Preconditioning and the Limit of the Compressible to the Incompressible Flow Equations for Finite Difference Schemes

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Abstract

We consider the use of preconditioning methods to accelerate the convergence to a steady state for both the incompressible and compressible fluid dynamic equations. We also consider the relation between them for both the continuous problem and the finite difference approximation. In particular we examine how preconditioning can be used to improve the accuracy of the steady state numerical approximation especially for small mach numbers. The analysis relies on the inviscid equations. We find necessary properties of the artificial viscosity, or Roe matrix, in order to have convergence as the Mach number goes to zero. An application to flow around an airfoil is presented.

1 Introduction

It is well known, that it is difficult to solve the compressible equations for low Mach numbers. For an explicit scheme this is easily seen by inspecting the time steps. For stability, the time step must be chosen inversely proportional to the largest eigenvalue of the system which, for slow flows, is approximately the speed of sound, c. However, other waves are convected at the fluid speed, u , which is much slower. Hence, these waves don't change very much over a time step. Thus, thousands of time steps may be required to reach a steady state. Should one try a multigrid acceleration one finds that the same disparity in wave speeds slows down the multigrid acceleration. With an implicit method an ADI factorization is usually used so that one can easily invert the implicit factors. The use of ADI introduces factorization errors which again slows down the convergence rate when there are wave speeds of very different magnitudes [15] .

For small Mach numbers it can be shown ([8],[10],[12]) that the incompressible equations approximate the compressible equations. Hence, one needs to justify the computational use of the compressible equations for low Mach flows. We present several reasons why one would still use the compressible equations even though the Mach number of the flow is small.

There are many highly efficient compressible codes available that could used for such problems especially in complicated geometries. For low speed aerodynamic problems at high angle of attack most of the of the flow consists of a low Mach number flow. However, there are localized regions containing shocks. In many problems thermal effects are important and the energy equation is coupled to the other equations. Then, the compressible equations must be used even for low Mach number flows.

Thus, for both the incompressible and compressible equations we consider systems of the form

$$w_t + f_x + g_y = 0.$$

This system is written in conservation form though for some applications this is not necessary. Our analysis will be based on the linearized equations so that the conservation form does not appear in the analysis though it does appear in the final numerical approximation. This system is now replaced by

$$\mathbf{P}^{-1} w_t + f_x + g_y = 0,$$

or in linearized form

$$\mathbf{P}^{-1} w_t + A w_x + B w_y = 0, \tag{1}$$

with A and B constant matrices.

In order for this system to be equivalent to the original system, in the steady state, we demand that $\mathbf{P}^{-1}$ have an inverse. This only need be true in the flow regime under consideration. We shall see later that frequently $\mathbf{P}$ is singular at stagnation points and also along sonic lines. Thus, we will consider strictly subsonic flow without a stagnation point. For transonic flow it is necessary to smooth out the singularity in a neighborhood of the sonic line. We also assume that the Jacobian matrices $A = \frac{\partial f}{\partial w}$ and $B = \frac{\partial g}{\partial w}$ are simultaneously symmetrizable . In terms of the 'symmetrizing' variables we also demand that $\mathbf{P}$ be positive definite.

Assuming the steady state has a unique solution, it does not matter which system we march to a steady state. We shall later see that for the finite difference approximations the steady state solutions are not necessarily the same and usually the preconditioned system leads to a better behaved steady state.

2 Incompressible equations

We first consider the incompressible inviscid equations in primitive variables.

$$\begin{aligned} u_x + v_y &= 0 \\ u_t + uu_x + vu_y + p_x &= 0 \\ v_t + uv_x + vv_y + p_y &= 0 \end{aligned}$$

We consider generalizations of Chorin's pseudo-compressibility method [3]. Using the preconditioning suggested in [18] (with $\alpha = 1$) we have

$$\begin{aligned} \frac{1}{\beta^2} p_t + u_x + v_y &= 0 \\ \frac{u}{\beta^2} p_t + u_t + uu_x + vu_y + p_x &= 0 \\ \frac{v}{\beta^2} p_t + v_t + uv_x + vv_y + p_y &= 0 \end{aligned} \tag{2}$$

We can also write this in matrix form using

$$\mathbf{P}^{-1} = \begin{pmatrix} 1/\beta^2 & 0 & 0 \\ u/\beta^2 & 1 & 0 \\ v/\beta^2 & 0 & 1 \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} \beta^2 & 0 & 0 \\ -u & 1 & 0 \\ -v & 0 & 1 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 1/\beta^2 & 0 & 0 \\ u/\beta^2 & 1 & 0 \\ v/\beta^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & 1 & 0 \\ 1 & u & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & v & 0 \\ 1 & 0 & v \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_y = 0 \tag{3}$$

Multiplying by $\mathbf{P}$ we rewrite this as

$$w_t + \mathbf{P}Aw_x + \mathbf{P}Bw_y = 0.$$

We also define

$$D = \omega_1 A + \omega_2 B \qquad -1 \le \omega_1, \omega_2 \le 1 \tag{4}$$

where ω_1, ω_2 are the Fourier transform variables in the x and y directions respectively. The speeds of the waves are now given by the roots of $det(\lambda I - \mathbf{P}A\omega_1 - \mathbf{P}B\omega_2) = 0$. Let $q = u\omega_1 + v\omega_2$. Then the eigenvalues of $\mathbf{P}D$ are

$$d_0 = q \quad , \quad d_\pm = \pm\beta \tag{5}$$

and so the 'acoustic' speed is isotropic.

The spatial derivatives involve symmetric matrices, i.e. D is a symmetric matrix but $\mathbf{P}$ is not symmetric. Thus, while the original system was symmetric hyperbolic the preconditioned system is no longer symmetric. In [18] it is shown that as long as

$$\beta^2 > (u^2 + v^2)$$

then the equations can be symmetrized. On the other hand the eigenvalues are most equalized if $\beta^2 = (u^2+v^2)$ [18]. Hence, we wish to choose β^2 slightly larger than u^2+v^2. However, numerous calculations verify, that in general, a constant β is the best for the convergence rate.

