Decompositions of First-order Hyperbolic Systems

Phil Roe †, E. Turkel ††

† Department of Aerospace Engineering
University of Michigan
Ann Arbor Michigan

†† School of Mathematical Sciences
Sackler Faculty of Exact Sciences
Tel-Aviv University
Tel-Aviv, Israel

Abstract

We write the linearized Euler equations as a first order symmetric hyperbolic system in both two and three space dimensions. We then multiply the time derivative by a positive definite matrix to give a modified symmetric hyperbolic system. We then consider two canonical forms, the convective equation and the wave equation. We can then decompose, for both two and three space dimensions, the preconditioned Euler system into convective and wave portions.

1 Introduction

We shall consider first-order systems of partial differential equations. Within this class we seek to identify canonical forms to which the systems may be reduced. Our analysis is restricted to the linear case with constant coefficients.

In one space dimension, any strongly hyperbolic $(m \times m)$ system of the form

$$\partial_t \boldsymbol{u} + A \partial_x \boldsymbol{u} = 0 \tag{1}$$

is reducible to m scalar convection equations of the form

$$\partial_t (\boldsymbol{\ell_k} \cdot \boldsymbol{u}) + \lambda_k \partial_x (\boldsymbol{\ell_k} \cdot \boldsymbol{u}) = 0 \tag{2}$$

where ℓ_k is a left eigenvector of A, and λ_k is the corresponding eigenvalue. This follows since for a strongly hyperbolic system A is diagonalizable. Thus, the canonical form is simply the familiar linear advection equation

$$\partial_t u + a \partial_x u = 0$$

In two dimensions we consider first the steady-state equations

$$A \partial_x \boldsymbol{u} + B \partial_y \boldsymbol{u} = 0 \tag{3}$$

When A and B are symmetric it is known [1] that this equation can be diagonalised by a congruence transformation if and only if some real linear combination of A and B is positive definite. In that case the canonical form is again linear advection

$$a \partial_x u + b \partial_y u = 0. \tag{4}$$

If the above condition is not met, then we shall show that some of the eigenvalues of $A^{-1}B$ appear in conjugate pairs, and a second canonical form appears which is a (2×2) system, equivalent under scaling and axis transformations to the Cauchy-Riemann equations

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \partial_y \begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{5}$$

The general two-dimensional system can then be written in a form that is block diagonal, with a mixture of scalar and (2×2) blocks. When the steady solution is sought by marching in a time-like variable, we show that a preconditioning matrix P can always be found, such that this block structure is preserved in the transient solution of

$$P^{-1} \partial_t \boldsymbol{u} + A \partial_x \boldsymbol{u} + B \partial_y \boldsymbol{u} = 0$$

The scalar blocks have then the canonical form

$$\partial_t u + a\partial_x u + b\partial_y u = 0. \tag{6}$$

The (2×2) blocks, although they behave elliptically in the steady state, behave hyperbolically in the transient phase, and the associated variables follow a wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \tag{7}$$

Although both of these equations are hyperbolic, considered as time-dependent problems, only (6) is associated with any distinct direction (namely $dy/dx = b/a$). By contrast, (7), even during the transient phase, is associated with characteristic surfaces, $(x - x_0)^2 + (y - y_0)^2 - (t - t_0)^2 = const$, rather than with individual directions. In the steady state, the solution is clearly elliptic, satisfying Laplace's equation, and has no uniquely distinguished directions.

In three dimensions, we have found three canonical forms, which are

$$\partial_t u + a\partial_x u + b\partial_y u + c\partial_x u = 0, \tag{8}$$

and (3×3) systems equivalent to either

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}, \tag{9}$$

or

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^2 \psi = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}, \tag{10}$$

In this paper we will only present the decomposition and not discuss any applications. However, to provide motivation, one possible application would be to numerical approximations of these equations. Then one could consider applying quite different numerical techniques to the different subproblems that appear (see e.g. [4]) .

We will begin by presenting the complete analysis for the general two-dimensional problems, and then apply this to the special case of the Euler equations. In the following section we shall treat, less rigorously, the general three-dimensional case, but our results for the Euler equations are precise.

2 The Two-dimensional Case

2.1 General Steady-state Problems

We consider

$$A\partial_x \boldsymbol{u} + B\partial_y \boldsymbol{u} = 0 \tag{11}$$

with A and B symmetric.

