

# Choice of Variables and Preconditioning for Time Dependent problems

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*We consider the use of low speed preconditioning for time dependent problems. These are solved using a dual time step approach. We consider the effect of this dual time step on the parameter of the low speed preconditioning. In addition, we compare the use of two sets of variables, conservation and primitive variables, to solve the system. We show the effect of these choices on both the convergence to a steady state and the accuracy of the numerical solutions for low Mach number steady state and time dependent flows.*

## Introduction

Methods for preconditioning the low speed Euler and Navier-Stokes equations have been available for about twenty years. In spite of these years of development and analysis there still exists difficulties with the robustness of these techniques. This manifests itself in that some of the parameters of the preconditioning matrix are problem dependent. Because of the stagnation points it is necessary to prevent the parameters from becoming too small. Unfortunately, the cutoff is frequently large and problem dependent. In addition the theory of preconditioning is mainly based on properties of the inviscid Euler equations with artificial viscosity even though most of the applications are for viscous flows.

The preconditioning changes the time dependent behavior of the system and so is only directly useful for steady state calculations. To overcome this difficulty a dual time step method has been used by many investigators.<sup>3-6,20</sup> In this approach the solution at the next physical time step is determined as a steady state problem to which preconditioning is applicable. We shall analyze time dependent effects on the preconditioning parameters. We consider

$$\mathbf{P}^{-1} \frac{\partial w}{\partial \tau} + \frac{\partial w}{\partial t} + \mathbf{R} = 0 \quad (1)$$

where  $t$  is the physical time,  $\tau$  is an artificial time,  $\mathbf{R}$  denotes the residual for the steady Navier-Stokes equations,  $w$  refers to a general set of unknowns and  $\mathbf{P}$  refers to the preconditioning matrix. In the next section we will discuss in more details the options for choosing  $w$ . Equation (1) is advanced in artificial time by a Runge-Kutta (RK) method until a steady state in  $\tau$  is reached. We replace the physical time derivative by a backward difference formula (BDF). The general formula for BDF

schemes can be presented as

$$\frac{\partial w}{\partial t} \sim \frac{c_t w^{n+1} - F(w^n, w^{n-1}, \dots)}{\Delta t}$$

where  $c_t$  is constant which depends on the choice of BDF scheme. Let superscript 0 denote the last artificial time step,  $k$  the most recent stage of RK,  $n$  the last physical time step and  $n+1$  the next physical time step. A typical stage of the RK is

$$w^{k+1} = w^0 - \alpha_k \Delta \tau \mathbf{P} \left\{ R^k + \frac{c_t w^{n+1} - F(w^n, \dots)}{\Delta t} \right\}$$

where  $\alpha_k$  are the stage coefficients of the RK scheme. In practice, only the inviscid portion of  $R^k$  is updated at each stage. The viscous portion is updated, for a 5 stage scheme, only on the odd stages. Because  $w^{n+1}$  is not known, we replace it by  $w^{k+1}$ , i.e. current stage of RK. We reformulate this as

$$w^{k+1} = w^0 - \alpha_k \Delta \tau \mathbf{P} \left\{ R^k + \frac{c_t w^k - F(w^n, \dots)}{\Delta t} \right\} - \alpha_k c_t \Delta \tau \mathbf{P} \left( \frac{w^{k+1} - w^k}{\Delta t} \right)$$

We apply residual smoothing to the term inside the curly brackets. This is done so that the residual smoothing operates on a difference that vanishes in the steady state. Collecting terms we have

$$(I + \alpha_k c_t \frac{\Delta \tau}{\Delta t} \mathbf{P}) w^{k+1} = w^0 - \alpha_k \Delta \tau \mathbf{P} \left\{ R^k + \frac{c_t w^k - F(w^n, \dots)}{\Delta t} \right\} + \alpha_k c_t \frac{\Delta \tau}{\Delta t} \mathbf{P} w^k \quad (2)$$

The space discretization consists of a central difference formula plus a matrix valued artificial dissipation using second and fourth differences. We now describe

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the artificial viscosity due to the second order differencing. We express the dissipation in terms of derivatives rather than differences for presentation only. We consider the equation

$$\frac{\partial w}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = \text{visc terms}$$

where  $F$ ,  $G$ , and  $H$  represent the fluxes along  $x$ ,  $y$ , and  $z$  directions. In the  $x$  direction, we supplement the residual by the artificial viscosity as follows:

$$R = \frac{\partial F}{\partial x} - h_x \frac{\partial}{\partial x} (\epsilon_2 \mathbf{P}^{-1} |\mathbf{P}A| \frac{\partial w}{\partial x}) \quad (3)$$

where  $A$  is the Jacobian of  $F$  with respect to  $w$ ,  $h_x$  is the mesh spacing, and  $\epsilon_2$  is a constant. The absolute value of the matrix is found by diagonalizing the matrix, taking absolute values of the eigenvalues, with appropriate cut-offs to avoid singularities. A similar procedure is used for the  $y$  and  $z$  directions.

### Choice of Variables

We consider the sets of variables defined by

$$\begin{aligned} w_c &= (\rho, \rho u, \rho v, \rho w, E) \\ Q &= (p, u, v, w, T) \end{aligned} \quad (4)$$

$$w_0 = (p, u, v, w, S), \quad d\hat{w}_0 = \left( \frac{dp}{\rho c}, du, dv, dw, dS \right)$$

We shall refer to  $Q$  as the primitive variables and  $w_c$  as the conservation variables. The flux  $F$  and the physical time derivatives are evaluated in conservation variables so that the correct shock jumps are obtained. One implementation of (2) is to use the conservation variables throughout the equation. As the Mach number decreases to zero, the density usually becomes constant and so the conservation variables become less accurate. For almost incompressible flow the primitive variables are more appropriate.