We wish to stress that β has the dimensions of a speed. Therefore, β cannot be a universal constant. There are papers that claim that $\beta = 1$ or $\beta = 2.5$ are optimal. Such claims cannot be true in general. It is simple to see that if one nondimensionalizes the equation then β gets divided by a reference velocity. Hence, the optimal 'constant' β depends on the dimensionalization of the problem and in particular depends on the inflow conditions. In many calculations the inflow mass flux is equal to 1 or else $p + (u^2+v^2)/2 = 1$. Such conditions will give an optimal β close to one. However, if one chooses the incoming mass flux as ten then the optimal β will be larger.

We next define the Bernoulli function

$$H = p + (u^2+v^2)/2.$$

Bernoulli's theorem states that for steady inviscid flow H is constant along streamlines. We now multiply the second equation of (2) by u and the third equation of (2) by v and add these two equations. If $\beta^2 = u^2+v^2$, the result is

$$H_t + uH_x + vH_y = 0. \tag{6}$$

Thus, by altering the time dependence of the equations we have constructed a new equation in which H is convected along streamlines. Furthermore, if H is a uniform constant both initially and at inflow then H will remain constant for all time. On the numerical level this will usually not be true because of the introduction of an artificial viscosity or because of upwinding. For viscous flow, (6) is replaced by

$$H_t + uH_x + vH_y = \frac{1}{Re}(u\Delta u + v\Delta v)$$

We note that these relationships for H follow from the momentum equations and do not depend on the form of the continuity equation. Hence, we consider the following generalization of (2)

$$\begin{aligned}
\frac{1}{\beta^2}p_t + aH_t + u_x + v_y &= 0\\
\frac{\alpha u}{\beta^2}p_t + u_t + uu_x + vu_y + p_x &= 0\\
\frac{\alpha v}{\beta^2}p_t + v_t + uv_x + vv_y + p_y &= 0
\end{aligned}$$

where, a is a free parameter. The eigenvalues of $\mathbf{PD}$ are independent of the parameter a and are given by (5). For $a = 0, \alpha = 1$ we recover our original scheme. For $a = -1$ the time derivative of the pressure no longer appears in the continuity equation. For general β we have

$$\mathbf{P}^{-1} = \frac{1}{\beta^2}\begin{pmatrix} (a+1) & au & av \\ \alpha u & \beta^2 & 0 \\ \alpha v & 0 & \beta^2 \end{pmatrix},$$

$$\mathbf{P} = \frac{1}{d}\begin{pmatrix} \beta^2 & -au & -av \\ -\alpha u & 1+a-\frac{a\alpha v^2}{\beta^2} & \frac{a\alpha uv}{\beta^2} \\ -\alpha v & \frac{a\alpha uv}{\beta^2} & 1+a-\frac{a\alpha u^2}{\beta^2} \end{pmatrix}$$

where $d = 1 + a - a\alpha\frac{u^2+v^2}{\beta^2}$ and we require that $d \geq 0$. If $\beta^2 = u^2 + v^2$ and $\alpha = 1$ then

$$\mathbf{P} = \begin{pmatrix} u^2+v^2 & -au & -av \\ -u & 1+\frac{au^2}{u^2+v^2} & \frac{auv}{u^2+v^2} \\ -v & \frac{auv}{u^2+v^2} & 1+\frac{av^2}{u^2+v^2} \end{pmatrix}$$

In [19] an analogy to the symmetric preconditioning of van Leer, Lee and Roe was constructed for the incompressible equations. If we choose $a = 1, \alpha = 1$ we get this preconditioning of van Leer et. al . , i.e. $\mathbf{P}$ is symmetric .

These examples show that the preconditioning is not unique. If fact, since the determinant of the transpose of a matrix is equal to the determinant of the original matrix it follows that the transpose of $\mathbf{P}$ is also a preconditioner with the same eigenvalues for the preconditioned system. In general, these various systems will have similar eigenvalues but different eigenvectors for the preconditioned system. Calculations show that the system given by $\mathbf{P}$ in (2) is more robust and converges faster than that with the transpose preconditioner. This shows that it is not sufficient to consider just the eigenvalues but that the eigenvectors are also of importance. However, even when $\mathbf{P}$ is symmetric $\mathbf{P}$D is not symmetric and so the eigenvectors of the preconditioned system do not form an orthogonal basis.

We next examine some general form that the preconditioner can have. For this analysis it is easier to use streamwise coordinates as suggested in [20] and so $v = 0$. Let u_* be some normalization of the velocity components, then

$$A = \begin{pmatrix} 0 & u_* & 0 \\ u_* & u & 0 \\ 0 & 0 & u \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & u_* \\ 0 & 0 & 0 \\ u_* & 0 & 0 \end{pmatrix}$$

Then the "convective" eigenvector for the non-preconditioned system is

$$\begin{pmatrix} 0 \\ \omega_2 \\ -\omega_1 \end{pmatrix}.$$

The "acoustic" eigenvectors are given by

$$\begin{pmatrix} \frac{-u\omega_1+\sqrt{(u\omega_1)^2+4u_*^2}}{2} \\ u_*\omega_1 \\ u_*\omega_2 \end{pmatrix}, \quad \begin{pmatrix} \frac{-u\omega_1-\sqrt{(u\omega_1)^2+4u_*^2}}{2} \\ u_*\omega_1 \\ u_*\omega_2 \end{pmatrix}.$$

