.

(5) In the Euclidean space of dimension $\min(m, n)$, there exist vectors $x_1, ..., x_m, y_1, ..., y_n$, such that $|x_k| \leq 1$, $|y_\ell| \leq 1$, and $\langle x_k, y_\ell \rangle = c_k \ell$ for all $k, \ell$.

**Proof.** Clearly, (3) $\Rightarrow$ (2) $\Rightarrow$ (1). We show that (1) $\Rightarrow$ (5). The algebra $\mathcal{A}$ can be considered as a real linear space with inner product $\langle x, y \rangle = \text{Re} \{ (x^* y) \}$ for $x, y \in \mathcal{A}$ (the corresponding quadratic form may be degenerate but that does not change the facts); then we have $\langle A_k A_{k'} \rangle \leq 1$, $\langle B_\ell B_{\bar{\ell}} \rangle \leq 1$, and $\langle A_k, B_{\bar{k}} \rangle = c_k \ell$ for all $k, \ell$. Thus, all the requirements of part (5) are satisfied except $\dim H \geq \min(m, n)$. The dimension of the space has to be at most $\min(m, n)$. This can be easily achieved by the orthogonal projection of the vectors $x_k$ onto the subspace generated by the vectors $y_\ell$, or in the other way around.

We show that (5) $\Rightarrow$ (4). One has to make the vectors to be unit vectors. For this, it is sufficient that instead of $x_k$ and $y_\ell$ one should take $x_k + y_\ell$ and $y_\ell + y_\ell$, respectively, where $x_k$ and $y_\ell$, are chosen to be orthogonal to each other and to the vectors $x_k \otimes y_\ell$, then $\langle x_k + y_\ell, y_k + y_\ell \rangle = c_k \ell$, and for a proper choice of the lengths of the vectors $x_k, y_\ell$, we obtain $|x_k + y_\ell| = 1, |y_\ell + y_\ell| = 1$. Of course, this construction may require one to go into a space of larger dimension; but afterwards one can restrict oneself to the space generated by the vectors $x_k, y_\ell$; this dimension does not exceed $m+n$.

Thus (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (5) $\Rightarrow$ (4). It remains to prove that (4) $\Rightarrow$ (3). Consider the vectors $x_k, y_\ell$ from $\mathbb{R}^{m+n}$. We consider the Clifford algebra $\mathcal{C}(m+n)$. According to what has been proved in Sec. 1, there exists a state $\varphi$ on the algebra $\mathcal{C}(m+n) \otimes \mathcal{C}(m+n)$ such that

560
\[ f(X(x) \otimes X(y)) = \langle x, y \rangle, \]
\[ f(I \otimes X(y)) = 0, \]
\[ f(I \otimes X(x)) = 0 \]
for all \( x, y \in \mathbb{R}^{m+n} \). Setting \( A_x = X(x) \otimes \mathbf{1}, B_y = I \otimes X(y) \), we obtain \( f(A_x B_y) = C_{x,y} \), \( f(A_x) = 0, f(B_y) = 0 \). The operators \( X(x) \) generate in \( C(m+n) \) a subalgebra isomorphic to \( C(m') \) with some \( m' \leq m \); similarly, \( X(y) \in C(n') \subset C(m+n), n' \leq n \). We restrict the state \( f \) from the algebra \( C(m+n) \otimes C(m+n) \) to the algebra \( C(m') \otimes C(n') \). We take a faithful representation of the algebra \( C(m') \) into the space \( H_m \) whose dimension is equal to \( 2^m \) for even \( m' \) and to \( 2^{m'-1} \) for odd \( m' \); we represent similarly \( C(n') \) into \( H_n \); the tensor product gives a faithful representation of \( C(m') \otimes C(n') \) into \( H = H_m \otimes H_n \). The state \( f \) on \( C(m') \otimes C(n') \) is realized by a density matrix in \( H \).

All the requirements of part (3) hold.

The proved Theorem 2.1 reduces the question of the quantum realizability of a correlation matrix to the simpler question of the existence of finite dimensional vectors, but does not give an explicit solution for this problem. For the simplest case when \( m = 2 \) and \( n = 2 \) an explicit solution is given by the following theorem.

**Theorem 2.2.** A \( 2 \times 2 \) matrix \( C \) with real elements is a quantum realized correlation matrix if and only if \( |C_{k \ell}| \leq 1 \) for all \( k, \ell \) and if at least one of the following two inequalities holds:

\[ 0 \leq \sum_{k \ell} \left( C_{k \ell}^2 - C_{k \ell} C_{k \ell} - C_{k \ell} C_{k \ell} - C_{k \ell} C_{k \ell} \right) \leq \frac{1}{2} \left( \sum_{k \ell} C_{k \ell}^2 \right)^2 - \frac{1}{2} \sum_{k \ell} C_{k \ell}^2 - 2 \sum_{k \ell} C_{k \ell} C_{k \ell} C_{k \ell} C_{k \ell} , \]

\[ 0 \leq 2 \max_{k \ell} C_{k \ell}^2 - \left( \max_{k \ell} C_{k \ell}^2 \right)^2 + 2 \sum_{k \ell} C_{k \ell} C_{k \ell} C_{k \ell} C_{k \ell} . \]

The proof is elementary (in fact, one has four vectors in the plane) but cumbersome; we shall not give it. The same refers to the following question, giving the explicit solution of the dual problem.