According to a result of Au-Yeung [1], this equation can be diagonalized by a congruence transformation if and only if some real linear combination of A and B is positive definite. In the present context, this condition has a physical interpretation. Suppose that we add a time-dependent term $\partial_t u$ to (11). Then the speeds with which waves propagate in a direction θ are the eigenvalues of $A\cos\theta + B\sin\theta$. If this matrix is positive definite for $\theta = \theta^*$, then the domain of influence of a disturbance at the origin lies in the half-plane $x\cos\theta^* + y\sin\theta^* > 0$. Therefore the domain of influence does not enclose the origin, and tangents to it from the origin may be drawn. These are the characteristic directions for the steady solution. This situation arises whenever the flow speed is greater than the speed of any signal relative to the medium (supersonic case).

To deal with the subsonic case, we begin by attempting a formal diagonalization of (11). We premultiply the equation by a matrix S, and effect a change of variable $Qv = u$, so that

$$(SAQ)\partial_x \boldsymbol{v} + (SBQ)\partial_y \boldsymbol{v} = 0. \tag{12}$$

We require that SAQ, SBQ be representable as

$$\begin{aligned} SAQ &= \Lambda_A^\#, \\ SBQ &= \Lambda_B^\#, \end{aligned}$$

where $\Lambda_A^\#, \Lambda_B^\#$ denote matrices having the same block diagonal structure. (Note that matrices sharing such a structure form a closed group under multiplication and inversion.) Then

$$(\Lambda_A^\#)^{-1}\Lambda_B^\# = (Q^{-1}A^{-1}S^{-1})(SBQ) = Q^{-1}A^{-1}BQ$$

so that Q must represent a similarity transformation that block diagonalises $A^{-1}B$. (If A happens to be singular we may diagonalize $B^{-1}A$ instead). At the present stage, the matrix S is still arbitrary. We may first attempt a complete diagonalisation

$$A^{-1}B = R\Lambda L \qquad \text{where} \qquad R = L^{-1}$$

If this succeeds, we can take $S = LA^{-1}$ and $Q = R$, to derive

$$\partial_x(L\boldsymbol{u}) + \Lambda\partial_y(L\boldsymbol{u}) = 0.$$

representing a set of characteristic equations for the characteristic variables $\boldsymbol{v_k} = \boldsymbol{\ell_k} \cdot \boldsymbol{u}$ propagating along directions $dy/dx = \lambda_k$.

Suppose this fails in that

$$\det(A^{-1}B - \lambda I) = 0$$

has a pair of conjugate roots

$$\lambda_{1,2} = \lambda_R \pm i\lambda_I.$$

Then we will also find conjugate eigenvectors

$$\begin{aligned} \mathbf{r}_{1,2} &= \mathbf{r}_R \pm i\mathbf{r}_I, \\ \ell_{1,2} &= \ell_R \pm i\ell_I. \end{aligned}$$

That is,

$$A^{-1}B = \begin{bmatrix} \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \boldsymbol{r}_1 & \boldsymbol{r}_2, & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \end{bmatrix} \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \ell_1 & \cdot & \cdot \\ \cdot & \cdot & \ell_2 & \cdot & \cdot \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \end{bmatrix} \tag{13}$$

Now introduce the complex matrices

$$J = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & & & \\ i & -i & & & \\ & & \sqrt{2} & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix}, \quad J^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i & & & \\ 1 & -i & & & \\ & & \sqrt{2} & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix} \tag{14}$$

and rewrite the factorization

$$A^{-1}B = (RJ^T)((J^T)^{-1}\Lambda J^{-1})(JL) = R^{\#}\Lambda^{\#}L^{\#}$$

Then it can be verified that $R^{\#}, L^{\#}$ are real matrices, and that $\Lambda^{\#}$ is the real symmetric matrix

$$\Lambda^{\#} = (J^T)^{-1}\Lambda J^{-1} = \begin{bmatrix} \lambda_R & \lambda_I & & & \\ \lambda_I & -\lambda_R & & & \\ & & \lambda_3 & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix} \tag{15}$$