We now consider preconditioners of the form

$$\mathbf{P} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

Given D (4, we want the eigenvalues of $\mathbf{P}D$ to be $\omega_1 u$, $\pm u$. This gives us three relations for the four unknowns:

$$a = \frac{u^2}{u_*^2}$$

$$(b+c)u_* + du = 0$$

$$u^2 d - bcu_*^2 = u^2$$

The values suggested in [18] are $b = 0, c = -\frac{u}{u_*}, d = 1$ while the values suggested in [20] are $b = c = -\frac{u}{u_*}, d = 2$. We next present the eigenvectors of $\mathbf{P}D$ in terms of the elements of $\mathbf{P}$. We exclude the case $\omega_2 = 0, \omega_1 = \pm 1$ as in this case $\mathbf{PD}$ has a double eigenvalue u and the eigensystem completely changes. Then the "convective" eigenvector is

$$\begin{pmatrix} 0 \\ \omega_2 \\ -(1 + b\frac{u_*}{u})\omega_1 \end{pmatrix}.$$

The "acoustic" eigenvectors are given by

$$\begin{pmatrix} \frac{u^2}{u_*} - bu\omega_1^2 - \frac{u^2\omega_1}{u_*} + bu\omega_1 \\ u_*(a + b\omega_1^2) - u_*(b+c)\omega_1 \\ (u_* b\omega_1 + u)\omega_2 \end{pmatrix}, \quad \begin{pmatrix} \frac{u^2}{u_*} - bu\omega_1^2 + \frac{u^2\omega_1}{u_*} - bu\omega_1 \\ u_*(a + b\omega_1^2) + u_*(b+c)\omega_1 \\ (u_* b\omega_1 - u)\omega_2 \end{pmatrix}.$$

We note that the convective eigenvector is the same as before the preconditioning for the choice $b = 0$. The two acoustic eigenvectors are orthogonal to each other if we choose $b = 0$ and $c^2 = \frac{u^2(1 - \frac{u^2}{u_*^2})}{u_*^2}$. This is similar, but not identical, to the choice suggested in [18]. There is no way to make the convective eigenvector normal to both acoustic eigenvectors for preconditioners of the form (7).

3 Compressible equations

The time dependent Euler equations can be written as

$$\begin{aligned} p_t + up_x + vp_y + \rho a^2(u_x + v_y) &= 0 \\ u_t + uu_x + vu_y + \frac{p_x}{\rho} &= 0 \\ v_t + uv_x + vv_y + \frac{p_y}{\rho} &= 0 \\ S_t + uS_x + vS_y &= 0 \end{aligned} \tag{8}$$

where a is the speed of sound given by $a^2 = \frac{\gamma p}{\rho}$.

The form of this system is unchanged if we nondimensionalize the equations. From now on we shall assume that u, v, p, ρ are nondimensional quantities where the dimensional variables are nondimensionalized by u_*, p_*, ρ_* with $p_* = \rho_* u_*^2$. Following [8] we define $\epsilon = \frac{u_*}{a_*} = M_*$. If the fluid is isentropic then

$$p = \frac{\rho^\gamma}{\gamma\epsilon^2} \quad , \quad a = \frac{\rho^{\frac{\gamma-1}{2}}}{\epsilon} \quad . \tag{9}$$

Hence, as ϵ goes to zero the speed of sound, a, goes to infinity and so the first equation in (8) reduces to $u_x + v_y = 0$.

It was pointed out in ([18], [19]) that these equations can be symmetrized by using $\frac{dp}{\rho a}$ as the independent variable rather than dp . Hence, we define ϕ by $d\phi = \frac{dp}{\rho a}$. For isentropic flow both p and a are functions only of the density and so using (9) this can be integrated explicitly. This gives $\phi = \frac{\rho^{\frac{\gamma-1}{2}}}{\frac{\gamma-1}{2}\epsilon}$. As the Mach number goes to zero ϕ tends to infinity and therefore, Gustafsson and Stoor [8] subtract a constant and define

$$\phi = \frac{\rho^{\frac{\gamma-1}{2}} - 1}{\frac{\gamma-1}{2}\epsilon} \quad . \tag{10}$$

This amounts to specifying the constant in the integration of $d\phi$ from dp. They then prove, using energy methods, that for the linearized equations

$$a\phi_x \to \frac{\partial p_{incompressible}}{\partial x} \quad .$$

Hence, ϕ and all its derivatives behave as $O(M)$ as $M \to 0$. Since $\rho \to 1$ and using the definition of $d\phi$ this is equivalent to

$$dp_{compressible} \to dp_{incompressible}. \tag{11}$$

We consider preconditionings that are a generalization of (3) :

$$\begin{pmatrix} \frac{a^2}{\beta^2} & 0 & 0 & 0 \\ \frac{\alpha a u}{\beta^2} & 1 & 0 & 0 \\ \frac{\alpha a v}{\beta^2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{dp}{\rho a} \\ du \\ dv \\ dS \end{pmatrix}_t + \begin{pmatrix} u & a & 0 & 0 \\ a & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \begin{pmatrix} \frac{dp}{\rho a} \\ du \\ dv \\ dS \end{pmatrix}_x + \begin{pmatrix} v & 0 & a & 0 \\ 0 & v & 0 & 0 \\ a & 0 & v & 0 \\ 0 & 0 & 0 & v \end{pmatrix} \begin{pmatrix} \frac{dp}{\rho a} \\ du \\ dv \\ dS \end{pmatrix}_y = 0$$

The nonpreconditioned case corresponds to $\beta^2 = a^2, \alpha = 0$. Let $q = u\omega_1 + v\omega_2$ then the eigenvalues of $\mathbf{P}$D are given by

$$d_0 = q \quad (double)$$

$$d_\pm = 1/2\left[(1-\alpha+\beta^2/a^2)q \pm \sqrt{((1-\alpha+\beta^2/a^2)^2q^2 + 4(1-q^2/a^2)\beta^2}\right] \quad .$$

If we consider the special case $\alpha = 1 + \beta^2/a^2$ we find that the 'acoustic' eigenvalues are given by

$$d_\pm = \sqrt{(1-q^2/a^2)\beta^2} \quad .$$

Hence, these eigenvalues are isotropic in the limit of M going to zero. However, this eigenvalue vanishes at the sonic line and so the matrix is singular. In general, if we demand that the acoustic eigenvalues be isotropic then we have a singularity at the sonic line where the eigenvalues cannot be isotropic. The two ways out of this difficulty are either to smooth the formulas near the singular line or else to give up on isotropy. This difficulty is not a property of the preconditioning just presented but applies equally to all preconditioners.