If we change from conservation variables to  $Q$  variables we replace (3) by

$$\begin{aligned} R_Q &= \frac{\partial F}{\partial x} - h_x \frac{\partial}{\partial x} (\epsilon_2 \Gamma^{-1} |\mathbf{P}_Q A_Q| \frac{\partial Q}{\partial x}) \\ \mathbf{P}_Q &= \frac{\partial Q}{\partial w_c} \mathbf{P}_c \frac{\partial w_c}{\partial Q} \\ \Gamma^{-1} &= \frac{\partial w_c}{\partial Q} \mathbf{P}_Q^{-1} \quad \Gamma = \mathbf{P}_Q \frac{\partial Q}{\partial w_c} = \frac{\partial Q}{\partial w_c} \mathbf{P}_c \end{aligned} \quad (5)$$

Multiplying (2) by  $\frac{\partial Q}{\partial w_c}$  and substituting  $\Delta w_c =$

$\frac{\partial w_c}{\partial Q} \Delta Q$ , we get

$$\begin{aligned} (I + \alpha_k c_t \frac{\Delta \tau}{\Delta t} P_Q) Q^{k+1} &= Q^0 \\ &- \alpha_k \Delta \tau \Gamma \left\{ R_Q^k + \frac{c_t w_c^k - F(w_c^n, \dots)}{\Delta t} \right\} \\ &+ \alpha_k c_t \frac{\Delta \tau}{\Delta t} P_Q Q^k \end{aligned} \quad (6)$$

After each stage  $w_c^{k+1}$  is calculated using the nonlinear relation between  $w_c$  and  $Q$ . Note that  $Q$  only appears in the artificial time terms and the artificial viscosity. After the artificial time derivative approaches zero the resultant equation is in conservation form including the physical time derivative. If the physical time derivative would also be transformed to  $Q$  variables then we might lose the conservation form and hence the correct jump conditions at a shock. Preconditioning destroys conservation in the midst of the pseudo-time iteration process. However, when the pseudo-time derivative approaches zero, the algorithm should recover the conservation form.

Another possibility is to consider a mixture of conservation and  $Q$  variables. When evaluating the artificial viscosity we use  $Q$  variables as given by (5). However, when updating the variables we revert to  $w_c$  variables. This would be equivalent to (6) if the relation between  $w_c$  and  $Q$  variables were linear. We then get

$$\begin{aligned} (I + \alpha_k c_t \frac{\Delta \tau}{\Delta t} \mathbf{P}) w_c^{k+1} &= w_c^0 \\ &- \alpha_k \Delta \tau \mathbf{P} \left\{ R_Q^k + \frac{c_t w_c^k - F(w_c^n, \dots)}{\Delta t} \right\} \\ &+ \alpha_k c_t \frac{\Delta \tau}{\Delta t} \mathbf{P} w_c^k \end{aligned} \quad (7)$$

This is the same as (2) except that the artificial viscosity in  $w_c$  variables is replaced by  $R_Q$  based on  $Q$  variables. Hence, the two steady states (within the time dependent problem) are different while (6) and (7) have the same numerical steady state.

Computations demonstrate that the variables used in the artificial viscosity have a much larger effect than the choice of variables used to update the solution. Hence, we shall concentrate on comparing (2) with (6) and less on the mixed formulation (7). We shall further see that we can efficiently solve these linear systems.

### Low Speed Preconditioning

In the above description  $\mathbf{P}$  is a preconditioning operator based on the conservation variables, which is a full matrix and is difficult to analyze. Instead, we consider the entropy variables  $w_0$ . Now, the entropy equation decouples from the other variables. Furthermore, the Jacobian matrix is sparse. The simplest preconditioner in

$w_0$  variables is given by, see.<sup>9,10,12,14</sup>

$$\mathbf{P}_0^{-1} = \begin{pmatrix} \frac{1}{\beta^2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\beta$  is a parameter which should be of the order of the Mach number as to approximately equalize all the eigenvalues of  $P_0 A_0$ .

Let  $c^2 = \frac{\gamma p}{\rho}$ ,  $q^2 = u^2 + v^2 + w^2$  and  $\hat{q}^2 = \frac{(\gamma-1)q^2}{2}$ . Then the Jacobians that connect these variables are

$$\frac{\partial w_0}{\partial w_c} = \begin{pmatrix} \hat{q}^2 & (1-\gamma)u & (1-\gamma)v & (1-\gamma)w & \gamma-1 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 & 0 \\ -\frac{w}{\rho} & 0 & 0 & \frac{1}{\rho} & 0 \\ \hat{q}^2 - c^2 & (1-\gamma)u & (1-\gamma)v & (1-\gamma)w & \gamma-1 \end{pmatrix}$$

$$\frac{\partial w_c}{\partial w_0} = \begin{pmatrix} \frac{1}{c^2} & 0 & 0 & 0 & -\frac{1}{c^2} \\ \frac{u}{c^2} & \rho & 0 & 0 & -\frac{u}{c^2} \\ \frac{v}{c^2} & 0 & \rho & 0 & -\frac{v}{c^2} \\ \frac{w}{c^2} & 0 & 0 & \rho & -\frac{w}{c^2} \\ \frac{h}{c^2} & \rho u & \rho v & \rho w & -\frac{M^2}{2} \end{pmatrix}$$

where  $h = \frac{c^2}{\gamma-1} + \frac{q^2}{2}$ . The preconditioner  $\mathbf{P}_c$  in conservation variables is then given by  $\mathbf{P}_c = \frac{\partial w_c}{\partial w_0} \mathbf{P}_0 \frac{\partial w_0}{\partial w_c}$ . To calculate  $\mathbf{P}_c$  times a vector  $\vec{x}$  we do it in stages.

