Sixth Order Accurate Finite Difference Schemes for the Helmholtz Equation

I. Singer, E. Turkel
Department of Mathematics
Tel Aviv University
Tel Aviv, Israel

Abstract

We develop and analyze finite difference schemes for the two-dimensional Helmholtz equation. The schemes which are based on nine-point approximation have a sixth-order accurate local truncation order. The schemes are compared with the standard five points pointwise representation which has second-order accurate local truncation error and a nine points fourth-order local truncation error scheme based on the Padé approximation. Numerical results are presented on a model problem approximated with the developed schemes.

1 Introduction

Many physical phenomena such as acoustics, elasticity and electromagnetic waves are governed in the frequency domain by the Helmholtz equation

$$\nabla^2 u + k^2 u = F.$$

In this study we analyze finite difference approximations on uniform grids, denoting h for the grid-size. Besides the standard finite difference scheme which is of order $O\left(h^2\right)$, high order schemes with fourth order of accuracy were developed in [2]. These schemes, which can be used for variable k, are based on the Padé approximation and were examined in [3]. In [3] another approach was introduced to achieve higher order accuracy where the Helmholtz equation is used to replace higher order derivatives in the truncation error by lower order derivatives which can be approximated on a nine point stencil. In [5] this approach is used for a constant value of k to achieve a scheme with sixth order of accuracy. In this study we use this approach and improve it to get a family of sixth order of accuracy schemes that are easier to use in applications, such as the Helmholtz equation in an unbounded domain which is solved using PML [6], [5].

Another approach to get sixth-order of accuracy scheme is presented in [1]. The approach there is to find optimal coefficients for a class of wave equations. The specific scheme presented there for the Helmholtz equation is a special case, to order $O(h^6)$, of the family of schemes developed here.

2 Finite Difference Schemes

In two dimensions the Helmholtz equation becomes:

$$u_{xx} + u_{yy} + k^2 u = F \tag{1}$$

Let $\phi_{i,j}$ be a numerical approximation to $u(x_i, y_j)$, and $F_{i,j}$ be a known function. We wish to have symmetric stencil in both directions x and y. One scheme having these properties has the form

$$A_0 \phi_{i,j} + A_s \sigma_s + A_c \sigma_c = B_0 F_{i,j} + B_s \Gamma_s + B_c \Gamma_c \tag{2}$$

where

$$\sigma_s = \phi_{i,j+1} + \phi_{i+1,j} + \phi_{i,j-1} + \phi_{i-1,j}$$

is the sum of the values of the mid-side points and

$$\sigma_c = \phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j-1} + \phi_{i-1,j+1}$$

is the sum of the values at the corner points. Similarly for F we have

$$\Gamma_s = F_{i,j+1} + F_{i+1,j} + F_{i,j-1} + F_{i-1,j}$$

and

$$\Gamma_c = F_{i+1,j+1} + F_{i+1,j-1} + F_{i-1,j-1} + F_{i-1,j+1}.$$

3 Pointwise Representation

Expanding the standard approximation for the second derivative

$$D_{xx}\phi = \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{h^2}$$

in a Taylor series, we get for every sufficiently smooth u

$$D_{xx}u = u_{xx} + \frac{h^2}{12}u_{xxxx} + \frac{h^4}{360}u_{xxxxxx} + O\left(h^6\right).$$

Adding a similar approximation for u_{yy} we get

$$\begin{aligned}(D_{xx} + D_{yy})\, u &= u_{xx} + u_{yy} \\ &+ \frac{h^2}{12}\left(u_{xxxx} + u_{yyyy}\right) + \frac{h^4}{360}\left(u_{xxxxxx} + u_{yyyyyy}\right) + O\left(h^6\right). \quad (3)\end{aligned}$$

Hence, we get the representation

$$(D_{xx} + D_{yy})\, u = u_{xx} + u_{yy} + O\left(h^2\right)$$

and for the Helmholtz equation

$$(D_{xx} + D_{yy})\, u = F - k^2 u + O\left(h^2\right).$$

Using our notation,

$$(D_{xx} + D_{yy})\, u = \frac{\sigma_s - 4\phi_{i,j}}{h^2}. \quad (4)$$

Multiplying by h^2 leads to the standard five point pointwise representation, (2), (see [3])

$$\begin{aligned}A_0 &= -4 + (kh)^2, \quad A_s = 1, \quad A_c = 0 \\ B_0 &= h^2, \qquad\qquad B_s = B_c = 0.\end{aligned} \quad (5)$$