We next present the van Leer-Lee-Roe preconditioning for general nonaligned flow in $(\frac{dp}{\rho a}, du, dv, dS)$ variables [20].

$$\mathbf{P}_V = \begin{pmatrix} \frac{\tau}{\beta^2}M^2 & -\frac{\tau}{\beta^2}u/a & -\frac{\tau}{\beta^2}v/a & 0 \\ -\frac{\tau}{\beta^2}u/a & (\frac{\tau}{\beta^2}+1)\frac{u^2}{u^2+v^2} + \tau\frac{v^2}{u^2+v^2} & (\frac{\tau}{\beta^2}+1)\frac{uv}{u^2+v^2} & 0 \\ -\frac{\tau}{\beta^2}v/a & (\frac{\tau}{\beta^2}+1)\frac{uv}{u^2+v^2} & (\frac{\tau}{\beta^2}+1)\frac{v^2}{u^2+v^2} + \tau\frac{u^2}{u^2+v^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\beta = \begin{cases} \sqrt{1-M^2}, & M < 1, \\ \sqrt{M^2-1}, & M \geq 1; \end{cases} \quad , \quad \tau = \begin{cases} \sqrt{1-M^2}, & M < 1, \\ \sqrt{1-M^{-2}}, & M \geq 1. \end{cases}$$

At the sonic line $\beta = 0$ and $\tau = 0$ and the preconditioning matrix becomes singular.

4 Convergence to the incompressible equations

We have previously referred to several papers ([8],[10], [12]), that prove the convergence of the compressible equations to the incompressible equations, for isentropic flow, as the Mach number goes to zero. For nonisentropic flow there are no formal proofs. However, it is clear that for viscous flows that the boundary condition on the temperature, adiabatic or isothermal is very important, see [14] .

All these results refer to the time dependent physical equations. Once preconditioning is introduced time accuracy is lost and one can only discuss convergence of the steady state solutions. In this case one would hope that the time dependent preconditioned compressible equations converge to some time dependent preconditioning of the incompressible equations. In addition, one would also want this to be true on the numerical level. Thus, one would want to solve the preconditioned compressible equations by some numerical technique, on a fixed mesh and compare that with the solution of the incompressible equations on the same mesh. Mathematically, we have two limit processes occurring: the Mach number going to zero and the mesh size going to zero. These two limits need not commute. If one first converges the mesh size and then the Mach number it is equivalent to the convergence proofs for the analytic case. The more interesting problem is to converge the Mach number and then converge the mesh, i.e. we use a fix mesh as the Mach number is reduced.

In particular this requires a careful study of the viscosities introduced by the scheme. We first consider an upwinding scheme. For the compressible case the Riemann solver should depend on the preconditioned problem. One would then need to show that this Riemann problem converges to a Riemann problem for some preconditioning of the incompressible equations. We next consider a central difference scheme with a scalar viscosity. In this case a high order even difference of some quantity is added separately to each equation, e.g. for the incompressible equations: pressure for the continuity equation, u and v for the momentum equations. For the compressible equations one normally adds a density difference to the continuity equation. In such a case it is obvious that the numerical scheme for the compressible equations cannot converge to the numerical scheme for the incompressible equations. Furthermore, for low Mach number flows the density is almost constant and so the higher order difference of the density does not add much of a viscosity to the continuity equation. As such, we conclude that the artificial viscosity for the compressible continuity equation should be based on pressure and not density (at least for low Mach numbers).

By convergence of the compressible equations to the incompressible equations we are merely verifying what happens to the difference equations as the Mach number goes to zero. This gives neccessary but not sufficient conditions for the convergence of the solution of the numerical approximation to the preconditioned compressible equations to the numerical solution of the incompressible equations. We first consider a central difference approximation together with a scalar artificial viscosity for the nondimensionalized preconditioned inviscid equations. For the incompressible equations in nonconservative form we consider the preconditioned system

$$\begin{aligned}
p_t + \beta^2(u_x + v_y) &= h\left[(K_1 p_x)_x + (K_2 p_y)_y\right] \\
\frac{u}{\beta^2} p_t + u_t + u u_x + v u_y + p_x &= h\left[(K_1 u_x)_x + (K_2 u_y)_y\right] \\
\frac{v}{\beta^2} p_t + v_t + u v_x + v v_y + p_y &= h\left[(K_1 v_x)_x + (K_2 v_y)_y\right]
\end{aligned} \tag{12}$$

where each space derivative is approximated by a central difference with spacing h in each direction. We can also consider higher order accurate artificial viscosities but this doesn't change any of the properties that are of interest for the convergence to the incompressible limit. The time derivatives are replaced by a multistage scheme. K_1, K_2 are proportional to the largest eigenvalues of the coefficient matrix in the respective direction.

We next consider the same scheme for the preconditioned compressible inviscid equations, under the assumption that $p = p(\rho)$. It easier to analyze the convergence for the nonsymmetrical form since the pressure, p, converges and not $\frac{dp}{\rho a}$, see (11). Then for the compressible equations we have

$$\begin{aligned} p_t + \beta^2 \left[(\rho u)_x + (\rho v)_y) \right] &= h \left[(K_1 p_x)_x + (K_2 p_y)_y \right] \\ \frac{u}{\beta^2} p_t + u_t + u u_x + v u_y + \frac{p_x}{\rho} &= h \left[(K_1 u_x)_x + (K_2 u_y)_y \right] \\ \frac{v}{\beta^2} p_t + v_t + u v_x + v v_y + \frac{p_y}{\rho} &= h \left[(K_1 v_x)_x + (K_2 v_y)_y \right] \end{aligned} \tag{13}$$

Comparing (12) with (13) it is obvious that if $\rho \to 1$ as $M \to 0$ then (12) converges to (13). It is crucial for both the time derivative and the artificial viscosity in the compressible continuity equation to be pressure based rather than density based. The preconditioning of the momentum equations is not important for this convergence.