$$y_1 = \frac{\gamma-1}{c^2} \left[ \frac{q^2}{2} x_1 - (u x_2 + v x_3 + w x_4) + x_5 \right] \text{ and}$$

$$\vec{z} = \begin{pmatrix} 1 \\ u \\ v \\ w \\ h \end{pmatrix}$$

Then

$$\mathbf{P}_c \vec{x} = \vec{x} + (\beta^2 - 1) y_1 \vec{z} \quad (8)$$

$$\mathbf{P}_c^{-1} \vec{x} = \vec{x} + \left( \frac{1}{\beta^2} - 1 \right) y_1 \vec{z}$$

In (7) we need to evaluate  $(I + d \cdot \mathbf{P}_c)^{-1}$  times a vector where  $d = \alpha_k c_t \frac{\Delta \tau}{\Delta t}$ . Then

$$(I + d \cdot \mathbf{P}_c)^{-1} \vec{x} = \frac{\vec{x} + e y_1 \vec{z}}{1 + d} \quad e = \frac{(1 - \beta^2) d}{1 + \beta^2 d}$$

For the primitive variables,  $Q$ , we have

$$\mathbf{P}_Q = \begin{pmatrix} \beta^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{(\beta^2-1)T}{c_p p} & 0 & 0 & 0 & 1 \end{pmatrix}$$

where  $c_p = \frac{\gamma R}{\gamma-1}$ . Let

$$\vec{\zeta} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \frac{(\gamma-1)T}{\gamma p} \end{pmatrix}$$

then for (6)

$$(I + d \cdot \mathbf{P}_Q)^{-1} \vec{x} = \frac{\vec{x} + e x_1 \vec{\zeta}}{1 + d} \quad e = \frac{d(1 - \beta^2)}{1 + d \cdot \beta^2}$$

Note,  $\Gamma \frac{\partial w_c}{\partial Q} = \mathbf{P}_Q$ .

We need to choose  $\beta^2$  and the pseudo-time step. When we ignore the correction term and use an explicit formula it requires that the pseudo-time step also include a physical time step contribution. The amplification matrix in pseudo-time for  $\hat{w}_0$  variables, for the two dimensional Euler equation in generalized coordinates is given by

$$G(\theta) = \mathbf{P}_0 \left( \omega_0 \frac{c_t \text{Vol}}{\Delta t} + \omega_1 A + \omega_2 B \right) \quad (9)$$

where  $\text{Vol}$  is the volume of the cell and  $A, B$  are the Jacobian matrices of the inviscid flux vectors in the two generalized coordinate space dimensions.  $A$  and  $B$  are symmetric in  $\hat{w}_0$  variables and so this is a symmetric hyperbolic system where the physical  $t$  derivative is treated as another space derivative relative to the marching direction  $\tau$ . We denote the surface area of the cell as  $S_{ij}$  where the first subscript refers to the direction of the normal and the second is the projection of that normal in each direction. Define the contravariant velocity components as

$$\begin{aligned} U &= u S_{xx} + v S_{xy} + w S_{xz} \\ V &= u S_{yx} + v S_{yy} + w S_{yz} \\ W &= u S_{zx} + v S_{zy} + w S_{zz} \end{aligned} \quad (10)$$

In two space dimensions

$$\begin{aligned} G(\theta) &= \omega_0 \frac{c_t \mathbf{P}_0 \Delta t}{\text{Vol}} + \omega_1 \begin{pmatrix} \beta^2 U & \beta^2 c S_{xx} & \beta^2 c S_{xy} \\ c S_{xx} & U & 0 \\ c S_{xy} & 0 & U \end{pmatrix} \\ &+ \omega_2 \begin{pmatrix} \beta^2 V & \beta^2 c S_{yx} & \beta^2 c S_{yy} \\ c S_{yx} & V & 0 \\ c S_{yy} & 0 & V \end{pmatrix} \end{aligned}$$

In three dimensions let

$$D = \sqrt{\left( \frac{c_t \text{Vol}}{\Delta t} \right)^2 + U^2 + V^2 + W^2}.$$

Then choose

$$\omega_0 = \frac{c_t Vol}{\Delta t D} \quad \omega_1 = \frac{U}{D} \quad \omega_2 = \frac{V}{D} \quad \omega_3 = \frac{W}{D}$$

and so  $\omega_0^2 + \omega_1^2 + \omega_2^2 + \omega_3^2 = 1$ . This makes  $G(\theta)$  in (9) a convex combination of the Jacobian matrices. Define

$$\begin{aligned} \hat{U} &= \frac{S_{xx}U + S_{yx}V + S_{zx}W}{D \cdot Vol^2} \\ \hat{V} &= \frac{S_{xy}U + S_{yy}V + S_{zy}W}{D \cdot Vol^2} \\ \hat{W} &= \frac{S_{xz}U + S_{yz}V + S_{zz}W}{D \cdot Vol^2} \\ \hat{q}^2 &= \hat{U}^2 + \hat{V}^2 + \hat{W}^2 \end{aligned}$$

Thus  $\hat{U}, \hat{V}, \hat{W}$  are combinations of the velocity components  $u, v, w$  that depend on the geometry metrics. This combination comes from the analysis and does not necessarily have any physical interpretation. This gives

$$G = \begin{pmatrix} \beta^2 D & \beta^2 c \hat{U} & \beta^2 c \hat{V} & \beta^2 c \hat{W} \\ c \hat{U} & D & 0 & 0 \\ c \hat{V} & 0 & D & 0 \\ c \hat{W} & 0 & 0 & D \end{pmatrix}$$