**Theorem 2.3.** Assume that there is a given \( 2 \times 2 \) matrix \( \gamma \) with real elements, and let

\[ M = \sup \sum_{k \ell} Y_{k \ell} C_{k \ell} , \]

where the supremum is taken over all quantum realized \( 2 \times 2 \) correlation matrices \( C \). Then

1. if \( \prod_{k \ell} Y_{k \ell} > 0 \), then \( M - \sum_{k \ell} \left| Y_{k \ell} \right| \);  
2. if \( \prod_{k \ell} Y_{k \ell} < 0 \) and \( \left( \min_{k \ell} \left| Y_{k \ell} \right| \right) \left( \sum_{k \ell} \left| Y_{k \ell} \right| \right)^2 \leq \frac{1}{2} \), then \( M = \sum_{k \ell} \left| Y_{k \ell} \right| - 2 \min_{k \ell} \left| Y_{k \ell} \right| \);  
3. if \( \prod_{k \ell} Y_{k \ell} < 0 \) and \( \left( \min_{k \ell} \left| Y_{k \ell} \right| \right) \left( \sum_{k \ell} \left| Y_{k \ell} \right| \right)^2 \leq \frac{1}{2} \), then \( M = \sqrt{\sum_{k \ell} Y_{k \ell}^2} + \left( \prod_{k \ell} \left| Y_{k \ell} \right| \right) \left( \sum_{k \ell} Y_{k \ell}^2 \right) \).
3. Representation of Extremal Correlations

As one can easily see, the set $\text{Cor}(m,n)$ of all quantum-realized $m \times n$ correlation matrices, introduced in the previous section, is a closed, bounded, centrally symmetric, convex body in the space of $m \times n$ matrices. We consider the set $\bigcup_{\epsilon} \text{Cor}(m, n)$ of all extremal (in other terminology: extreme) points of the set $\text{Cor}(m, n)$; by a known general theorem, from the geometry of convex sets (see, at least [10]), $\text{Cor}(m, n)$ is the closed convex hull of the set $\bigcup_{\epsilon} \text{Cor}(m, n)$.

Let $C$ be an $m \times n$ matrix. By a $C$-system of vectors we mean any collection of vectors $x_1, \ldots, x_m, y_1, \ldots, y_n$ lying in some Euclidean space and satisfying the conditions $|x_k| \leq 1$, $|y_l| \leq 1$, $\langle x_k, y_l \rangle = c_{kl}$ for all $k, l$.

Clearly, a $C$-system of vectors exists if and only if $C \in \text{Cor}(m, n)$. By the rank of a $C$-system of vectors we mean the dimension of the linear hull of the vectors $x_1, \ldots, x_m, y_1, \ldots, y_n$.

**Lemma 3.1.** Let $C \in \bigcup_{\epsilon} \text{Cor}(m, n)$. Then all $C$-systems of vectors are isometric to each other and all have the following properties:

1. $|x_k| = 1$, $|y_l| = 1$ for all $k, l$.
2. The linear hull of the vectors $x_1, \ldots, x_m$ coincides with the linear hull of the vectors $y_1, \ldots, y_n$.
3. For any quadratic form $Q$, defined on the linear hull of the vectors $x_1, \ldots, x_m, y_1, \ldots, y_n$, from the inequalities $Q(x_i) = 0, \ldots, Q(x_m) = 0, Q(y_i) = 0, \ldots, Q(y_n) = 0$, there follows that $Q = 0$ identically. (a) The rank $\gamma$ of any $C$-system of vectors satisfies the inequalities $\gamma \leq m$, $\gamma \leq n$, and $\gamma \leq \frac{1}{2} + \sqrt{\frac{1}{4} + 2(m + n)}$.

**Proof.** (a). For any $C$-system of vectors the property (1) holds. Indeed, assume, for example, $|x_1| < 1$, then for sufficiently small vectors $\Delta x$ we obtain $|x_1 + \Delta x| \leq 1$ and $|x_1 - \Delta x| \leq 1$. Defining the matrix $AC$ by the equalities

$$(AC)_{kl} = \begin{cases} \langle \Delta x, y_l \rangle & \text{for } k = 1, \\ 0 & \text{otherwise}, \end{cases}$$

we obtain $C - AC \in \text{Cor}(m, n)$ and $C + AC \in \text{Cor}(m, n)$, which for $AC \neq 0$ contradicts the assumption that $C$ is an extremal point.

(b) For any $C$-system of vectors, every vector $y_l$ lies in the subspace spanned by the vectors $x_1, \ldots, x_m$. Indeed, otherwise, replacing $y_l$ by its projection onto the mentioned subspace, we would obtain a $C$-system of vectors, contradicting what has been proved at part (a). In the same way, every vector $x_k$ lies in the subspace spanned by the vectors $y_1, \ldots, y_n$.