For the incompressible equations in conservative form we multiply the first equation in (12) by u and add it to the second and third equations. However, we do not change the artificial viscosity. Then

$$\begin{aligned} p_t + \beta^2 (u_x + v_y) &= h \left[(K_1 p_x)_x + (K_2 p_y)_y \right] \\ \frac{2u}{\beta^2} p_t + u_t + (u^2 + p)_x + (uv)_y &= h \left[(K_1 u_x)_x + (K_2 u_y)_y \right] \\ \frac{2v}{\beta^2} p_t + v_t + (uv)_x + (v^2 + p)_y &= h \left[(K_1 v_x)_x + (K_2 v_y)_y \right] \end{aligned} \tag{14}$$

For the compressible equations in conservative form we have two choices. One choice is to multiply the first equation in (13) by u and the second by ρ and add the two. The spatial derivatives are then in conservation form. However, the time derivative is of the form ρu_t rather than $(\rho u)_t$ and the artificial viscosity terms are not in conservation form. Hence, we instead choose to apply the preconditioning directly to the conservative form. The resultant preconditioned compressible equations in conservative form is

$$\begin{aligned} p_t + \beta^2 \left[(\rho u)_x + (\rho v)_y) \right] &= h \left[(K_1 p_x)_x + (K_2 p_y)_y \right] \\ \frac{2u}{\beta^2} p_t + (\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y &= h \left[(K_1 (\rho u)_x)_x + (K_2 (\rho u)_y)_y \right] \\ \frac{v}{\beta^2} p_t + (\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y &= h \left[(K_1 (\rho v)_x)_x + (K_2 (\rho v)_y)_y \right] \end{aligned} \tag{15}$$

Note that (15) is not equivalent to (13).

We now consider how to construct a general artificial viscosity that will enable us to reach the incompressible limit without the difficulties exhibited in the previous section. Consider

$$\mathbf{P}^{-1} w_t + f_x + g_y = h \left[(Q_1 w_x)_x + (Q_2 w_y)_y \right] \quad . \tag{16}$$

We wish to find the dependence of $\mathbf{P}$ and Q_i on the Mach number as $M \to 0$ so that we get the proper convergence. We therefore consider the isentropic equations based on $w = (\phi, u, v)$ (see (10)). This has the symmetric form

$$\mathbf{P}^{-1} w_t + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{11} & a_{23} & a_{33} \end{pmatrix} w_x + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{11} & b_{23} & b_{33} \end{pmatrix} w_y = h \left[(Q_1 w_x)_x + (Q_2 w_y)_y \right] \quad .$$

As $M \to 0, a_{12}$ and $b_{13} = O(1/M)$ while $d\phi = O(M)$ while all other quantities are bounded. Hence, the leading terms in the first equation are all $O(1/M)$ while they are $O(1)$ for the second and third equations. Multiplying the first equation by M and taking the limit we get $u_x + v_y$ for the space derivatives on the left hand side. Using $d\phi = O(M), du = O(1), dv = O(1)$ we see that a necessary condition for convergence as $M \to 0$ is that P^{-1}, Q_1, Q_2 all have the form

$$\mathbf{P}^{-1}, \mathbf{Q_1}, \mathbf{Q_2} \sim \begin{pmatrix} O(1/M^2) & O(1/M) & O(1/M) \\ O(1/M) & O(1) & O(1) \\ O(1/M) & O(1) & O(1) \end{pmatrix} \quad . \tag{17}$$

If we would have considered the variables (p, u, v) instead of (ϕ, u, v) then all the terms would have been of order 1. However, the matrices A, B would not be symmetric. Hence, each set has its advantages and we shall consider both sets of variables.

As an example, consider the equations in $(\frac{dp}{\rho c}, du, dv)$ variables. Then

$$\mathbf{A_{comp}} = \begin{pmatrix} u & c & 0 \\ c & u & 0 \\ 0 & 0 & u \end{pmatrix}, \quad \mathbf{B_{comp}} = \begin{pmatrix} v & 0 & c \\ 0 & v & 0 \\ c & 0 & v \end{pmatrix} \quad .$$

Since these equations are nondimensionalized $c = O(1/M)$. Let

$$\mathbf{D} = A\omega_1 + B\omega_2 = \begin{pmatrix} Rq\omega_1 & \omega_1 c & \omega_2 c \\ \omega_1 c & Rq\omega_1 & 0 \\ \omega_2 c & 0 & Rq\omega_1 \end{pmatrix},$$

where $q^2 = u^2 + v^2$, $R = \frac{u\omega_1 + v\omega 2}{q}$ and we define $\mathbf{P^{-1}_{comp}}$ and $\mathbf{P_{comp}}$ by

$$\mathbf{P^{-1}} = \begin{pmatrix} \frac{2}{M^2} - 1 & \frac{u}{qM} & \frac{v}{qM} \\ \frac{u}{qM} & 1 & 0 \\ \frac{v}{qM} & 0 & 1 \end{pmatrix}, \quad \mathbf{P} = \frac{1}{1-M^2} \begin{pmatrix} M^2 & -\frac{Mu}{q} & -\frac{Mv}{q} \\ -\frac{Mu}{q} & 1 - M^2 + \frac{u^2}{q^2} & \frac{uv}{q} \\ -\frac{Mv}{q} & \frac{uv}{q} & 1 - M^2 + \frac{v^2}{q^2} \end{pmatrix}.$$

By inspection $\mathbf{P^{-1}_{comp}}$ has the form desired of (17). To find the form of the artificial viscosity we return to the numerical approximation for

$$\mathbf{P}^{-1} w_t + f_x + g_y = h \left[(Q_1 w_x)_x + (Q_2 w_y)_y \right] \quad .$$