Now we write the governing equations in the form (12) with $S = (R^\#)^{-1}A^{-1}$ and $Q = (L^\#)^{-1}$; the result, after partial simplification, is

$$(R^\#)^{-1}(L^\#)^{-1}\partial_x \boldsymbol{v} + \Lambda^\# \partial_y \boldsymbol{v} = 0$$

Here,

$$\begin{aligned}
(R^\#)^{-1}(L^\#)^{-1} &= (L^\# R^\#)^{-1} \\
&= (JLRJ^T)^{-1} \\
&= (JJ^T)^{-1} \\
&= \begin{bmatrix} 1 & 0 & & & \\ 0 & -1 & & & \\ & & 1 & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix}.
\end{aligned}$$

Thus, when expressed in terms of variables $v = L^\# u$, the equations decouple into $(m-2)$ advection equations, and the 2×2 symmetric elliptic system

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) \partial_x \left(\begin{array}{c} u \\ v \end{array}\right) + \left(\begin{array}{cc} \lambda_R & \lambda_I \\ \lambda_I & -\lambda_R \end{array}\right) \partial_y \left(\begin{array}{c} u \\ v \end{array}\right) = 0. \tag{16}$$

If there is more than one conjugate pair of eigenvalues, the treatment above can easily be extended by adding additional (2×2) blocks to the matrix J, so we have

<u>THEOREM</u> Any $(m \times m)$ first order constant coefficient hyperbolic system in two dimensions decomposes into $(m - 2k)$ convection equations and k independent (2×2) symmetric elliptic systems.

<u>REMARK</u> The $(m \times 2)$ blocks that appear in $R^\#$ provide basis vectors for the subspaces inhabited by the elliptic subproblems; the $(2 \times m)$ blocks that appear in $L^\#$ are operators that project an arbitrary vector into these subspaces.

2.2 The Steady Euler Equations

Take the vector of unknowns to be $\boldsymbol{u} = (dp/(\rho a), du, dv, dp - a^2 d\rho)^T$, and adopt coordinates aligned with the flow direction. Then A, B are the symmetric matrices

$$A = \begin{bmatrix} M & 1 & 0 & 0 \\ 1 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We then find that the equation $A^{-1}B = R\Lambda L$ becomes

$$A^{-1}B = \begin{bmatrix} 0 & 0 & \frac{M}{M^2-1} & 0 \\ 0 & 0 & \frac{-1}{M^2-1} & 0 \\ \frac{1}{M} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} M & M & 0 & 0 \\ -1 & -1 & 1 & 0 \\ \sqrt{M^2-1} & -\sqrt{M^2-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{M^2-1}} & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{M^2-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2M} & 0 & \frac{1}{2\sqrt{M^2-1}} & 0 \\ \frac{1}{2M} & 0 & \frac{-1}{2\sqrt{M^2-1}} & 0 \\ \frac{1}{M} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For supersonic flow, the transformation $\boldsymbol{v} = L\boldsymbol{u}$ produces four real characteristic equations, which are, in primitive variables,

$$\begin{aligned} \partial_x S &= 0 \qquad dS = dp - a^2 d\rho \\ \partial_x H &= 0 \qquad dH = dp + \frac{\rho}{2}(u^2) = dp + \rho q\, dq \\ (\beta\partial_x + \partial_y)(\beta dp + \rho u dv) &= 0 \\ (\beta\partial_x - \partial_y)(\beta dp - \rho u dv) &= 0 \end{aligned}$$

where $\beta^2 = M^2 - 1$.

If $M^2 < 1$, two of these characteristic equations break down. We may then have recourse to the alternative factorization $R^{\#}\Lambda^{\#}L^{\#}$, which is

$$A^{-1}B = \begin{bmatrix} 0 & 0 & \frac{M}{M^2-1} & 0 \\ 0 & 0 & -\frac{1}{M^2-1} & 0 \\ \frac{1}{M} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \sqrt{2}M & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 1 & 0 \\ 0 & -\sqrt{2(1-M^2)} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{-1}{\sqrt{1-M^2}} & 0 & 0 \\ \frac{-1}{\sqrt{1-M^2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}M} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2(1-M^2)}} & 0 \\ \frac{1}{M} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From this factorization we recover the same streamwise characteristic equations as in the supersonic case. It is interesting to note that the entropy characteristic corresponds to a shared eigenvector of the matrices A, B, whereas the enthalpy characteristic does not. The two acoustic characteristics collapse into the elliptic system deriving from (16), which is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x \begin{pmatrix} \frac{\beta_*}{\rho u} dp \\ dv \end{pmatrix} + \begin{pmatrix} 0 & \frac{-1}{\beta_*} \\ \frac{-1}{\beta_*} & 0 \end{pmatrix} \partial_y \begin{pmatrix} \frac{\beta_*}{\rho u} dp \\ dv \end{pmatrix} = 0. \tag{17}$$

where $\beta_*^2 = 1 - M^2$.