(c) All $C$-systems of vectors are isometric to each other. Indeed, let $x_1', \ldots, x_m', y_1', \ldots, y_n'$ be a $C$-system and assume that $x_1, \ldots, x_m, y_1, \ldots, y_n$ is also a $C$-system. We consider their direct (orthogonal) sum: $x_k = \frac{1}{2} (x_k' + x_k'')$, $y_l = \frac{1}{2} (y_l' + y_l'')$. We have $|x_k|^2 = \frac{1}{2} (|x_k'|^2 + |x_k''|^2)$, $|y_l|^2 = \frac{1}{2} (|y_l'|^2 + |y_l''|^2)$, $\langle x_k', y_l' \rangle = \frac{1}{2} (\langle x_k', y_l' \rangle + 2 \langle x_k'', y_l'' \rangle)$, $\langle x_k', y_l'' \rangle = \frac{1}{2} (\langle x_k', y_l' \rangle + 2 \langle x_k'', y_l'' \rangle)$.

From where it is clear that $x_1', \ldots, x_m', y_1', \ldots, y_n'$ is again a $C$-system. According to what has
been proved at part (b), each vector \( y^*_l \) is a linear combination of the vectors \( x^*_k \): 
\[
y^*_l = \sum_k a_{kl} x^*_k.
\]
We have \( \frac{1}{\sqrt{2}} (y^*_l \otimes y^*_l) = \frac{1}{\sqrt{2}} \sum_k a_{kl} x^*_k \otimes \sum_k a_{kl} x^*_k \), whence \( y^*_l = \sum_k a_{kl} x^*_k \) and
\[
y^*_l = \sum_k a_{kl} x^*_k.
\]
Then
\[
\langle y^*_l, y^*_l \rangle = \sum_k a_{kl}^2 = \sum_k a_{kl}^2 c_{kl} = \sum_k a_{kl} c_{kl};
\]
similarly, \( \langle y^*_l, y^*_l \rangle = \sum_k a_{kl}^2 c_{kl} \); thus, \( \langle y^*_l, y^*_l \rangle = \langle y^*_l, y^*_l \rangle \) for all \( \ell, \ell \). In the same way one proves that \( \langle x^*_k, x^*_k \rangle = \langle x^*_k, x^*_k \rangle \) for all \( k, k \). Finally, \( \langle x^*_k, y^*_l \rangle = c_{kl} \).

(d) We show part (3). We assume the opposite: there exists a form \( \mathcal{Q} \), not identically zero and vanishing at \( x^*_k \cdot y^*_l \). On the subspace generated by these vectors, we consider a new metric \(|\| \cdot \|_\mathcal{Q} \), defined by the equality \( |z|^2 = |z|^2 + \mathcal{Q}(z) \); \( \mathcal{Q} \) is chosen to be small so that the right-hand side be positive definite. We have \( |x^*_k|^2 = 1, |y^*_l|^2 = 1 \). We define a matrix \( \Delta \mathcal{C} \) in terms of the symmetric bilinear form, corresponding to the quadratic form \( \mathcal{Q} : (\Delta \mathcal{C})_{kl} = \mathcal{Q}(x^*_k, y^*_l) \), then \( \langle x^*_k, y^*_l \rangle = (\mathcal{C} + \mathcal{E} \Delta \mathcal{C})_{kl} \). We can see that \( \mathcal{C} + \mathcal{E} \Delta \mathcal{C} \in \text{Cor}(m,n) \) for all small \( \mathcal{E} \), both positive and negative. For \( \Delta \mathcal{C} \neq 0 \), this contradicts the assumption that \( \mathcal{C} \) is an extremal point.

(e) Part (4) follows from part (3), already proved. Indeed, the space of quadratic forms on an \( n \)-dimensional space has dimension \( \frac{1}{2}(n+1) \); part (3) shows that any of these forms is uniquely defined by a collection of \( m+n \) of its values; thus, \( \frac{1}{2} \leq \mathcal{E}(1+1) \leq m+n \), whence \( \mathcal{E} \leq -\frac{1}{2} + \sqrt{\frac{1}{4} (2m+n+1)} \). The inequalities \( \mathcal{E} \leq \mathcal{E} \leq \mathcal{E} \) are obvious from the previously proved part (2). The lemma is proved.

Assume that \( \mathcal{C} \) is an \( m \times n \) matrix with real elements. By an operator representation of the \( \mathcal{C} \) correlations we mean any collection \( (H, W, A_1, \ldots, A_m, B_1, \ldots, B_n) \), consisting of a Hilbert space \( H \) of finite or countable dimension, a density matrix \( W \) in \( H \), and the Hermitian operators \( A_1, \ldots, A_m, B_1, \ldots, B_n \) in \( H \), which satisfy the conditions:

the spectrum of each of the operators \( A_k, B_l \) lies on \([-1,1]\), \( A_k B_l = B_l A_k \), \( Tr(A_k B_l) = \mathcal{C}_{kl} \) for all \( k, l \).

Clearly, an operator representation of the \( \mathcal{C} \) correlations exists if and only if \( \mathcal{C} \in Cor(m,n) \).

We define the obvious manner the unitary equivalence of two such representations (for this we then use that not only \( A_k, B_l \) should "coincide," but also \( W \)).

If the density matrix \( W \) is one-dimensional, then such a representation is said to be pure.

If in \( H \) there exists a projection \( P \) commuting with all \( A_k, B_l \) and such that \( PW = W \), then one can define in an obvious manner a subrepresentation of the given representation. If, however, there is no such \( P \) (other than the identity, of course), then the representation is said to be nondegenerate. Clearly, every operator representation of the \( \mathcal{C} \) correlations contains a unique nondegenerate subrepresentation.