For constant $\mathbf{P}$ we want $\mathbf{P}Q_1 = |\mathbf{P}A|$. Hence, $Q_1 = \mathbf{P^{-1}}|\mathbf{P}A|, Q_2 = \mathbf{P^{-1}}|\mathbf{P}B|$. Then

$$(\mathbf{P^{-1}}|\mathbf{PD}|)_{comp} = q \begin{pmatrix} \frac{S+R}{M^2} & \frac{|u|R}{qM} & \frac{|v|R}{qM} \\ \frac{|u|R}{qM} & \frac{u^2|R|}{q^2} + \frac{v^2 S}{q^2(1-M^2)} & \frac{|uv|R}{q^2} - \frac{|uv|S}{q^2(1-M^2)} \\ \frac{|v|R}{qM} & \frac{|uv|R}{q^2} - \frac{|uv|S}{q^2(1-M^2)} & \frac{v^2|R|}{q^2} + \frac{u^2 S}{q^2(1-M^2)} \end{pmatrix},$$

where $q^2 = u^2 + v^2$, $R = \frac{u\omega_1 + v\omega_2}{q}$, $S = \sqrt{(1 - M^2)(1 - M^2 R^2)}$.

Letting $M \to 0$ we get $S \to 1$. To get the right limit of $(\mathbf{P^{-1}}|\mathbf{PD}|)_{comp}$ we must first multiply each element by the appropriate factor of M (or equivalently go to (p,u,v) variables). We then get

$$(\frac{dp}{\rho c}, du, dv)_{comp} \to (\frac{dp}{q}, du, dv)_{incomp}$$

and

$$(\mathbf{P^{-1}}|\mathbf{PD}|)_{comp} \rightarrow (\mathbf{P^{-1}}|\mathbf{PD}|)_{incomp} = q\begin{pmatrix} |R|+1 & \frac{|uR|}{q} & \frac{|vR|}{q} \\ \frac{|uR|}{q} & \frac{u^2|R|+v^2}{q^2} & \frac{|uv|(|R|-1)}{q^2} \\ \frac{|vR|}{q} & \frac{|uv|(|R|-1)}{q^2} & \frac{v^2|R|+u^2}{q^2} \end{pmatrix} .$$

Changing from the variables $(\frac{dp}{q}, du, dv)_{incomp}$ to $(dp, du, dv)_{incomp}$ we get the same viscosity matrix given by Arnone and Turkel [1].

We next point out that the conditions on the matrix $(\mathbf{P^{-1}}|\mathbf{PD}|)_{\mathbf{comp}}$ are not satisfied by the non-preconditioned Roe matrices. Furthermore, even reasonable preconditioners need not satisfy these conditions. Consider, for example, the one dimensional system

$$\mathbf{P^{-1}}w_t + Aw_x = h(Qw_x)_x.$$

A reasonable choice is $\mathbf{P^{-1}} = |A|$ i.e. $\mathbf{P} = |A^{-1}|$. In this case all the wave speeds of $\mathbf{P}A$ are ± 1. Now

$$Q = \mathbf{P}^{-1}|\mathbf{P}A| = |A|\left||A|^{-1}A\right| = |A| \sim \begin{pmatrix} O(\frac{1}{M}) & O(1) & O(1) \\ O(1) & O(\frac{1}{M}) & O(1) \\ O(1) & O(1) & O(1) \end{pmatrix},$$

i.e. Q is the nonpreconditioned Roe matrix which does not have the desired property. We therefore conclude that for an upwind difference scheme the Riemann solver should be based on the preconditioned system and not the original scheme. In [4] plots are shown to illustrate the greatly improved accuracy for low Mach number flows when the Riemann solver is based on the preconditioning. For central difference schemes with a scalar artificial viscosity the viscosity in the continuity equation should include pressure terms. For a matrix viscosity $Q_1 = \mathbf{P^{-1}}|\mathbf{P}A|, Q_2 = \mathbf{P^{-1}}|\mathbf{P}B|$. Characteristics in the boundary conditions these should be based on the characteristics of the modified system and not the physical system. When using multigrid it is better to transfer the residuals based on the preconditioned system to the next grid since these residuals are more balanced than the physical residuals. However, conservation of the variables is now lost. Preconditioning is even more important when using multigrid than with an explicit scheme. With the original system the disparity of the eigenvalues greatly affects the smoothing rates of the slow components and so slows down the multigrid method, [13]. As indicated above there are accuracy difficulties at low Mach numbers [21]. Some of these can be alleviated by preconditioning the dissipation terms. For very small Mach numbers there is also a difficulty with round off errors as $\frac{p}{u^2+v^2} \rightarrow \infty$. Several people have suggested subtracting out a constant pressure from the dynamic pressure. A more detailed analysis [7] suggests replacing the pressure p by $\tilde{p}$ where $p = \frac{p_0+\epsilon\tilde{p}}{\epsilon^2}$ and ϵ is a representative Mach number.

We conclude from the above remarks that the steady state solution of the preconditioned system may be different from that of the physical system. Thus, on the finite difference level the preconditioning can improve the accuracy as well as the convergence rate.

5 Computational Results

We now present a calculation for two dimensional flow around an airfoil to demonstrate the previous theory. The discretization is based on the multistage time method coupled with a central difference approximation as described in [9].