These are the Cauchy-Riemann equations written in Prandtl-Glauert coordinates $(x, y/\beta_*)$.

This decomposition differs from that found by Ta'asan [3, 4]. He employed, as canonical variables, (S, H, u, v) and his decomposition was not completely block diagonal, containing some additional coupling terms. The variables which emerge from the present treatment are (S, H, p, v) and there is no coupling between blocks.

2.3 Preconditioning

Suppose that the time-dependent two-dimensional problem obeys

$$\partial_t \boldsymbol{u} + A\partial_x \boldsymbol{u} + B\partial_y \boldsymbol{u} = 0 \tag{18}$$

where A and B are symmetric. To find a decomposition of this equation as it stands would require the three-dimensional treatment given in the next section. However, there are occasions, when only the steady state is of interest, but it is convenient to compute it by following some time-like evolution. In such cases (18) may be replaced (sometimes advantageously [5, 6, 7]) by

$$P^{-1}\partial_t \boldsymbol{u} + A\partial_x \boldsymbol{u} + B\partial_y \boldsymbol{u} = 0 \tag{19}$$

where P is a 'preconditioning matrix'. To retain well-posedness, P must be positive definite. In this section we point out that P can always be chosen to preserve the canonical block-diagonal form of the steady state equations.

As in the steady case, premultiply (19) by $(R^\#)^{-1}A^{-1}$ and make the change of variable $\boldsymbol{v} = L^\# \boldsymbol{u}$; this gives

$$(R^\#)^{-1}A^{-1}P^{-1}(L^\#)^{-1}\partial_t \boldsymbol{v} + (JJ^T)^{-1}\partial_x \boldsymbol{v} + \Lambda^\# \partial_y \boldsymbol{v} = 0.$$

For the decomposition to be preserved, it is neccessary and sufficient that

$$(R^\#)^{-1}A^{-1}P^{-1}(L^\#)^{-1} = D^\#$$

where $D^\#$ is block diagonal of the same form as the matrices that multiply the space derivatives. Solving for P gives

$$P = (R^\# A D L^\#)^{-1}. \tag{20}$$

where the only constraint on $D^\#$ is that it shall be positive definite to ensure well-posedness.

To apply this analysis to the Euler equations we take

$$D^\# = \begin{bmatrix} P & Q & 0 & 0 \\ R & S & 0 & 0 \\ 0 & 0 & \frac{1}{M} & 0 \\ 0 & 0 & 0 & \frac{1}{M} \end{bmatrix}. \tag{21}$$

This assigns the fluid velocity u to each of the scalar waves. We require $\Delta = PS - QR > 0$. After carrying through the required operations we find that

$$(22) \qquad P^{-1} = \begin{bmatrix} \frac{1}{M^2} - P\frac{\beta_*^2}{M} & \frac{1}{M} & Q\beta_* & 0 \\ \frac{1}{M} & 1 & 0 & 0 \\ R\beta_* & 0 & -SM & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

and also

$$(23) \qquad P = \begin{bmatrix} -\frac{S}{\Delta}\frac{M}{\beta_*^2} & \frac{S}{\Delta}\frac{1}{\beta_*^2} & -\frac{Q}{\Delta}\frac{1}{\beta_*} & 0 \\ \frac{S}{\Delta}\frac{1}{\beta_*^2} & 1 - \frac{S}{\Delta}\frac{1}{M\beta_*^2} & \frac{Q}{\Delta}\frac{1}{M\beta_*} & 0 \\ -\frac{R}{\Delta}\frac{1}{\beta_*} & \frac{R}{\Delta}\frac{1}{M\beta_*} & -\frac{P}{\Delta}\frac{1}{M} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is the most general preconditioner that preserves the canonical decomposition. It may be noted that $\det P = (\beta_*^2\Delta)^{-1}$. The preconditioner of van Leer *et. al.* [7] corresponds to the special case $P = S = -1/(M\beta_*)$, $Q = R = 0$, $\Delta = 1/(M^2\beta_*^2)$.