An operator representation of the \( \mathcal{C} \) correlations is said to be factorial if the von Neumann algebra, generated by the operators \( A_1, \ldots, A_m, B_1, \ldots, B_n \), is factorial.
Having two operator representations of the $C$-correlations

$$\left( H^{(1)}, W^{(1)}, A^{(1)}, \ldots, A^{(w)}, B^{(1)}, \ldots, B^{(v)} \right), \quad i = 1, 2,$$

and given positive coefficients $\alpha_1, \alpha_2$, such that $\alpha_1 \cdot \alpha_2 = 1$, we can construct the direct sum of the given representations:

$$H = H^{(1)} \oplus H^{(2)}, \quad W = \alpha_1 W^{(1)} \oplus \alpha_2 W^{(2)}$$

$$A_K = A_K^{(1)} \oplus A_K^{(2)}, \quad B_k = B_k^{(1)} \oplus B_k^{(2)}.$$

In addition, instead of $\alpha_1 W^{(1)} \oplus \alpha_2 W^{(2)}$ one can take any density matrix $\tilde{W}$ in $H = H_1 \oplus H_2$ such that $p^{(1)} \tilde{W} p^{(1)} = \alpha_1 W^{(1)} \oplus 0, \quad p^{(2)} \tilde{W} p^{(2)} = 0 \oplus \alpha_2 W^{(2)}$ (here $p^{(1)}, p^{(2)}$ are the projections onto $H_1 \oplus 0, 0 \oplus H_2$, respectively); any representation obtained in this manner will be called a pseudodirect sum of the two given representations.

The operator representation of the $C$-correlations is said to be Clifford if all the anticommutators $[A_K, A_{K'}, B_k, B_{k'}] = \epsilon_{K K'} A_K A_{K'} + B_k B_{k'}$ are scalar (i.e., multiples of the identity operator).

As shown by Theorem 2.1, Clifford representations exist for all $C \in \text{Cor}(m, n)$.

**Theorem 3.1.** Let $C \in \text{ExCor}(m, n)$. Then any nondegenerate operator representation of the $C$-correlations is Clifford.

**Theorem 3.2.** Let $C \in \text{ExCor}(m, n)$ and assume that the rank of any $C$-system of vectors is an even number. Then:

1. All the pure, nondegenerate operator representations of $C$-correlations are unitarily equivalent.
2. Any nonpure, nondegenerate operator representation of $C$-correlations decomposes into the direct sum of a finite or countable family of pure nondegenerate representations.

**Theorem 3.3.** Let $C \in \text{ExCor}(m, n)$ and assume that the rank of any $C$-system of vectors is an odd number. Then:

1. There exists exactly two unitarily nonequivalent, pure, factorial, nondegenerate operator representations of $C$-correlations.
2. Any nonpure, factorial, nondegenerate operator representation of $C$-correlations decomposes into a direct sum of a finite or a countable number of unitarily equivalent to each other, pure, factorial, nondegenerate representations.
3. Any nonfactorial, nondegenerate operator representation of $C$-correlations decomposes into a pseudodirect sum of two factorial, nondegenerate representations.

Proof of Theorem 3.1. The algebra of operator in $H$ can be considered as a real linear space with inner product $\langle X, Y \rangle = \text{Re} \, \text{Tr}(X^* Y W)$. Moreover, one has to take into account that the equality $\langle X, X \rangle = 0$ means only $X W = 0$ and not necessarily $X = 0$. We have

$$\langle A_{K K'}, A_{K K'} \rangle \leq 1, \quad \langle B_k, B_{k'} \rangle \leq 1, \quad \langle A_{K K'}, B_{k k'} \rangle = c_{K K'};$$

thus, we have obtained a $C$-system of vectors. By virtue of Lemma 3.1, all such systems are isometric. Let $x_1, x_m, y_1, \ldots, y_n$ be a $C$-system of vectors, lying in $\mathbb{R}^r$, where $r$ is the rank of any $C$-system. From the mentioned lemma we know that every vector $x_\alpha$
Is a linear combination of vectors \( y_\ell \), \( x_\ell = \sum \alpha_{\ell \ell} y_\ell \). In view of the isometry we obtain from here \( A_\ell W' = \sum \alpha_{\ell \ell} B_\ell W' \). Taking into account the nondegeneracy of the representation, we can see that the closed linear hull of the vectors of the form \( B_{\ell 1} \cdots B_{\ell \nu} W \psi \) ( \( \nu \) runs through \( 1, \ldots, n \); \( \psi \) runs through \( \mathcal{H} \)) coincides with the space \( \mathcal{H} \). Indeed, it is invariant not only relative to all \( B_{\ell \ell} \) but also relative to all \( A_\ell \). In fact, \( A_\ell B_{\ell 1} \cdots B_{\ell \nu} W \psi = B_{\ell 1} \cdots B_{\ell \nu} A_\ell W \psi = \sum \alpha_{\ell \ell} B_{\ell 1} \cdots B_{\ell \nu} B_{\ell \ell} W \psi \).