4 Fourth Order Accurate Scheme

Differentiating (1) twice with respect to x and y we get

$$\begin{aligned}u_{xxxx} + u_{xxyy} &= F_{xx} - \left(k^2 u\right)_{xx} \\ u_{yyyy} + u_{xxyy} &= F_{yy} - \left(k^2 u\right)_{yy}.\end{aligned} \quad (6)$$

Adding these equations we get

$$u_{xxxx} + u_{yyyy} = F_{xx} + F_{yy} - \left(\left(k^2 u\right)_{xx} + \left(k^2 u\right)_{yy} \right) - 2u_{xxyy}. \qquad (7)$$

Inserting into (3) and neglecting all $O\left(h^6\right)$ terms we conclude

$$\left(D_{xx} + D_{yy}\right) u = F - k^2 u + \qquad (8)$$
$$\frac{h^2}{12} \left(F_{xx} + F_{yy} - \left(\left(k^2 u\right)_{xx} + \left(k^2 u\right)_{yy} \right) - 2u_{xxyy} \right) + O\left(h^4\right).$$

In order to preserve the $O\left(h^4\right)$ approximation, we can use an $O\left(h^2\right)$ approximation for $F_{xx} + F_{yy} - \left(\left(k^2 u\right)_{xx} + \left(k^2 u\right)_{yy} \right) - 2u_{xxyy}$. We choose $F_{xx} + F_{yy} \sim \left(D_{xx} + D_{yy}\right) F$, $u_{xxyy} \sim D_{xx} D_{yy} u$. One can choose to directly difference $\left(k^2 u\right)_{xx} + \left(k^2 u\right)_{yy}$ with second order accuracy. Instead we shall assume that k is constant. Then $\left(k^2 u\right)_{xx} + \left(k^2 u\right)_{yy} = k^2 \left(u_{xx} + u_{yy}\right) = k^2 \left(F - k^2 u\right)$ giving

$$\left(D_{xx} + D_{yy} + \frac{h^2}{6} D_{xx} D_{yy} \right) u + \left(1 - \frac{(kh)^2}{12} \right) k^2 u \qquad (9)$$
$$= F \left(1 - \frac{(kh)^2}{12} \right) + \frac{h^2}{12} \left(D_{xx} + D_{yy}\right) F + O\left(h^4\right).$$

Using (4) and

$$D_{xx} D_{yy} u = \frac{\sigma_c - 2\sigma_s + 4\phi_{i,j}}{h^4}$$

in (9) we get an approximation which is similar to the EB approximation in [3]

$$A_0 = -\frac{10}{3} + (kh)^2 \left(1 - \frac{(kh)^2}{12} \right), \qquad A_s = \frac{2}{3}, \qquad A_c = \frac{1}{6} \qquad (10)$$
$$B_0 = \left(\frac{2}{3} - \frac{k^2}{12} \right) h^2, \qquad B_s = \frac{h^2}{12}, \qquad B_c = 0.$$

5 Sixth Order Accurate Scheme

To achieve sixth order accurate schemes we have to assume k is a constant, since we need more derivatives of the Helmholtz equation. We also need a fourth order accurate approximation for $D_{xx}+D_{yy}$. Using this assumption and differentiating (6) twice with respect to x and y we get

$$\begin{aligned} u_{xxxxxx}+u_{xxxxyy} &= F_{xxxx}-k^2u_{xxxx} \\ u_{yyyyyy}+u_{xxyyyy} &= F_{yyyy}-k^2u_{yyyy} \end{aligned} \tag{11}$$

and

$$u_{xxxxyy}+u_{xxxxyy}=F_{xxyy}-k^2u_{xxyy}. \tag{12}$$

Using (7,11,12) we get

$$\begin{aligned} u_{xxxxxx}+u_{yyyyyy} &= F_{xxxx}+F_{yyyy}-F_{xxyy} \\ &\quad -k^2\left(F_{xx}+F_{yy}-k^2\left(u_{xx}+u_{yy}\right)-3u_{xxyy}\right). \end{aligned}$$

Inserting into (3) we get

$$\begin{aligned} &\left(D_{xx}+D_{yy}\right)u=F-k^2u \\ &+\frac{h^2}{12}\left(F_{xx}+F_{yy}-k^2\left(u_{xx}+u_{yy}\right)-2u_{xxyy}\right) \\ &+\frac{h^4}{360}\left(F_{xxxx}+F_{yyyy}-F_{xxyy}-k^2\left(F_{xx}+F_{yy}-k^2\left(u_{xx}+u_{yy}\right)-3u_{xxyy}\right)\right)+O\left(h^6\right). \end{aligned}$$