We can easily symmetrize  $G$  but this is not necessary in our case. The eigenvalues are  $\lambda_0 = D$  and

$$\lambda_{\pm} = \frac{\beta^2 + 1}{2} D \pm \sqrt{\left(\frac{\beta^2 - 1}{2}\right)^2 D^2 + \beta^2 c^2 \hat{q}^2} \quad (11)$$

The analysis so far is based on inviscid equations: therefore  $\beta$  should be designated as  $\beta_{inv}$ , which is chosen such that  $\beta c \hat{q} = \lambda_0 = D$ . For low Mach numbers  $\beta^2$  is small and so  $\lambda_{\pm} \sim \frac{1 \pm \sqrt{5}}{2} D$ . We then choose  $\beta_{inv}$  as a term that depends on  $D$  plus a cutoff to prevent  $\beta$  from becoming too small. This cutoff depends on a global quantity  $M_{ref}$ . We choose

$$\beta_{inv}^2 = K_1 \frac{D^2}{c^2 \hat{q}^2} + K_2 M_{ref}^2 \quad (12)$$

We stress that  $\beta^2$  depends on  $\frac{c_t Vol}{\Delta t}$  even when using an implicit method for the time derivative term.  $K_1$  and  $K_2$  are user defined constants. In most cases  $K_2$  can be chosen as zero since the physical time derivative term prevents  $\beta$  from becoming too small. This is in contrast to steady state problems where this term is crucial.

We also account for the viscous effects in  $\beta^2$  but this is only used to reduce the time step rather than directly changing the  $\beta$  used in the preconditioning or the artificial viscosity.<sup>15</sup> For exterior problems we find that in the farfield where there are large volumes, the preconditioning is turned off locally due to the time derivative.

Similarly, when  $\Delta t_{phys}$  is small enough then preconditioning is turned off globally.

From (11) we define  $\lambda_{inv} = \lambda_+$ . We scale  $\lambda_{vis}$  by the volume so it also has dimensions  $\frac{1}{time}$ . We choose the total time step by

$$\Delta \tau = \frac{1}{\lambda_{inv} + a \lambda_{vis} + \frac{bc_t}{\Delta t}} \quad (13)$$

where  $a$  and  $b$  are constants. Typical values are  $a = 4-5$  and  $b = 9-25$ . These are determined semi-empirically based on numerical experiments. Straightforward linear analysis shows that using a simple implicit formula in each stage of the Runge-Kutta formula does not allow the use of a larger time step. Hence, even in this case we choose  $b = 25$ . In order to reduce this value of  $b$  one would need an explicit-implicit type formula as derived in<sup>1</sup>. Note that even when  $b = 0$ ,  $\Delta \tau$  depends on  $\Delta t_{phys}$  through  $\lambda_{inv}$  which is a function of  $\beta^2$ . However, for the explicit treatment of the time derivative we need to add a term to the time step that depends on  $\frac{c_t}{\Delta t}$  even when preconditioning is not used (i.e.  $\beta = 1$ ). Our choice for  $\beta$  and  $\Delta \tau$  is different than that of Venkateswaran and Merkle.<sup>18</sup> In addition the parameters of the residual smoothing should now depend on the physical time derivative term.

## Results

The governing equations are solved using a finite volume central difference code augmented by a matrix artificial viscosity.<sup>2,8</sup> The equations are advanced in pseudo-time by a five stage Runge-Kutta scheme accelerated by residual smoothing and multigrid.<sup>2,16</sup> The parameters for all the cases are identical with the exception that without preconditioning the explicit CFL in the residual smoothing is 3.75, while with preconditioning CFL-explicit is 3.25. For all preconditioning cases  $\beta_{min}^2 = M_{\infty}^2$ . The dual time-stepping uses a second order BDF formula stabilized with the algorithm of Melson and Sanetrik.<sup>6</sup> Early work on dual stepping and preconditioning is presented in.<sup>7,20</sup> Turkel et al.<sup>13</sup> compared results from two different preconditioning schemes for steady state problems.

### NACA4412

We first consider a steady state case, turbulent flow around a NACA4412 airfoil. A mesh with  $257 \times 81$  grid points, constructed using Wigton's<sup>19</sup> grid generation procedure and displayed in Fig. 1, is used for these computations. The inflow conditions are  $\alpha = 13.87^\circ$  and  $M_{\infty} = 0.2, 0.05$  and  $0.01$ . The Reynolds number is  $1.52 \times 10^6$  and the Baldwin-Lomax turbulence model is used.