We solve the equation in conservation form based on a hybrid system

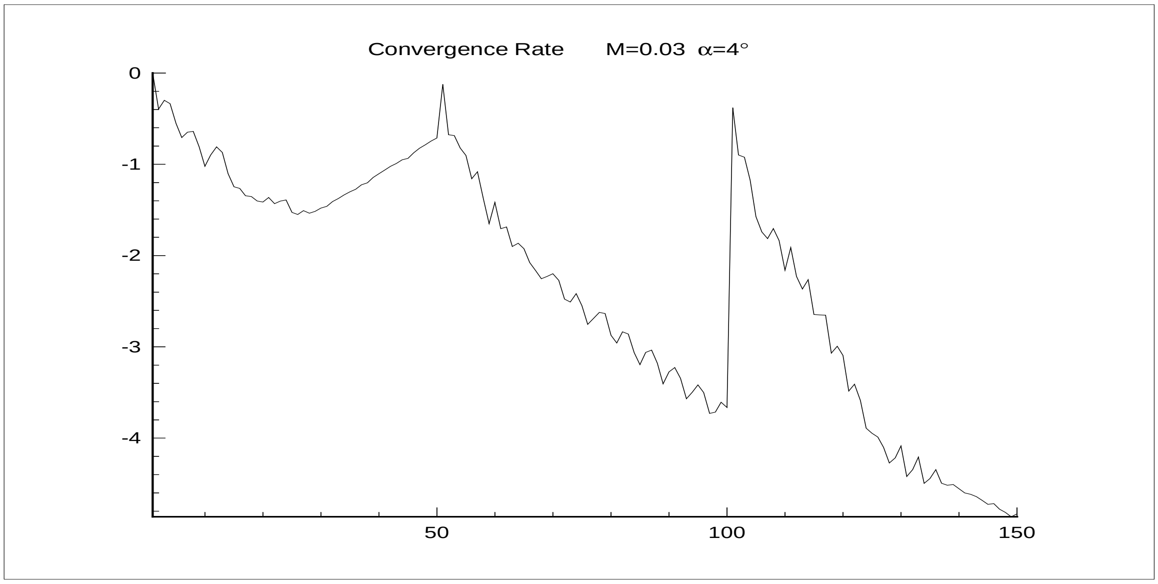

Figure 1: Convergence Rate for $M_\infty = 0.03$ and $\alpha = 4^\circ$

$$W = \begin{pmatrix} p' \\ \rho u \\ \rho v \\ E' \end{pmatrix}$$

$$F = \begin{pmatrix} \rho u \\ \rho u^2 + p' \\ \rho uv \\ \rho H' u \end{pmatrix}, \quad G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p' \\ \rho H' v \end{pmatrix}, \quad Q = \begin{pmatrix} p' \\ \rho u \\ \rho v \\ H' \end{pmatrix}$$

$$\frac{\partial w}{\partial t} = \mathbf{P} \left[\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right]$$

where $p' = p - p_\infty$, $E' = c_p \rho (T - T_\infty) - (p - p_\infty) + \frac{\rho(u^2+v^2)}{2} = E + p_\infty + \frac{\gamma p_\infty}{(\gamma-1)\rho_\infty}\rho$ and $\rho H' = E' + p'$. We subtract the constants to keep the quantities in scale. Then

$$\mathbf{P} = \mathbf{I} \quad + \quad \Delta \cdot \begin{pmatrix} 1 - \frac{1}{\Delta} & -\frac{u}{G+h_\infty} & -\frac{v}{G+h_\infty} & \frac{1}{G+h_\infty} \\ -B_2 & \frac{uB_2}{G+h_\infty} & \frac{vB_2}{G+h_\infty} & -\frac{B_2}{G+h_\infty} \\ -B_3 & \frac{uB_3}{G+h_\infty} & \frac{vB_3}{G+h_\infty} & -\frac{B_3}{G+h_\infty} \\ -B_4 & \frac{uB_4}{G+h_\infty} & \frac{vB_4}{G+h_\infty} & -\frac{B_4}{G+h_\infty} \end{pmatrix}$$

where $h = c_p T$, $G = \frac{u^2+v^2}{2}$, $\Delta = \frac{(G+h_\infty)\beta^2}{h}$,

$$\begin{aligned} B_1 &= \frac{1}{\beta^2} - \frac{1}{(\gamma-1)h} \\ B_2 &= (B_1 + \frac{\alpha}{\beta^2})u \\ B_3 &= (B_1 + \frac{\alpha}{\beta^2})v \\ B_4 &= B_1 H' + \frac{\alpha(u^2+v^2)}{\beta^2} \end{aligned}$$