Using the Taylor series for $D_{xx}D_{yy}$ we find

$$D_{xx}D_{yy}u=u_{xxyy}+\frac{h^2}{12}\left(u_{xxxxyy}+u_{xxyyyy}\right)+O(h^4).$$

So we get

$$\begin{aligned} &\left(D_{xx}+D_{yy}+\frac{h^2}{6}D_{xx}D_{yy}\right)u=F-k^2u+\frac{h^2}{12}\left(F_{xx}+F_{yy}-k^2\left(u_{xx}+u_{yy}\right)\right)+ \\ &\frac{h^4}{360}\left(F_{xxxx}+F_{yyyy}+4F_{xxyy}-k^2\left(F_{xx}+F_{yy}-k^2\left(u_{xx}+u_{yy}\right)+2u_{xxyy}\right)\right)+O\left(h^6\right). \end{aligned} \tag{13}$$

Using $D_{xx}D_{yy}u$ for u_{xxyy} and $F - k^2u$ instead of $u_{xx} + u_{yy}$ and rearranging we get

$$\left(D_{xx} + D_{yy} + \frac{h^2}{6}\left(1 + \frac{(kh)^2}{30}\right) D_{xx}D_{yy}\right) u$$
$$+ \left(1 - \frac{(kh)^2}{12}\left(1 - \frac{(kh)^2}{30}\right)\right) k^2 u$$
$$= F\left(1 - \frac{(kh)^2}{12}\left(1 - \frac{(kh)^2}{30}\right)\right) + \frac{h^2}{12}\left(1 - \frac{(kh)^2}{30}\right)(F_{xx} + F_{yy})$$
$$+ \frac{h^4}{360}\left(F_{xxxx} + 4F_{xxyy} + F_{yyyy}\right) + O\left(h^6\right).$$

For arbitrary F we need a fourth order accuracy approximation for $F_{xx} + F_{yy}$ and a second order accuracy approximation for $F_{xxxx} + 4F_{xxyy} + F_{yyyy}$. This is easily done, but requires more than nine points and we cannot put the approximation in the form (2). In many problems $F = 0$, and we can write in our notation

$$A_0 = -\frac{10}{3} + (kh)^2\left(\frac{46}{45} - \frac{(kh)^2}{12} + \frac{(kh)^4}{360}\right) \tag{14}$$
$$A_s = \frac{2}{3} - \frac{(kh)^2}{90}, \qquad A_c = \frac{1}{6} + \frac{(kh)^2}{180}.$$

We call this approximation EB-6.

6 Divergence Form

The last two schemes developed have a difficulty. In some problems we want to use the schemes in an unbounded domain, with a PML (Perfectly Matched Layer) to absorb the outgoing waves at infinity [6]. In the PML we need to solve a variable coefficient problem

$$\frac{\partial}{\partial x}(Au_x) + \frac{\partial}{\partial y}(Bu_y) + Ck^2u = F \tag{15}$$

where A, B, C are functions of x, y. We have not found a formula which keeps the self-adjoint form and is also fourth order accurate for non-constant A, B

and C. In the PML A, B, C are variable and in the interior of the domain, $A = B = C = 1$. We want to use in the interior a symmetric stencil, and automatically switch to a second order accurate stencil in the PML. In [4] we show that one can use a lower order formula in the PML and still retain the global high accuracy in the physical domain.

We start with the standard second order three point symmetric approximation

$$D_x (Au_x)_j = \frac{A_{i+\frac{1}{2},j} (u_{i+1,j} - u_{i,j}) - A_{i-\frac{1}{2},j} (u_{i,j} - u_{i-1,j})}{h^2}.$$

We can construct a more general divergence free form by averaging this in the j direction. So we take $\alpha D_x (Au_x)_j + \frac{1-\alpha}{2}(D_x (Au_x)_{j+1} + D_x (Au_x)_{j-1})$ and a similar formula for $D_y (Bu_y)$. The approximation to $k^2 u$ can be a general nine point formula. Thus, for $A = B = 1$ we get

$$A_0 = -4\alpha + (1 - 4\beta_s - 4\beta_c) (kh)^2 \tag{16}$$
$$A_s = 2\alpha - 1 + \beta_s (kh)^2, \qquad A_c = 1 - \alpha + \beta_c (kh)^2.$$

This approximation is guaranteed to be $O(h^2)$ for all values of α, β_s, β_c. Choosing $\alpha = 1, \beta_s = 0, \beta_c = 0$ recovers the pointwise representation (5). We wish to construct higher order approximations that use this limited subset of coefficients. Unfortunately this cannot be done for either (10) or (14).