When the preconditioned equations are written in Cartesian coordinates not aligned with the flow, it can be shown that terms may appear which depend on the flow direction even though the flow velocity vanishes. It can also be shown that a necessary and sufficient condition to avoid this phenomenon is that $P_{22} = P_{33}$, which implies

$$(24) \qquad \beta_*^2(M\Delta + A) = D.$$

The wavespeeds in the direction θ are given by

$$\det[A\cos\theta + B\sin\theta - P^{-1}\lambda]$$

which gives a twice repeated root $\lambda = M\cos\theta$, together with the quadratic term

$$(25) \qquad \Delta\beta^2\lambda^2 + [(P - S)\beta^2\cos\theta + (Q + R)\beta\sin\theta]\lambda = \beta^2\cos^2\theta + \sin^2\theta.$$

To keep this symmetrical with respect to the streamline, we must take $Q + R = 0$, and we consider first the case $Q = R = 0 \Rightarrow \Delta = AD$. The wavespeeds in the x-direction are then $1/P$ and $-1/S$, and in the y-direction $\pm 1/(\beta_*\sqrt{PS})$. It can be shown that the best condition number is obtained from the preconditioner of van Leer *et. al.*, but unfortunately this does not satisfy the condition (24) at low Mach numbers. We require, in fact, that as $M \to 0$, then

$$(S + 1/M)(P - 1/M) = -1/M^2.$$

This is the graph of a hyperbola in the SP plane. We are only concerned with the branch that passes through the origin (the other branch leads to imaginary wavespeeds), and only with the half of that branch for which P, S are both negative (the other half does not make P positive definite). The condition number based solely on the 'acoustic' waves is

$$K = \max\left(S/P, P/S\right)$$

which is unity at the origin and degrades away from it. Taking into account that the flow velocity is M, and that $|P| > |S|$,we actually have

$$K = \max\left(\frac{S}{P}, \frac{P}{S}, \frac{1}{MS}, MP\right).$$

The point on the hyperbola that minimises this is

$$P = \frac{-1}{M} \qquad S = \frac{-1}{2M}$$

leading to K=2.

Of course, the preconditioner does not have to satisfy (24) away from $M = 0$, and can be blended in with the van Leer *et.al.* preconditioner, by some such choice as

$$P = \frac{-1}{M\beta_*}, \qquad S = \frac{-1}{M * \beta_*(1+\beta_*)}$$

which leads to simple preconditioner

$$(26) \qquad P = \begin{bmatrix} \frac{M^2}{\beta_*} & \frac{-M}{\beta_*} & 0 & 0 \\ \frac{-M}{\beta_*} & \frac{1+\beta_*}{\beta_*} & 0 & 0 \\ 0 & 0 & \beta_*(1+\beta_*) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} \frac{1+\beta_*}{M^2} & \frac{1}{M} & 0 & 0 \\ \frac{1}{M} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta_*(1+\beta_*)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Rather strikingly, these are identical with those given by van Leer *et.al.*, except for the factors $(1 + \beta_*)$ in the $()_{33}$ elements.

3 The Three-dimensional Case

3.1 General Steady-state Problems

The most common block diagonal forms in three-dimensional problems appear to have blocks that are either scalar or else (3×3). The possibility of (2×2) blocks is ruled out, apart from exceptional cases, by an anlysis of Gilquin, Laurens and Rosier [2], who considered the general (2×2) symmetric system in multi-dimensions. They only require that the system be strongly hyperbolic (i.e. combinations of the coefficient matrices can always be diagonalized). They then prove that such systems can always be decomposed into a convective part (possibly with speed zero) and a *two dimensional* elliptic part. Therefore, (2×2) blocks are incapable of representing three-dimensional elliptic systems.

To see better what happens in three space dimension we consider the specific case of the Euler equations. In addition we add a preconditioner of the time derivative as described previously.