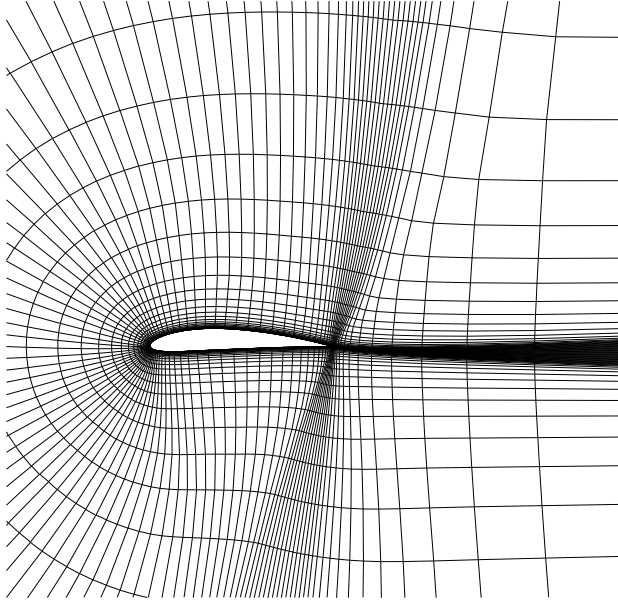


Fig. 1 Partial view of grid for NACA4412

We compare the convergence rate without preconditioning versus preconditioning based on conservation variables and primitive ( $p, u, v, w, T$ ) variables. For all preconditioned cases we use  $K_2 = 1.0$ . We show a portion of the grid in figure 1. Every second grid line is shown to increase the clarity. There are 50 iterations on two coarser grids and 500 five stage RK iterations on the finest grid. In figures 2 and 3 we compare the residual as well as the drag history for  $M_\infty = 0.2$ . We stress that the only difference between the two preconditioned results is the artificial viscosity, (Eqns. 2 and 7). In one case it is evaluated directly on the conservation variables. In the other case we take differences of the primitive variables, multiply by  $\Gamma^{-1}|\mathbf{P}\mathbf{A}|$  in primitive variables and transform back to conservation variables. We see in the following figures that in the beginning of the computation both approaches give the same rate of convergence and accuracy. However, for small residuals an artificial viscosity based on the conservation variables stalls. When the artificial viscosity is based on the primitive variables convergence continues to decrease further. Our explanation for this phenomena is that for the conservation variables the density is fairly constant. Hence, at low residual values the contribution from density derivatives is negligible. For the first few iterations the preconditioning slows the convergence rate of the residual. However, the drag converges faster even at the beginning using preconditioning. Overall convergence is improved significantly when preconditioning is used.

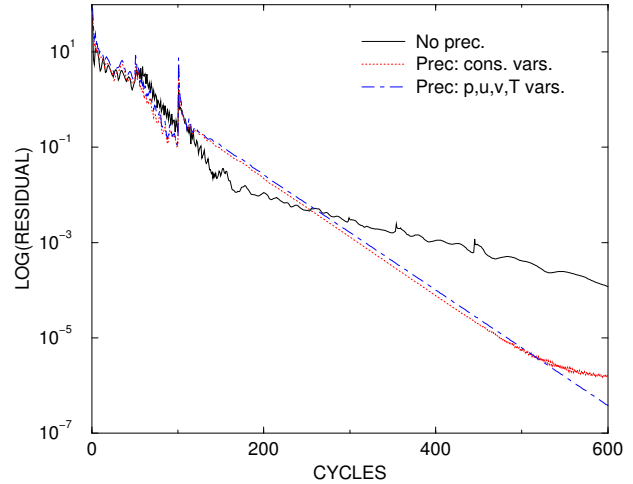


Fig. 2 NACA4412 -  $M_\infty = 0.2$ , residual history

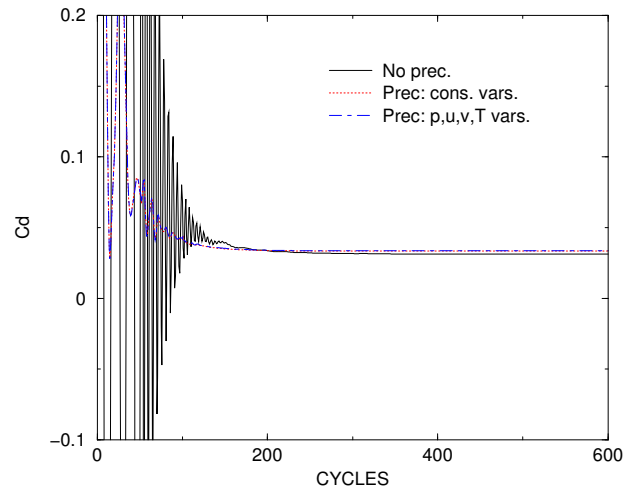
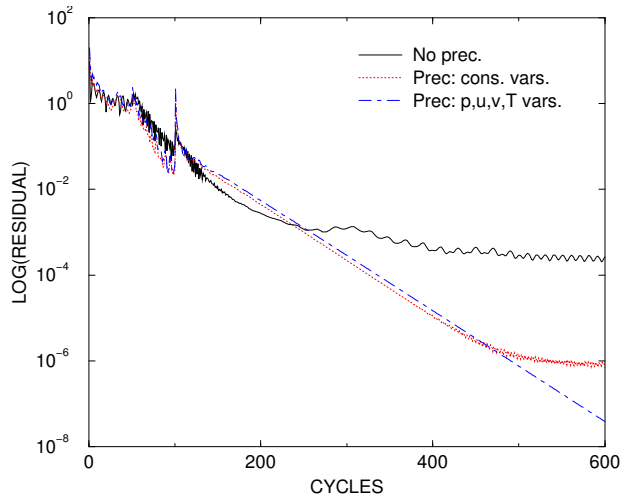
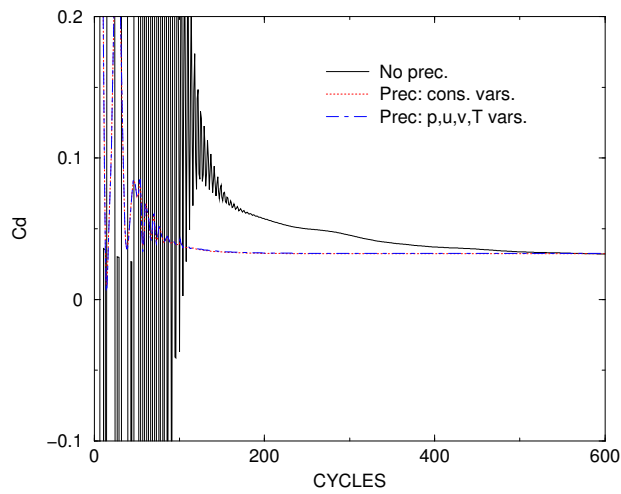