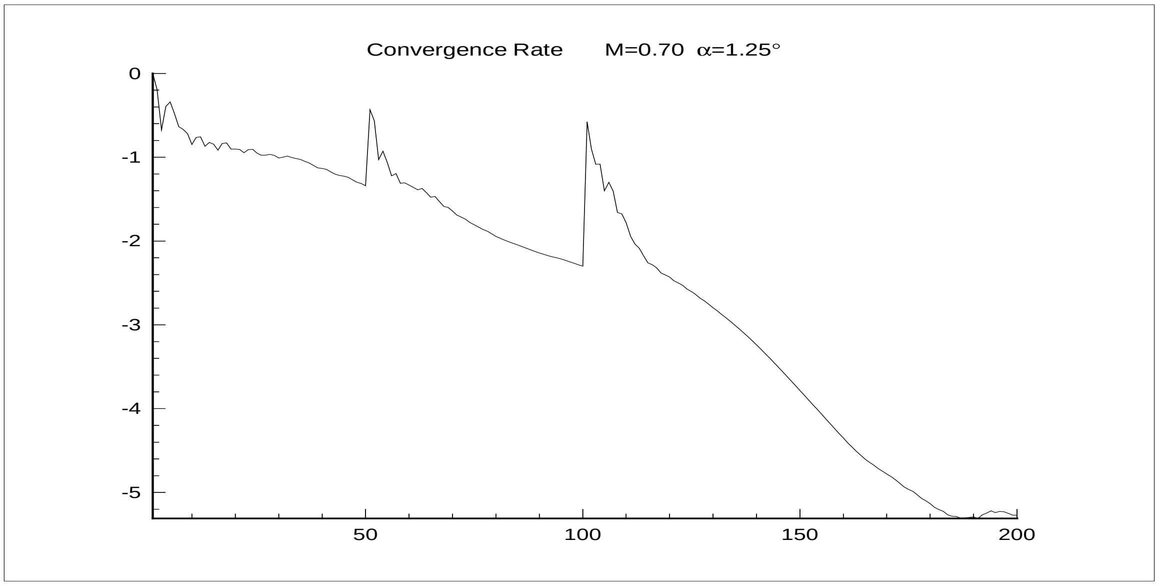

Figure 2: Convergence Rate for $M_\infty = 0.70$ and $\alpha = 1.25°$

We choose $\beta^2 = max\left[u^2+v^2, K(u_\infty^2+v_\infty^2)\right], \quad \alpha = 1, \quad K = 0.1$. We can evaluate this efficiently by defining $S = \Delta \cdot \left(d\rho - \frac{ud(\rho u)+vd(\rho v)-dE'}{G+h_\infty}\right) = \frac{\beta^2 dp}{(\gamma-1)h}$. Then

$$
\begin{aligned}
dp_{new} &= S \\
d(\rho u)_{new} &= d(\rho u)_{orig} + B_2 S \\
d(\rho v)_{new} &= d(\rho v)_{orig} + B_3 S \\
dE_{new} &= dE_{orig} + B_4 S
\end{aligned}
$$

These equations are given for the nondimensionalized variables. The nondimensionalization affects the convergence, In some codes, p and ρ are fixed in the far field. This implies that the speed of sound, a, is also bounded. As the Mach number goes to zero the pressure remains of order 1 while the velocities go to zero. Alternatively, one can nondimensionalize so that the velocities are of order 1 in the far field and then the pressure and speed of sound go to infinity, unless one subtracts an appropriate constant,

The boundary conditions at the far field boundary, for subsonic flow, are based on the one dimensional theory of characteristics in the direction normal to the boundary. The preconditioning changes the form of these characteristic variables. They are now given by

$$
\begin{aligned}
R_1 &= u - \frac{p'}{2\rho\beta^2}\left(u(1-\alpha-\frac{\beta^2}{c^2}) - \sqrt{(u(1-\alpha+\frac{\beta^2}{c^2}))^2 + 4(1-\frac{\alpha u^2}{c^2})\beta^2}\right) \\
R_2 &= u - \frac{p'}{2\rho\beta^2}\left(u(1-\alpha-\frac{\beta^2}{c^2}) + \sqrt{(u(1-\alpha+\frac{\beta^2}{c^2}))^2 + 4(1-\frac{\alpha u^2}{c^2})\beta^2}\right)
\end{aligned}
$$

where u is the component of the velocity normal to the boundary. This formulas simplify slightly if $\alpha = 1$ and more if $\alpha = 1 + \frac{\beta^2}{c^2}$. If we consider low Mach numbers then we can approximate these by

$$R_1 = u + \frac{p'}{\rho\beta}, \quad R_2 = u - \frac{p'}{\rho\beta}$$

which is the same as for the incompressible case. At solid boundaries the normal momentum equation is used which is not affected by the preconditioning.

The solution is advanced by an explicit Runge-Kutta method ([9]) accelerated using both residual smoothing and multigrid ([16]. We present two calculations for inviscid flow about a NACA 0012. The first calculation is for inflow conditions $M = 0.03, \alpha = 4°$. Without preconditioning the scheme does not converge while preconditioning dramatically increases the rate of convergence. The use of the preconditioning adds only a few percent to the total computational time. For viscous flows the computational time required for the preconditioning is negligible. In the second case we consider the same geometry but with an inflow of $M = 0.7, \alpha = 1.25°$. In this case we also see an increased rate of convergence for the preconditioned case but not as dramatic as before.

A Appendix

Let W denote the conservative variables $(\rho, m, n, E)^t$, with $m = \rho u, n = \rho v$, let w denote the primitive variables $(p, u, v, S)^t$ and let $\hat{w}$ denote $(p, u, v, T)^t$. Then $p = \rho RT$

$$\frac{\partial W}{\partial w} = \begin{pmatrix} \frac{1}{c^2} & 0 & 0 & -\frac{\rho}{C_p} \\ \frac{u}{c^2} & \rho & 0 & -\frac{m}{C_p} \\ \frac{v}{c^2} & 0 & \rho & -\frac{n}{C_p} \\ \frac{1}{\gamma-1} + \frac{M^2}{2} & m & n & -\frac{\rho(u^2+v^2)}{2C_p} \end{pmatrix}$$

$$\frac{\partial w}{\partial W} = \begin{pmatrix} \frac{(\gamma-1)(u^2+v^2)}{2} & -(\gamma-1)u & -(\gamma-1)v & \gamma-1 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{C_p[(\gamma-1)\frac{M^2}{2}-1]}{\rho} & -\frac{(\gamma-1)uC_p}{\rho c^2} & -\frac{(\gamma-1)vC_p}{\rho c^2} & \frac{(\gamma-1)C_p}{\rho c^2} \end{pmatrix}$$

$$\frac{\partial W}{\partial \hat{w}} = \begin{pmatrix} \frac{\rho}{p} & 0 & 0 & -\frac{\rho}{T} \\ \frac{m}{p} & \rho & 0 & -\frac{m}{T} \\ \frac{n}{p} & 0 & \rho & -\frac{n}{T} \\ \frac{E}{p} & m & n & -\frac{\rho(u^2+v^2)}{2T} \end{pmatrix}$$

$$\frac{\partial \hat{w}}{\partial W} = \begin{pmatrix} \frac{(\gamma-1)(u^2+v^2)}{2} & -(\gamma-1)u & -(\gamma-1)v & \gamma-1 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ -\frac{T}{\rho} + \frac{(\gamma-1)T(u^2+v^2)}{2p} & -\frac{(\gamma-1)uT}{p} & -\frac{(\gamma-1)vT}{p} & \frac{(\gamma-1)T}{p} \end{pmatrix} .$$

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