We can see now that for the operators \( X \), commuting with \( B_{\ell 1}, \ldots, B_{\ell n} \), the equality \( \langle X, X \rangle = 0 \) for \( X W = 0 \) implies \( X = 0 \); in fact \( X B_{\ell 1} \cdots B_{\ell \nu} W \psi = B_{\ell 1} \cdots B_{\ell \nu} X W \psi = 0 \). The same holds also for the operators which commute with \( A_{\ell 1}, \ldots, A_{\ell n} \).

From the above mentioned lemma we also know that \( |x_\ell| = 1 \); in view of the isometry we obtain from here \( \Re \text{Tr}(A_\ell^* A_\ell W) = 1 \), i.e., \( \text{Tr}((1 - A_\ell^* A_\ell) W) = 0 \); taking into account that \( 1 - A_\ell^* A_\ell \geq 0 \), we conclude that \( (1 - A_\ell^* A_\ell) W = 0 \). But \( 1 - A_\ell^* A_\ell \) commutes with \( B_{\ell 1}, \ldots, B_{\ell n} \); thus, \( A_\ell^* A_\ell \equiv 1 \) for all \( \ell \). Similarly, \( B_{\ell \ell}^\dagger = B_{\ell \ell} \) for all \( \ell \). Then we have

\[
(\alpha_{\ell 1} + \sum \alpha_{\ell \ell} B_{\ell \ell}) (A_\ell - \sum \alpha_{\ell \ell} B_{\ell \ell}) W = 0,
\]

i.e., \((1 - \sum \alpha_{\ell \ell} B_{\ell \ell}) W = 0\), whence \((\sum \alpha_{\ell \ell} B_{\ell \ell})^2 = 1\) for all \( \ell \).

Finally, we make use of part (3) of the above mentioned lemma. It shows that any quadratic form on \( \mathbb{R}^n \) is uniquely determined by its values at the points \( x_1, \ldots, x_m, y_1, \ldots, y_n \). This holds for quadratic forms with real values; but then this is automatically true also for forms with values in any linear space, in particular, for forms with operator values.

We define a quadratic form \( Q \) on the space \( \mathbb{R}^n \), having operator values, by the following equality

\[
Q (\sum \beta_{\ell} y_\ell) - (\sum \beta_{\ell} B_{\ell} y_\ell)^2 \quad \text{for any} \quad \beta_1, \ldots, \beta_n.
\]

This is well defined since for those \( \beta_\ell \) for which \( \sum \beta_{\ell} y_\ell = 0 \), we have \( \sum \beta_{\ell} B_{\ell} y_\ell = 0 \) and, therefore, \( \sum \beta_{\ell} B_{\ell} y_\ell = 0 \). The form \( Q \) has the properties

\[
Q (x_\ell) = B_{\ell \ell}^2 = 1,
\]

\[
Q (x_\ell) - Q (\sum \alpha_{\ell \ell} B_{\ell} y_\ell) = (\sum \alpha_{\ell \ell} B_{\ell \ell})^2 = 1 \quad (\text{for all } x_\ell, y_\ell).
\]

Thus, at the points \( x_1, \ldots, x_m, y_1, \ldots, y_n \), it coincides with the form \( Q_\ell (z) = |z|^2 \). In this case, these forms coincide identically: \( (\sum \beta_{\ell} y_\ell)^2 = (\sum \beta_{\ell} y_\ell) \) for any \( \beta_1, \ldots, \beta_n \).

From here, \( B_{\ell 1} B_{\ell 2} + B_{\ell 2} B_{\ell 1} = 2 \langle y_\ell, y_\ell \rangle \). Similarly, we obtain \( A_{\ell 1} A_{\ell 2} + A_{\ell 2} A_{\ell 1} = 2 \langle x_\ell, x_\ell \rangle \).

Proof of Theorem 3.2. From the given proof of Theorem 3.1 we can see how one constructs a Clifford representation: one selects a \( \mathbb{C} \) -system of vectors \( x_1, \ldots, x_m, y_1, \ldots, y_n \) in the space \( \mathbb{R}^n \), where \( \mathbb{R} \) is the ring of such a system, one constructs the tensor product \( \mathcal{C}(\mathbb{R}) \otimes C(\mathbb{R}) \) of two Clifford algebras, one considers some representation \( \mathcal{A} \) of the algebra \( C(\mathbb{R}) \otimes C(\mathbb{R}) \), and one introduces the operators \( A_{\ell 1} = a_1(x_\ell) \otimes 1 \), \( B_{\ell 2} = a_2(1 \otimes y_\ell) \). The density matrix \( \mathcal{W} \) defines on the algebra \( \mathcal{C}(\mathbb{R}) \otimes C(\mathbb{R}) \) a state \( \mathcal{W} \) such that \( \mathcal{W} (x_\ell \otimes X(y_\ell)) = \langle x_\ell, y_\ell \rangle \) and, consequently, \( \mathcal{W} (x_\ell \otimes X(y_\ell)) = \langle x_\ell, y_\ell \rangle \) for all \( x_\ell, y_\ell \in \mathbb{R}^n \). From this
it is clear that $f(A) = 1$, where the element $\hat{A}$ of the algebra $\mathbb{C}(\tau) \otimes \mathbb{C}(\tau)$ is defined in Sec. 1.