Fig. 3 NACA4412 -  $M_\infty = 0.2$ , drag history

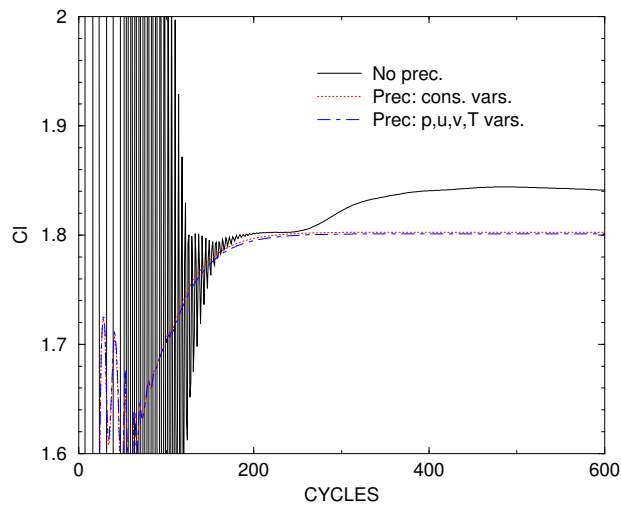
We next consider the same case but with an inflow Mach number 0.05. The results are given in figures 4 - 6. The improved convergence for the preconditioning is more evident for this case. The convergence for the conservation variables is reasonable until it bottoms out. The convergence with the primitive variables continues in a straight line. Note the difference in the final lift coefficient between the preconditioned and non-preconditioned schemes. This is a reflection of the decrease in accuracy without preconditioning for low Mach numbers.<sup>11,14</sup>



**Fig. 4** NACA4412 -  $M_\infty = 0.05$ , residual history

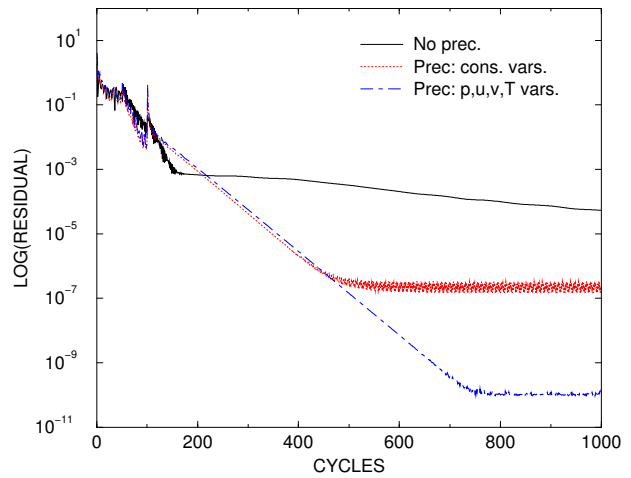


**Fig. 5** NACA4412 -  $M_\infty = 0.05$ , drag history

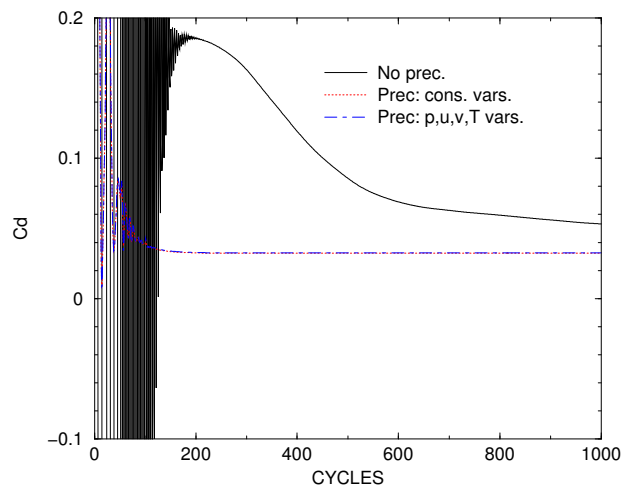


**Fig. 6** NACA4412 -  $M_\infty = 0.05$ , lift history

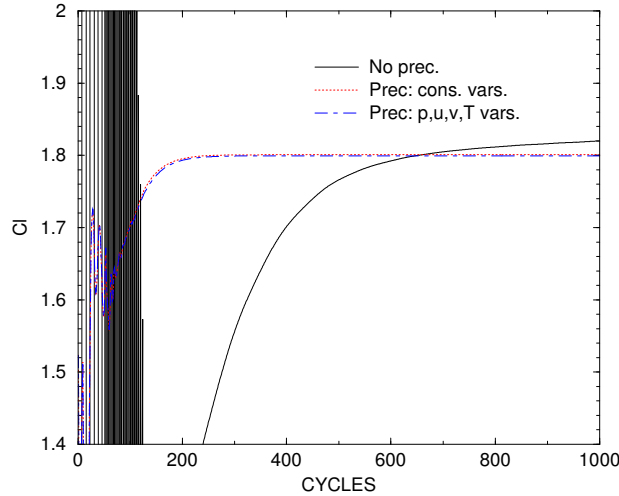
We finally show the results in figures 7 through 9 for the same case but an inflow Mach number 0.01. The difference in lift and drag between the preconditioned and non-preconditioned algorithms is now quite noticeable. The pattern of the convergence history is similar to the previous cases but is more dramatic. We now clearly see that using conservation variables in the artificial viscosity limits the convergence of the residual even with preconditioning.