3.2 Subsonic - three dimensional

In three space dimensions the preconditioned subsonic equations are

$$
\begin{aligned}
\frac{\beta+1}{M^2} p_t + \frac{\rho u}{M^2} u_t + u p_x + \rho a^2 (u_x + v_y + w_z) &= 0 \\
\frac{1}{\rho u} p_t + u_t + u u_x + \frac{p_x}{\rho} &= 0 \\
\frac{1}{\beta} v_t + u v_x + \frac{p_y}{\rho} &= 0 \\
\frac{1}{\beta} w_t + u w_x + \frac{p_z}{\rho} &= 0 \\
S_t + u S_x &= 0
\end{aligned}
$$

As always S and H are convected along the stream line. We now get three equations for the unknowns (p,v,w). Let $(dq, dr, ds) = (\frac{\beta}{\rho u} dp, du, dv)$.

$$
\begin{pmatrix} q \\ r \\ s \end{pmatrix}_t + \begin{pmatrix} -\beta u & 0 & 0 \\ 0 & \beta u & 0 \\ 0 & 0 & \beta u \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}_x + \begin{pmatrix} 0 & u & 0 \\ u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}_y + \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}_z = 0
$$

We next rescale x by defining $x = \beta x'$. We rewrite this equation in terms of x' and then drop the $'$. The effect is to eliminate the β term from the x derivative. Then in this modified coordinate system we get

$$
\begin{pmatrix} q \\ r \\ s \end{pmatrix}_t + \begin{pmatrix} -u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}_x + \begin{pmatrix} 0 & u & 0 \\ u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}_y + \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}_z = 0
$$

We now consider the three dimensional Laplace equation

$$
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0.
$$

Let $u = \psi_x$, $v = -\psi_y$ and $w = -\psi_z$. Then writing the second order scalar equation as a first order system we get

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_x + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_y + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_z = 0$$

However, by the existence of the potential ψ we also have the equation $v_z - w_y = 0$. Hence, to go from the system back to the second order Laplace equation for a potential ψ we have a compatibility requirement on the solution. Converting this into a time dependent system we get

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_x + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_y + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_z = 0$$

Thus the "acoustic" portion of the three dimensional Euler equation corresponds to the reduction of the Laplace equation (actually wave equation) to a first order system. However, in distinction to two dimensions, this system is no longer "elliptic". To see this we start with the system form for three space dimensions and convert back to a second order equation (assuming u is constant).

Then q satisfies the wave equation $\frac{\partial^2 q}{\partial t^2} = u^2 \Delta q$. However r and s do not obey a wave equation. Instead we get

$$\begin{aligned} \frac{\partial^2 r}{\partial t^2} &= u^2 (\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 s}{\partial y \partial z}) \\ &= u^2 (\Delta r + \frac{\partial}{\partial z}(\frac{\partial s}{\partial y} - \frac{\partial r}{\partial z})) \\ \frac{\partial^2 s}{\partial t^2} &= u^2 (\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial z^2} + \frac{\partial^2 r}{\partial y \partial z}) \\ &= u^2 (\Delta s - \frac{\partial}{\partial y}(\frac{\partial s}{\partial y} - \frac{\partial r}{\partial z})) \end{aligned}$$

Differentiate the equation for r by y and the equation for s by z and add we find that $\frac{\partial^2 (r_y + s_z)}{\partial t^2} = u^2 \Delta (r_y + s_z)$, i.e.$(r_y + s_z)$ also satisfies the wave equation. Since q satisfies a wave equation so does q_x and hence so does the divergence $q_x + r_y + s_z$. Define $\omega = r_z - s_y$ i.e. the z component of the vorticity. Differentiating the equation for r by z and the equation for s by y and subtracting we find that $\omega_t + u\omega_x =$, Thus, we have two "elliptic" variables, q and div(q,r,s) and one "hyperbolic" variable ω in addition to the previous two hyperbolic variables

S and H. Hence, if we solve this three dimensional system with an initial condition that the vorticity is zero then the vorticity remains zero for all time. In this case we can go from the first order system back to the second order wave equation. An equivalent way to view this is to remember that the components of the first order system are the gradients of a potential ψ. In order for this potential to exist there are compatibilty conditions. In particular we need that the vorticity component ω be zero. In two dimensions the system is purely elliptic since ω is always identically zero. Hence, on a differential level the first order equivalent to the Laplace equation has a hyperbolic part. It is not clear how to do this split between elliptic and hyperbolic portions within an algebraic context.