We prove part (1). The one-dimensional density matrix $\hat{W}$ corresponds to some unit vector $\Psi \in \mathcal{H}$. The nondegeneracy of the given representation means that $\Psi$ is a cycle vector for the representation $\pi$ of the algebra $\mathbb{C}(\tau) \otimes \mathbb{C}(\tau)$. Applying Lemma 1.3, we conclude that the representation $\pi \otimes \pi$ is irreducible. Thus, it is uniquely defined within the accuracy of a unitary equivalence. It remains to take into account that the equality $\langle A\Psi, \Psi \rangle = 1$ determines uniquely the vector $\Psi$ within the accuracy of a scalar factor.

We prove part (2). The representation $\hat{W}$ of the factor $\mathbb{C}(\tau) \otimes \mathbb{C}(\tau)$ has automatically (within the accuracy of a unitary equivalence) the form $\hat{W}(A) = \pi_1(A) \otimes I$, where $\pi_1$ is an irreducible representation in the space $H_1$, $I$ is the identity operator in $H_2$, and $H = H_1 \otimes H_2$. The density matrix $\hat{W}$ has the property $\text{Tr}((\pi_1(A) \otimes I) \hat{W}) = 1$. We know (Lemma 1.1) that the eigenvalue 1 of the operator $\pi_1(A)$ has multiplicity 1. From this there follows that $\hat{W} = \hat{W}_1 \otimes \hat{W}_2$, where $\hat{W}_1$ is a one-dimensional density matrix in $H_1$ corresponding to the one-dimensional eigenspace for $\pi_1(A)$, and $\hat{W}_2$ is some density matrix in $H_2$. We have $\hat{W}_2 = \sum_n \alpha_n \hat{W}_{2,n}$, where $\alpha_n > 0$, $\sum_n \alpha_n = 1$, $\hat{W}_{2,n}$ are one-dimensional density matrices, corresponding to orthogonal one-dimensional subspaces $H_{1,n}$. We obtain the desired decomposition: $H = \bigoplus_n (H_1 \otimes H_{2,n})$.

The proof of Theorem 3.3 differs little from the proof of Theorem 3.2; we leave it to the reader.

4. Comparison with the Classical Case

The classical case can be characterized by the fact that all the operators $A_1, \ldots, A_m, B_1, \ldots, B_N$ commute. Instead of the commuting operators and density matrices one can consider random variables. Thus, we are interested in matrices $C$ of dimension $m \times n$ represented in the following form:

$$C_{k\ell} = E A_k B_{\ell};$$

where $E$ is the mathematical expectation, $A_1, \ldots, A_m, B_1, \ldots, B_N$ are random variables with $|A_k| \leq 1$ and $|B_{\ell}| \leq 1$ with probability 1 for all $k, \ell$. It is easy to see that the set of such matrices is a convex polyhedron; its vertices are matrices of rank 1: $C_{k\ell} = \alpha_k \ell_{\ell}$, where all $\alpha_k, \ell_{\ell}$ are equal to $\pm 1$.

Proceeding in the vein of the previous section, we introduce for this polyhedron the notation $(\mathcal{N}_m(n, n))$ in connection with the fact that its extreme points correspond to the systems of vectors of rank 1. Of course, $\mathcal{N}_m(n, n) \subset \mathcal{N}_m(n, n)$. The noncoincidence of these sets is a fundamental fact; exactly this is responsible for the existence of the scientific orientation connected with Bell's inequalities.

In Bell's pioneering investigations one has indicated a point of $\mathcal{N}_m(2, 2)$ and a hyperplane separating this point from $\mathcal{N}_m(2, 2)$. In subsequent investigations one has gradually outlined the set $(\mathcal{N}_m(2, 2))$; its complete explicit description has appeared in [7] and [8]. Apparently, the investigation of the set $\mathcal{N}_m(2, 2)$, as well as that of $\mathcal{N}_m(n, n)$, has begun in the author's paper [9].

566
It is natural to ask how much larger is $\text{Cov}(m,n)$ than $\text{Cov}_m(m,n)$. Some attempt to approach this question can be noticed in [7], where to each of the Bell inequalities one associates a numerical characteristic $\gamma$, showing how strongly it is violated in the quantum theory; however, the latter is presented only by a spin correlation experiment according to Bohm's scheme. We propose to pose this problem in the following manner: which is the smallest number $K(m,n)$ having the property that

$$\text{Cov}(m,n) \leq K(m,n) \cdot \text{Cov}_4(m,n),$$

i.e., such that $\{1/K(m,n)\} \text{C}$ belongs to $\text{Cov}_4(m,n)$ for any matrix $\text{C}$ from $\text{Cov}(m,n)$. It is easy to see that $K(m,n)$ is an increasing function of $m, n$; there arises the question whether the limit $K = \lim_{m,n \to \infty} K(m,n)$ is finite or not.

As long as the set $\text{Cov}(m,n)$ is determined in terms of $\mathbb{C}^*$-algebras or in terms of operators in a Hilbert space, the question of the constant $K$ from a mathematical point of view is rather special. But Theorem 2.1 shows that the set $\text{Cov}(m,n)$ can be determined in a significantly simpler manner in terms of vectors in a Euclidean space. Then it becomes clear that the constant $K$ is nothing else but Grothendieck's well known constant $K_G$, investigated by mathematicians from 1956 up to now!