6.1 Fourth Order Divergence Form

To conserve the fourth order accuracy of the scheme achieved in section 4, we need an $O(h^2)$ approximation for the term $F_{xx} + F_{yy} - k^2 (u_{xx} + u_{yy}) - 2u_{xxyy}$ in (8). Instead of using $(F - k^2 u)$ as the approximation for $(u_{xx} + u_{yy})$ we can use $\left(D_{xx} + D_{yy} + \frac{h^2}{6} D_{xx} D_{yy}\right) u$, yielding

$$\left(D_{xx} + D_{yy} + \frac{h^2}{6} D_{xx} D_{yy}\right) \left(1 + \frac{(kh)^2}{12}\right) u + k^2 u$$
$$= F + \frac{h^2}{12} (D_{xx} + D_{yy}) F + O\left(h^4\right).$$

Rearranging, we get

$$\begin{aligned}
&\left(D_{xx}+D_{yy}+\frac{h^2}{6}D_{xx}D_{yy}\right)u \\
+&k^2\left(1+\frac{h^2}{12}\left(D_{xx}+D_{yy}\right)+\frac{h^4}{72}D_{xx}D_{yy}\right)u \\
=&F+\frac{h^2}{12}\left(D_{xx}+D_{yy}\right)F+O\left(h^4\right).
\end{aligned} \tag{17}$$

The term $\frac{h^4}{72}$ is fourth order, and can be multiplied by an arbitrary scalar, chosen as $\frac{\gamma}{2}$. The same treatment can be done on the right hand side of (17), choosing $\left(\frac{h^2}{12}\left(D_{xx}+D_{yy}\right)+\delta\frac{h^4}{144}D_{xx}D_{yy}\right)F$ (for arbitrary δ) instead of $\frac{h^2}{12}\left(D_{xx}+D_{yy}\right)F$, yielding

$$\begin{aligned}
A_0 &= -\frac{10}{3}+(kh)^2\left(\frac{2}{3}+\frac{\gamma}{36}\right) \\
A_s &= \frac{2}{3}+(kh)^2\left(\frac{1}{12}-\frac{\gamma}{36}\right), \quad A_c=\frac{1}{6}+(kh)^2\frac{\gamma}{144} \\
B_0 &= \left(\frac{2}{3}+\frac{\delta}{36}\right)h^2, \quad B_s=\left(\frac{1}{12}-\frac{\delta}{72}\right)h^2, \quad B_c=\frac{\delta}{144}h^2.
\end{aligned} \tag{18}$$

This approximation is the same as the Padé approximation developed in [2], which is also valid for non-constant k.

This scheme is of the form (16) with

$$\alpha=\frac{5}{6}, \qquad \beta_s=\frac{1}{12}-\frac{\gamma}{72}, \quad \beta_c=\frac{\gamma}{144}$$

Hence, when used for problems with variable coefficients the scheme will be second order accurate. In regions where the coefficients are constant the scheme will increase to fourth order accuracy.

6.2 Sixth Order Divergence Form

Rearranging (13) we get

$$\left(D_{xx} + D_{yy} + \frac{h^2}{6}\left(1 + \frac{(kh)^2}{30}\right) D_{xx}D_{yy} \right) u = \tag{19}$$
$$-k^2 u - \frac{k^2h^2}{12}\left(u_{xx} + u_{yy}\right) + \frac{k^4h^4}{360}\left(u_{xx} + u_{yy}\right) + F$$
$$+\frac{h^2}{12}\left(1 - \frac{k^2h^2}{30}\right)\left(F_{xx} + F_{yy}\right) + \frac{h^4}{360}\left(F_{xxxx} + 4F_{xxyy} + F_{yyyy}\right) + O\left(h^6\right).$$

To preserve the sixth order accuracy we need a fourth order approximation for the term $\frac{k^2h^2}{12}\left(u_{xx} + u_{yy}\right)$. Using (3,7) we get

$$u_{xx} + u_{yy} = \left(D_{xx} + D_{yy}\right) u - \frac{h^2}{12}\left(F_{xx} + F_{yy} - k^2\left(u_{xx} + u_{yy}\right) - 2u_{xxyy}\right) + O\left(h^4\right)$$

or

$$u_{xx} + u_{yy} =$$
$$\left(D_{xx} + D_{yy}\right) u - \frac{h^2}{12}\left(F_{xx} + F_{yy} - k^2\left(D_{xx} + D_{yy}\right) u - 2D_{xx}D_{yy}u\right) + O\left(h^4\right)$$
$$= \left(D_{xx} + D_{yy}\right)\left(1 + \frac{k^2h^2}{12}\right) u + \frac{h^2}{6} D_{xx}D_{yy}u - \frac{h^2}{12}\left(F_{xx} + F_{yy}\right) + O\left(h^4\right).$$