**Fig. 7** NACA4412 -  $M_\infty = 0.01$ , residual history



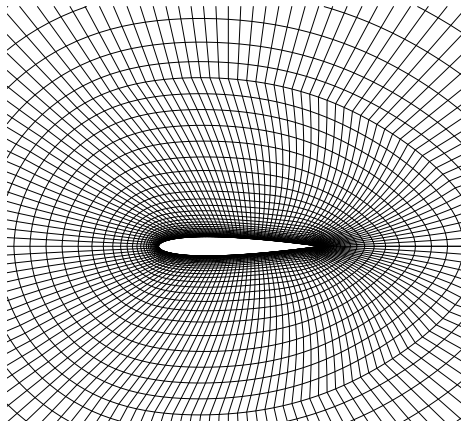
**Fig. 8** NACA4412 -  $M_\infty = 0.01$ , drag history



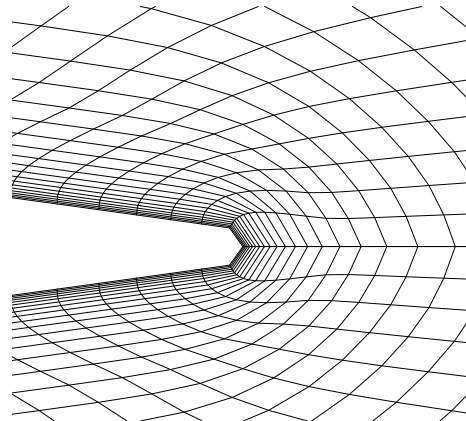
**Fig. 9** NACA4412 -  $M_\infty = 0.01$ , lift history

### NACA0012- high angle of attack

We next consider turbulent flow around a NACA0012 airfoil. An O mesh containing  $141 \times 61$  points (Fig. 10) is used for this configuration. An exploded view of the grid near the trailing edge is shown in Fig. 11. The inflow conditions are  $M_\infty = 0.1$ , an angle of attack of  $12^\circ$  and  $Re = 3 \times 10^6$ . A Spalart-Allmaras turbulence model is used. At this angle of attack the flow is still steady. We also consider an angle of attack of  $30^\circ$  where the flow is no longer steady.



**Fig. 10** Partial view of grid for NACA0012



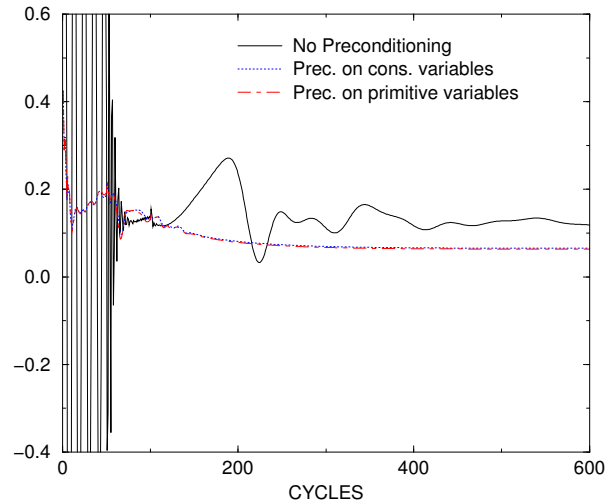
**Fig. 11** Grid near trailing edge of NACA0012

We begin with the steady state calculation for  $\alpha = 12^\circ$ . The convergence of the residual and drag coefficient are shown in Figs. 12 and 13. We see that without preconditioning the convergence stalls, as expected, with an inflow Mach number  $M_\infty = 0.1$ . Basing the preconditioning on the conservation variables (Eqn. 2) improves the convergence rate. However, using primitive variables in the artificial viscosity (Eqn. 6) yields a better asymptotic rate. Although not shown here, we observed that the variable used for the update ( Eqn. 6 versus Eqn. 7) has no effect on the convergence rate. When using  $w_c$  variables the residual is based on the density while when using  $Q$  variables the residual is based on the pressure. We see that the preconditioning yields significantly better convergence than the non-preconditioned scheme. Furthermore, the steady state is different when preconditioning is used. Based on earlier work of Turkel et al.,<sup>11</sup> we expect the preconditioned results to be more accurate.

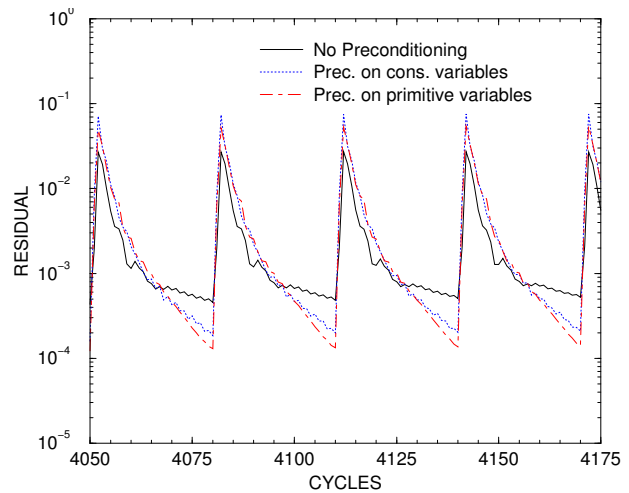
We next examine the convergence and accuracy for the time dependent case. We consider an angle of attack with  $\alpha = 30^\circ$  and inflow Mach number of  $M_\infty = 0.1$ . Since the flow is now time dependent we are only interested in the convergence rate within a physical time cycle. We use 30 cycles (5 stage Runge-Kutta with multigrid and residual smoothing) of pseudo-time stepping within each physical time cycle. In Fig. 14 we display the residual (density for  $w_c$  variables and pressure for  $Q$  variables) while in Fig. 15 we display the lift. In both cases we see that initially the convergence is best without any preconditioning. The residual stalls after a few cycles but the lift has already reached its new level to within graphical