3.3 Supersonic - three dimensional

In three dimensions the preconditioned equations are

$$\begin{aligned}
\frac{\beta M+1}{M^2}p_t + \frac{\rho a}{M}u_t + up_x + \rho a^2(u_x+v_y+w_z) &= 0\\
\frac{1}{\rho u}p_t + u_t + uu_x + \frac{p_x}{\rho} &= 0\\
\frac{M}{\beta}v_t + uv_x + \frac{p_y}{\rho} &= 0\\
\frac{M}{\beta}w_t + uw_x + \frac{p_z}{\rho} &= 0\\
S_t + uS_x &= 0
\end{aligned}$$

Solving for (p,u,v,w,S) we get

$$\begin{aligned}
p_t + \frac{\beta u}{M}p_x + \frac{\rho a^2 M}{\beta}(v_y+w_z) &= 0\\
u_t + uu_x - \frac{1}{\rho}\left(\frac{\beta}{M}-1\right)p_x - \frac{a}{\beta}(v_y+w_z) &= 0\\
v_t + \frac{\beta}{M}(uv_x + \frac{p_y}{\rho}) &= 0\\
w_t + \frac{\beta}{M}(uw_x + \frac{p_z}{\rho}) &= 0\\
S_t + uS_x &= 0
\end{aligned}$$

As before the system for (p,v,w) decouples from the rest. Written in vector form we get

$$(27)\quad \begin{pmatrix} \frac{\beta}{\rho u}dp \\ dv \\ dw \end{pmatrix}_t + \frac{\beta u}{M}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\beta}{\rho u}dp \\ dv \\ dw \end{pmatrix}_x + \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\beta}{\rho u}dp \\ dv \\ dw \end{pmatrix}_y$$

$$(28)\quad + \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\beta}{\rho u}dp \\ dv \\ dw \end{pmatrix}_z = 0$$

Attempting the same variables as before we find

$$\begin{aligned} S_t + uS_x &= 0 \qquad dS = dp - a^2 d\rho \\ H_t + uH_x &= 0 \qquad dH = dp + \frac{\rho}{2}(u^2) \\ (\partial_t + \frac{\beta u}{M}\partial_x + \frac{u}{M}\partial_y + \frac{u}{M}\partial_z)(\beta dp + \rho u(dv + dw)) &= \frac{\rho u^2}{M}(w_y + v_z) \\ (\partial_t + \frac{\beta u}{M}\partial_x - \frac{u}{M}\partial_y - \frac{u}{M}\partial_z)(\beta dp - \rho u(dv + dw)) &= \frac{\rho u^2}{M}(w_y + v_z) \end{aligned}$$

We see that these equations are no longer uncoupled due to the right hand side of $\frac{\rho u^2}{M}(w_y + v_z)$. To get the additional convective quantity define the vorticity (component) as $\omega = v_z - w_y$. Then we find from the equations for v and w, that $\omega_t + u\omega_x = 0$. So ω is convected. We are left with three equations in the form

$$w_t + w_x + Bw_y + Cw_z = 0$$

with

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Since the x direction reduces to the identity matrix we can introduce new variables $\tau = t$ and $\xi = x - t$. In terms of the new variables we have

$$W_\tau + BW_y + CW_z = 0 \quad , \qquad W = (q, r, s)^t$$

This is just the subsonic case considered above. Hence, q satisfies a wave equation and r, s satisfy

$$\begin{aligned}
\frac{\partial^2 r}{\partial t^2} &= \Delta r + \frac{\partial}{\partial z}(\frac{\partial s}{\partial y} - \frac{\partial r}{\partial z}) \\
\frac{\partial^2 s}{\partial t^2} &= \Delta s - \frac{\partial}{\partial y}(\frac{\partial s}{\partial y} - \frac{\partial r}{\partial z})
\end{aligned}$$

So, $r_y + s_z$ satisfies a two dimensional wave equation while $(s_y - r_x)_\tau = 0$ in the moving coordinate system. Hence, the vorticity component $s_y - r_x$ satisfies a one dimensional convection, in the steamwise direction, as we already have found.

Thus, the supersonic system consists of three convective equations for the entropy, enthalpy and vorticity that decouple. In addition we have a system of two equations that is a combined hyperbolic-elliptic system in the sense that it is an elliptic system in the y and z directions in a system moving in the x direction.

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