Inserting this fourth order approximation and $\left(D_{xx} + D_{yy} - \frac{\gamma}{2}h^2 D_{xx}D_{yy}\right) u$ as a second order accurate approximation to the term $\frac{k^4h^4}{360}\left(u_{xx} + u_{yy}\right)$ in (19) we get

$$\left(D_{xx} + D_{yy} + \frac{h^2}{6}\left(1 + \frac{(kh)^2}{30}\right) D_{xx}D_{yy} \right) u =$$
$$- k^2 u - \frac{(kh)^2}{12}\left(\left(D_{xx} + D_{yy}\right)\left(1 + \frac{k^2h^2}{12}\right) u + \frac{h^2}{6} D_{xx}D_{yy}u - \frac{h^2}{12}\left(F_{xx} + F_{yy}\right)\right)$$
$$+ \frac{(kh)^4}{360}\left(D_{xx} + D_{yy} - \frac{\gamma}{2}h^2 D_{xx}D_{yy}\right) u$$
$$+ F + \frac{h^2}{12}\left(1 - \frac{k^2h^2}{30}\right)\left(F_{xx} + F_{yy}\right) + \frac{h^4}{360}\left(F_{xxxx} + 4F_{xxyy} + F_{yyyy}\right) + O\left(h^6\right).$$

or

$$\left(1+\frac{(kh)^2}{12}+\frac{(kh)^4}{240}\right)(D_{xx}+D_{yy})\,u+$$
$$\frac{h^2}{6}\left(1+\frac{7}{60}(kh)^2+\frac{\gamma}{120}(kh)^4\right)D_{xx}D_{yy}u+k^2u=$$
$$F+\frac{h^2}{12}\left(1+\frac{(kh)^2}{20}\right)(F_{xx}+F_{yy})+\frac{h^4}{360}(F_{xxxx}+4F_{xxyy}+F_{yyyy})+O(h^6)$$

Assuming $F=0$, we achieve

$$\begin{aligned} A_0 &= -\frac{10}{3}+\frac{67}{90}(kh)^2+\frac{\gamma-3}{180}(kh)^4 \\ A_s &= \frac{2}{3}+\frac{2}{45}(kh)^2+\frac{3-2\gamma}{720}(kh)^4 \\ A_c &= \frac{1}{6}+\frac{7}{360}(kh)^2+\frac{\gamma}{720}(kh)^4. \end{aligned} \tag{20}$$

This approximation is in the form (16) with

$$\alpha=\frac{5}{6},\qquad \beta_s=\frac{2}{45}+\frac{3-2\gamma}{720}(kh)^2,\quad \beta_c=\frac{7}{360}+\frac{\gamma}{720}(kh)^2$$

and is sixth-order accurate for all values of γ. We call this formulation HO-6.

Choosing $\gamma=\frac{11}{12}$ and using the notations in [1], we get

$$\begin{aligned} a_1 &= -\frac{A_c}{A_0}=\frac{1440+168(kh)^2+11(kh)^4}{28800-6432(kh)^2+100(kh)^4} \\ a_2 &= -\frac{A_s}{A_0}=\frac{2880+192(kh)^2+7(kh)^4}{14400-3216(kh)^2+50(kh)^4}. \end{aligned}$$

Expanding in a Taylor series we get

$$\begin{aligned} a_1 &= \frac{1}{20}+\frac{17}{1000}(kh)^2+\frac{801}{200000}(kh)^4+O\left(h^6\right) \\ a_2 &= \frac{1}{5}+\frac{29}{500}(kh)^2+\frac{2549}{200000}(kh)^4+O\left(h^6\right) \end{aligned}$$

which is the same approximation, up to sixth order accuracy, achieved in [1]. The method developed in [1] describes a general methodology for deriving high order discretizations for a large class of wave equations. That methodology involves a type of variational principal. From that viewpoint it is more general than the present presentation which stresses the properties of The Helmholtz equation. The main point of [1] is that optimal coefficients of the linear discretization formula can be found by minimizing the L_2 norm of the error for plane waves, integrated over all directions of incidence. This general principal could find applications in other cases, such as at non-reflecting boundaries though these applications have not yet been explored.

7 Numerical results

We examine the behavior of the schemes developed on model problems. Both model problems are the Helmholtz equation in a square coupled with a Dirichlet boundary condition. Even though the numerical examples use the model problems, they illustrate the benefits to be expected for more realistic problems.