accuracy. The overall convergence rate for the residual is improved with preconditioning on conservative and primitive variables. The convergence of lift with preconditioning with an artificial viscosity based on conservation,  $w_c$  variables is slightly worse but is also reasonable. When the preconditioning uses primitive variables in the artificial viscosity then the initial convergence is slowed and one requires about 15-20 subiteration cycles for the lift to converge to graphical accuracy. In all cases the variables used to advance the solution to the next cycle had little effect (figure not shown here) on either the convergence rate of the subiterations or the on the accuracy. The major influence is due to the variables used in the artificial viscosity.

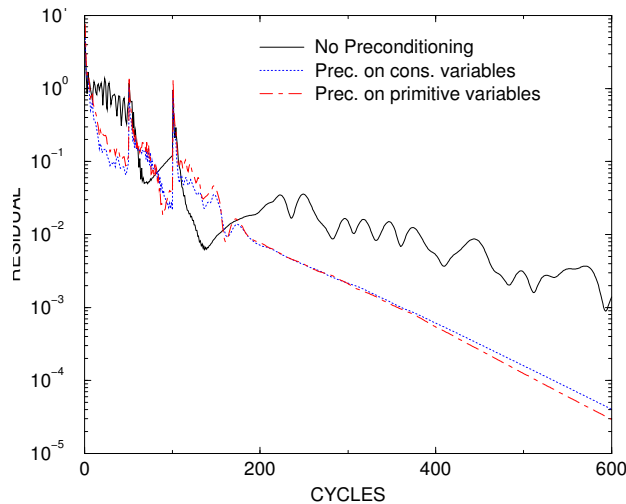
We stress that the use of preconditioning affects the accuracy of the solution and so changes the values of the lift and drag. In Fig. 16 we plot the time dependent history of the lift. We clearly see that for short times the two preconditioned results, with the artificial viscosity based on conservation or primitive variables, agree with each other but give a different lift than the non-preconditioned algorithm. Based on the results from low speed steady flows that demonstrated improved accuracy with preconditioning,<sup>11</sup> it is reasonable to assume that the unsteady solutions obtained here with preconditioning are more accurate compared to the un-preconditioned solutions.



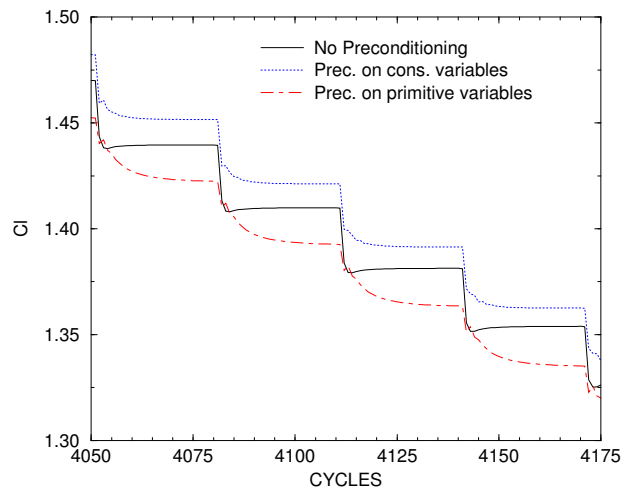
**Fig. 13**  $C_d$  for steady state NACA0012



**Fig. 14** Residual within subiterations for  $\alpha = 30^\circ$



**Fig. 12** Residual for steady state NACA0012



**Fig. 15**  $c_L$  within subiterations for  $\alpha = 30^\circ$



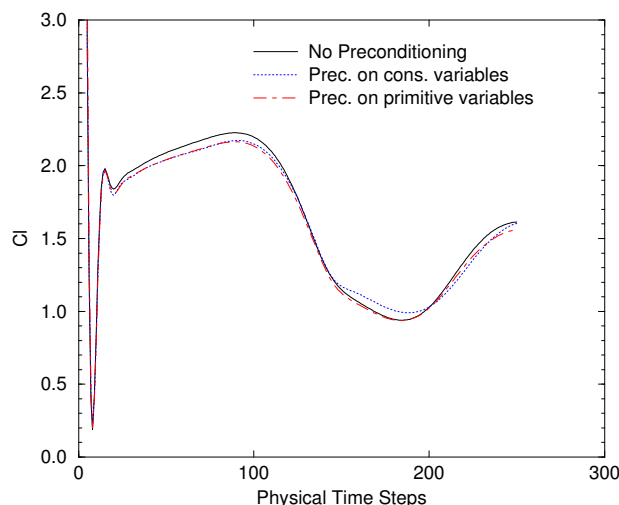


Fig. 16 Time dependent  $c_L$  for  $\alpha = 30^\circ$

## Concluding Remarks and Future Directions

A general preconditioning formulation for treating low speed flows with compressible flow equations is presented. Several alternative forms of preconditioning variables have been examined. The choice of the variables used to advance the solution in time had very little effect on convergence or accuracy. The convergence for low speed steady state problems is improved significantly when preconditioning is used. The primitive variables based preconditioner appears to be most effective for steady state flows. However, for time-dependent problems, preliminary results about the choice of preconditioners are somewhat inconclusive, and requires further testing.

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