Regarding the problem of the Grothendieck constant, we refer the reader to [11]. It is proved there that

$$K_G \leq \frac{\sqrt{3}}{2 \sqrt{2m(1+\sqrt{2})}} \approx 0.782.$$

The constant $K_G(k)$, considered there, differs from the above introduced $K(m,n)$. For given $m, n$, we select $k$ in such a manner that for any $C \in \text{Cov}(m,n)$ the rank of the $C$-system of vectors does not exceed $k$; then $K(m,n) \leq K_G(k)$. As shown in Sec. 3, $k$ can be chosen so that $2k \leq m, n \leq 2k$.

For small $n$, in [11] one has obtained: $K_G(2) = \sqrt{2} - 1; K_G(3) < 1.547; K_G(4) \approx 2.34$. We note that

$$\sqrt{2} \leq K(2,2) \leq K(2,n) \leq K_G(2) = \sqrt{2},$$

from where $K(2,n) = \sqrt{2}$ for all $n \geq 2$. In this sense one can assert that the quantum realized correlation matrix $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, well known among specialists in the Bell inequalities, is optimal (by its "nonclassical" character) not only among the $2 \times 2$ matrices but also among the $2 \times n$ matrices. We define the set $\text{Cov}_4(m,n)$ as the closed convex hull of the set of all $n \times m$ matrices $C$ of the form $C_{KL} = \begin{pmatrix} x_k & y_l \\ y_l & x_k \end{pmatrix}$, where $x_k, y_l$ are unit vectors of the $n$-dimensional and $m$-dimensional Euclidean space. In other words, $\text{Cov}_4(m,n)$ consists of matrices $C$ of the form $C_{KL} = \begin{pmatrix} A_K & B_L \\ B_L & A_K \end{pmatrix}$, where $A_K, B_L$ are random vectors with values in the $n$-dimensional and $m$-dimensional Euclidean space and $|A_K| \leq 1, |B_L| \leq 1$ with probability 1. We have

$$\text{Cov}_4(m,n) \subseteq \text{Cov}_4(n,m) \subseteq \text{Cov}_4(m,n).$$

If to the condition (1) of Theorem 2.1 one adds the requirement that the linear span of the observables $A_1, \ldots, A_m$ should have dimension at most $\gamma$, then the corresponding correlations matrices are in $\text{Cov}_4(m,n)$; this is clear from the proof of the mentioned theorem. In correlation experiments, related with the Bell in-
equalities, for the observables one applied frequently the projections of the spin onto certain directions and one makes use of particles spin \(1/2\). The linear span of these observables is not greater than three-dimensional (in fact all of them are represented by Hermitian \(2 \times 2\) matrices with zero trace). The correlation matrices obtained in this way belong to \(\mathcal{C}_\mathcal{V}(m,n)\) consequently, they exceed the classical ones at most \(K_\mathcal{V}(3)\) \(\leq \sqrt{5}/7\) times. If, however, as it is frequently done, we restrict ourselves to the projections of the spin onto directions lying in one selected plane, or one considers experiments with polarization photons and the observables are taken so that each of them is equal to \(-1\) in some state with plane polarization and to \(+1\) in another such state, then in a similar manner we obtain two-dimensional span of the observables, the set \(\mathcal{C}_\mathcal{V}(m,n)\) and the constant \(K_\mathcal{V}(2)\) \(=\sqrt{2}\).

In conclusion, we rewrite in our notation the explicit description of the sets \(\mathcal{C}_\mathcal{V}(m,n)\) and \(\mathcal{D}_\mathcal{V}(2,2)\) obtained in [7, 8]. For this, we use the notation \(\mathcal{D}_\mathcal{V}(m,n)\) for the set of those collections \((a, b, c)\), consisting of an \(m\)-vector \(a\), an \(n\)-vector \(b\), and an \(m \times n\)-matrix \(c\) which can be represented in the form \(a^* = \sum_k \alpha_k a_k\), \(b^* = \sum_k \beta_k b_k\), \(c_{kl} = \sum_i \sum_j \gamma_{ij} c_{ij} \delta_{k} \delta_{l}\); as before, the random variables \(a_k\), \(b_k\) are subjected to the conditions \(|a_k| \leq 1\), \(|b_k| \leq 1\).

The set \(\mathcal{D}_\mathcal{V}(2,2)\) is defined by the system eight double inequalities

\[-1 \leq |a_k + b_k| \leq 1 - |a_k - b_k|,\]

\[-1 \leq |c_{11} + c_{12} + c_{21} + c_{22} - 2c_{kk}| \leq 2 ;\]

where \(k = 1, 2\); \(l = 1, 2\).

The set \(\mathcal{C}_\mathcal{V}(2,2)\) can be defined by the single inequality

\[|c_{11} + c_{12} + c_{21} + c_{22}| \leq |c_{11} + c_{12} - c_{21} - c_{22}| + |c_{11} - c_{12} + c_{21} - c_{22}| + |c_{11} - c_{12} - c_{21} + c_{22}| \leq 4 .\]

One can get rid of the absolute value symbols and obtain 16 linear inequalities; from them eight inequalities are trivial and the other eight are known as the Bell/CH, CHSH, etc. inequalities (CHSH stands for Clauser, Horne, Shimony, Holt.)