The model problem used for both cases is

$$\Delta u + k^2 u = F(x, y)$$

in the square $[0, \pi] \times [0, \pi]$ with the boundary conditions

$$u(0, y) = u(\pi, y) = u(x, 0) = u(x, \pi) = 0.$$

We choose two different functions $F(x, y)$, which are tailored to have explicit exact solutions. We choose the exact solutions as

$$u(x, y) = \sin ix \sin jy \tag{21}$$

where i, j are positive integers, chosen as 1 and 2 respectively. In this model problem

$$F(x, y) = \left(k^2 - \left(i^2 + j^2\right)\right) \sin ix \sin jy$$

In the second problem we choose the exact solution as

$$\begin{aligned} u(x, y) &= \sinh \alpha x \sinh \alpha(\pi - x) \sinh \beta y \sinh \beta(\pi - y) \\ &= (\cosh \alpha\pi - \cosh \alpha(\pi - 2x))(\cosh \beta\pi - \cosh \beta(\pi - 2y)) \end{aligned} \tag{22}$$

where $\alpha = 0.5$,and $\beta = 0.7$. For this second problem

$$\begin{aligned} F(x,y) &= k^2 \sinh \alpha x \sinh \alpha (\pi - x) \sinh \beta y \sinh \beta (\pi - y) \\ &- \alpha^2 \cosh \alpha (\pi - 2x) \sinh \beta y \sinh \beta (\pi - y) \\ &- \beta^2 \cosh \beta (\pi - 2y) \sinh \alpha x \sinh \alpha (\pi - x) \end{aligned}$$

The problems are solved using the second order accurate PT scheme (5), the fourth order accurate schemes EB (10), and HO (18) with the parameter $\gamma = 2$, and the sixth order accurate schemes EB-6 (14) and HO-6 (20) with $\gamma = 2$ and $\gamma = -1$. We use a uniform grid size $h = \frac{\pi}{N}$ where N is the number of gridpoints in both x, y directions.

For the sixth-order schemes we need high order approximations for the function $F(x,y)$ and its derivatives. If the right hand side F is not known analytically then one can approximate the derivatives with high order finite differences. However, the fourth order derivatives require a larger stencil. This larger stencil is only for the known terms and does not affect the band width of the solver. Near boundaries the fourth order derivatives could be eliminated lowering the local accuracy but not the global accuracy. To simplify the model problem we use the exact values for $F_{xx}+F_{yy}$, and $F_{xxxx}+4F_{xxyy}+F_{yyyy}$, however this is not an important issue.We compare the numerical solution $\phi_{i,j}$ to $u(x_i, y_j)$ where u is the exact solution (21) or (22). The error vector e is

$$e_{i,j} = u(x_i, y_j) - \phi_{i,j}.$$

We measure the error in the l_∞ norm:

$$\|e\|_\infty = \max_{0 \le i,j \le N} e_{i,j}.$$

In the following table we present the l_∞ norm of the error for the first problem (21) with $k = 6.4$.

N	PT	EB	HO $\gamma = 2$	EB-6	HO-6 $\gamma = 2$	HO-6 $\gamma = -1$
4	2.20E-02	1.13E-02	3.80E-02	2.77E-03	2.39E-01	3.77E-03
8	5.92E-03	2.43E-03	2.39E-03	5.84E-05	2.81E-03	9.63E-05
16	1.51E-03	7.35E-05	1.49E-04	5.82E-07	4.27E-05	1.93E-06
32	3.79E-04	4.07E-06	9.29E-06	8.16E-09	6.65E-07	3.18E-08
64	9.49E-05	2.48E-07	5.80E-07	1.24E-10	1.04E-08	5.03E-10
128	2.37E-05	1.54E-08	3.63E-08	1.89E-12	1.62E-10	7.88E-12

In all the schemes the norm of the error decreases, as the number of gridpoints increases. We clearly see that the norm of the error behaves according to the order of the scheme. As we multiply N by 2 (divide h by 2) the norm of the error decreases by 4 in the PT scheme, by 16 in the EB and HO schemes and by 64 in the EB-6 and HO-6 schemes. Similar results are found in the second table, with the results for the second problem (22)

N	PT	EB	HO $\gamma = 2$	EB-6	HO-6 $\gamma = 2$	HO-6 $\gamma = -1$
4	4.34E-03	2.66E-04	5.03E-03	6.88E-04	5.74E-02	2.21E-01
8	1.90E-03	9.82E-04	4.14E-03	8.44E-05	2.43E-03	2.96E-03
16	2.74E-03	1.32E-05	1.21E-04	8.58E-07	5.25E-05	3.48E-05
32	2.78E-04	8.30E-07	7.77E-06	1.31E-09	9.06E-07	5.61E-08
64	6.51E-05	5.09E-08	4.87E-07	2.02E-10	1.43E-08	8.97E-09
128	1.61E-05	3.19E-09	3.04E-08	3.12E-12	2.24E-10	1.40E-10

We wish to computationally verify the order of the schemes, so we assume that for small values of h,

$$\|e\|_\infty \simeq C(k)\, N^{-r} = \frac{C(k)}{\pi^r} h^r$$

where r is the order of the scheme, i.e.

$$r = \begin{cases} 2 & \text{for PT scheme} \\ 4 & \text{for the schemes EB,HO} \\ 6 & \text{for the schemes EB-6,HO-6} \end{cases}$$

Hence, if $N = 2^l$

$$-\log_2 \left(\|e\|_\infty\right) \simeq l \cdot r - \log_2 C(k). \tag{23}$$

We calculate r, the order of accuracy, by measuring the slope of the curve. We verify (23) using Figs. 1-5 (in all the HO schemes $\gamma = 2$). All logarithms in the figures are to the base 2. Comparing figures **??** with figure 3 we see that there is no basic difference between the two cases. Hence, for the figures 4 through 6 we only consider case 2 and increase k. In [7] it is shown, that for the waveguide problem, the accuracy depends on the input boundary mode. The exact solution of the model problem, (22), does not depend on k. Hence, as k increases the error remains constant for the finer meshes. For small values of k our assumption (23) is correct. As k grows the accuracy of the scheme deteriorates and the assumption is correct only fine meshes, see also [3]. For coarse meshes, we enter the asymptotic range for kh constant, i.e. N behaves linearly in k for the point where the error begins to decrease with finer meshes.

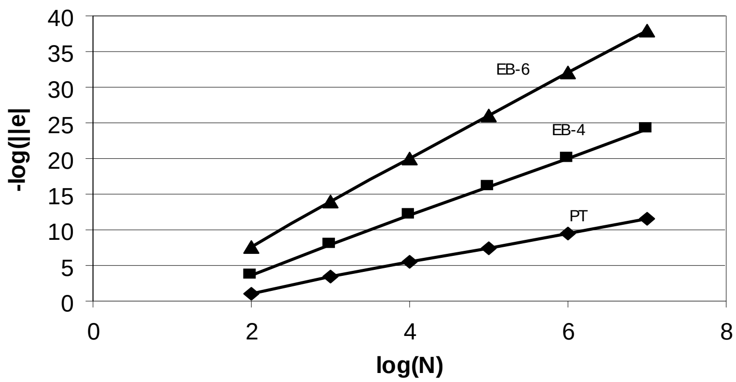

Figure 1: $-\log\left(\|e\|_{\infty}\right)$ in EB schemes, first problem, $k = 1.6$

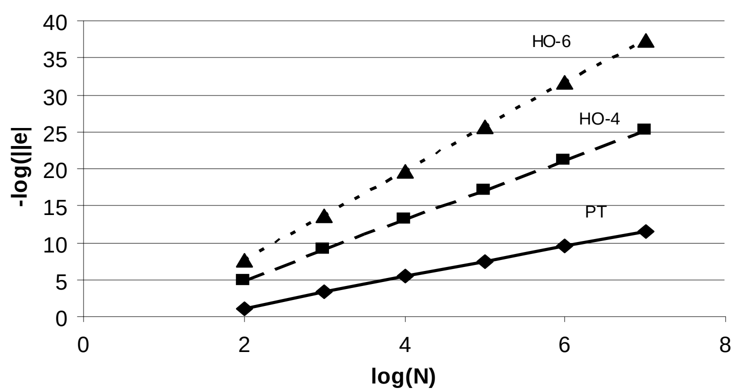

Figure 2: $-\log\left(\|e\|_{\infty}\right)$ in HO schemes, first problem, $k = 1.6$

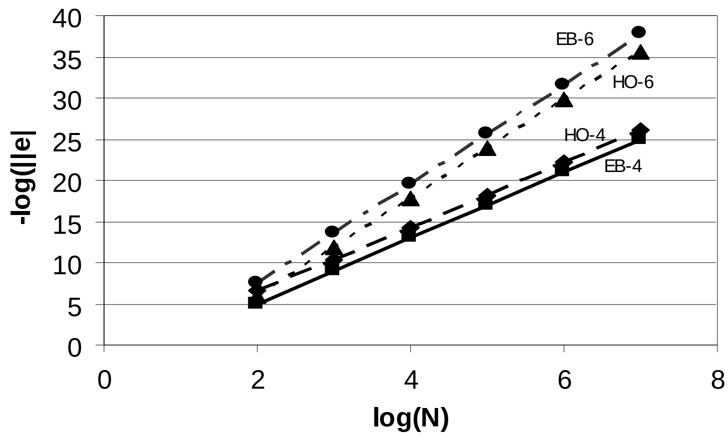

Figure 3: $-\log\left(\|e\|_{\infty}\right)$ in HO & EB schemes, second problem, $k = 1.6$