5. The Case of Three Domains

To three spatially separated domains there correspond three collections of observables \(A_1, \ldots, A_p; B_1, \ldots, B_p; C_1, \ldots, C_q\), and all the commutators \([A_k, B_k], [B_k, C_k], [C_k, A_k]\) vanish in every state \(\rho\) we define pair correlations \(\text{tr}(A_k B_k \rho), \text{tr}(B_k C_k), \text{tr}(C_k A_k)\) generating three matrices, and triple correlation \(\text{tr}(A_k B_k C_k \rho)\) generating a trivalent tensor. Here we shall consider only dual correlations. Clearly, each of the three correlation matrices must be quantum realized in the sense of the definition given before for the case of two domains; this, of course, is a necessary but not sufficient condition for the joint quantum realization of a triple \(\mathcal{V}\) of correlation matrices.

It turns out that for three domains the situation is entirely different from the case of two domains. Vectors in the Euclidean space are not suitable anymore and the quadratic operator inequalities do not work; in a series of cases the quantum boundaricos coincide unexpectedly with the classical ones. The fact is that the presence of a quantum correlation between two objects restricts their possible correlations with any other objects; and if two
objects are connected by a "full" correlation, then each of them emerges, with respect to any third object, as the classical one; in fact its state is "somehow known," i.e., in some sense it is subjected to measurement; and the measurement acts in a destructive manner on the quantum relations. What has been said is an attempt to comment on the precise statements, formulated below. Their proof will be published later.

Thus, in some C*-algebra there are given Hermitian elements $A_1, A_2, B_1, B_2, C_1, C_2$, and $A_1^2 = 1, B_1^2 = 1, C_1^2 = 1$. [A, B] = 0, [B, C] = 0, [C, A] = 0 for $k, l, m = 1, 2$. We introduce the notation

$$K_\varphi(A, B) = A_1 \beta_1 \cos \varphi + A_2 \beta_2 \sin \varphi - A_2 \beta_1 \sin \varphi + A_2 \beta_2 \cos \varphi;$$

$\varphi$ is an arbitrary parameter, running through $[0, 2\pi]$. In a similar manner one defines $K_\varphi(B, C)$ and $K_\varphi(C, A)$. Taking into account the results of Sec. 2, it is easy to see that the quantum bounds for $K_\varphi(A, B)$ are equal to $\pm 2$, i.e., $-2 \leq K_\varphi(A, B) \leq 2$. And the constants $-2$ and $2$ are sharp. The classical bounds for $K_\varphi(A, B)$ are

$$\pm 2 \max(|\cos \varphi|, |\sin \varphi|).$$

**Proposition 5.1.** For any $\varphi$, the quantum bounds for the observable

$$K_\varphi(A, B) + K_\varphi(B, C) + K_\varphi(C, A)$$

coincides with its classical bounds.

**Remark.** The mentioned classical bounds are easily computed; the upper and lower bounds are equal, respectively, to the largest and the smallest of the three numbers

$$-2 \cos \varphi + 4 \sin \varphi; \quad 6 \cos \varphi; \quad -2 \cos \varphi - 4 \sin \varphi.$$

**Proposition 5.2.** Under an appropriate choice of the C*-algebra $\mathcal{A}$ and observables $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathcal{A}$, we have: for any $\varphi \in [0, 2\pi]$ there exists a linear functional $\hat{f}$ on $\mathcal{A}$ satisfying the conditions

$$\hat{f}(1) = 1,$$

$$\hat{f}(\alpha A_1 + \alpha_2 A_2 + \beta B_1 + \beta_2 B_2 + \gamma C_1 + \gamma_2 C_2 + \gamma \cdot 1)^2 \geq 0$$

for any $\alpha_k, \beta_k, \gamma_k, \gamma_2, \gamma \cdot 1$ and such that

$$\hat{f}(K_\varphi(A, B) + K_\varphi(B, C) + K_\varphi(C, A)) = 6 \max(\cos(\varphi + \frac{\pi}{3}), \cos \varphi, \cos(\varphi - \frac{\pi}{3})).$$

**Remark.** We see that for some $\varphi$, the indicated value violates the boundaries mentioned in Proposition 5.1 (and in the remark following it): this is connected with the fact that the functional $\hat{f}$ is not a state. Thus, in the case of three domains, as opposed to the case of two domains, one cannot obtain sharp inequalities by using the positivity of the squares of only those observables which are linear combinations of the observables $A_k, B_k, C_{ml}$. We also note that the inner product of the unit vectors in the Euclidean space allows us to reach the boundary mentioned in Proposition 5.2; thus, they may violate the quantum boundaries, which again differs from the case of two domains.

**Proposition 5.3.** For any state $\hat{f}$ we have the inequality

$$(\frac{1}{2} \hat{f}(B_1 C_{ml})^2 + (\frac{1}{2} \hat{f}(K_\varphi(A, B))^2 \leq 1$$

for any $k, l, m$. 569
Remark. It \( f(\pi/4, (A|\Phi)) \) attains its maximal value, equal to \( ? \), then necessarily \( f(G, C_m) = 0 \).

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