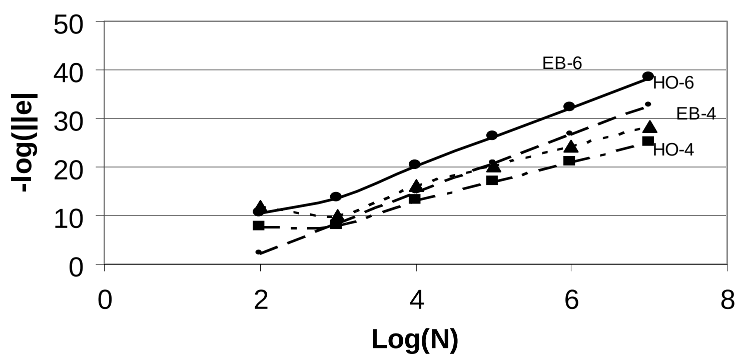

Figure 4: $-\log\left(\|e\|_{\infty}\right)$ in HO & EB schemes, second problem, $k = 6.4$

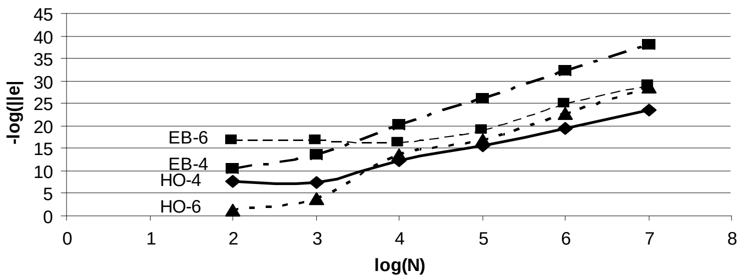

Figure 5: $-log\left(\|e\|_{\infty}\right)$ in HO & EB schemes, second problem, $k = 12.8$

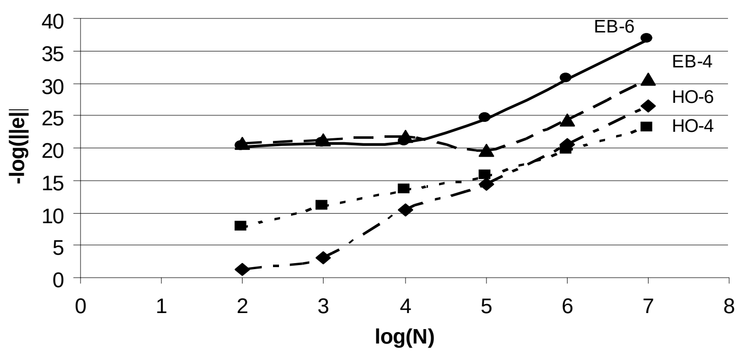

Figure 6: $-log\left(\|e\|_{\infty}\right)$ in HO & EB schemes, second problem, $k = 25.6$

References

[1] J.E. Caruthers, J.S. Steinhoff and R.C. Engels, *An Optimal Finite Difference Representation for a Class of Linear PDEs with Application to the Helmholtz Equation*, Journal of Computational Acoustics 7, 245-252, 1999.

[2] I. Harari and E. Turkel, *Accurate Finite Difference Methods for Time-Harmonic Wave Propagation*, Journal of Computational Physics 119, 252-270, 1995.

[3] I. Singer and E. Turkel, *High Order Finite Difference Methods for the Helmholtz Equation*, Computer Methods in Applied Mechanics and Engineering 163, 343-358, 1998.

[4] I. Singer and E. Turkel, *A Perfectly Matched Layer for the Helmholtz Equation in a Semi-infinite Strip*, Journal of Computational Physics 201, 439-465 (2004).

[5] S. Tsynkov and E. Turkel, *A Cartesian Perfectly Matched Layer for the Helmholtz Equation,* Artificial Boundary Conditions with Applications to CEM, Loic Tourvete editor, Novascience Publishing, 279-309, 2001.

[6] E. Turkel and A. Yefet, *Absorbing PML Boundary Layers for Wave-Like Equations,* Applied Numerical Mathematics 27, 533-557, 1998.

[7] E. Turkel, C. Farhat and U. Hetmaniuk, *Improved Accuracy for the Helmholtz Equation in Unbounded Domains* International Journal of Numerical Methods in Engineering 59, 1963-1988, 2004.